YOKOHAMA MATHEMATICAL
JOURNAL VOL. 27, 1979

A NON-UNIFORM RATE OF CONVERGENCE IN A LOCAL LIMIT THEOREM CONCERNING VARIABLES IN THE DOMAIN OF NORMAL ATTRACTION OF A STABLE LAW

By

SUJIT K. BASU, MAKOTO MAEJIMA and NISHITH K. PATRA

(Received December 20, 1978)

1. Introduction. Let $\{X_n\}$ be a sequence of independent and identically distributed (iid) random variables belonging to the domain of normal attraction of a stable distribution function (df) V_0 of index α, $1<\alpha<2$. We assume that $EX_1=0$. Thus there exists a constant $a>0$ such that $Z_n=(an^{\gamma})^{-1}\Sigma_{j=1}^{n}X_j$ converges in law to V_0, where $\gamma=\alpha^{-1}$.

Basu and Maejima [4] recently proved that if the df V_1 of X_1 is absolutely continuous with a probability density function (pdf) v_1 and its characteristic function (cf) $\omega_1(t)$ of X_1 is absolutely integrable in r-th power for some integer $r\geqq 1$, then for large n, the df V_n of Z_n is absolutely continuous with a pdf v_n such that

$$\sup_x (1+|x|^{\alpha})|v_n(x)-v_0(x)|=O(1) \tag{1.1}$$

as $n\to\infty$, v_0 being the pdf of V_0.

The present authors recently came across another result concerning the rate of convergence in such local limit theorems. From this result, due to Banys [1], it follows that if the df of V_1 of X_1 is absolutely continuous and further if

$$\int_{-\infty}^{\infty} x^2|v_1(x)-v_0(x)|dx<\infty\ , \tag{1.2}$$

then

$$\int_{-\infty}^{\infty} |v_n(x)-v_0(x)|dx=O(n^{1-2\gamma}) \tag{1.3}$$

as $n\to\infty$.

In this paper we study the consequence of the condition (1.2) on the pointwise convergence of v_n to v_0. In particular, we shall prove the following improved version of the non-uniform rate of convergence given by (1.1):

Theorem 1. *Let $\{X_n\}$ be a sequence of iid random variables each with an absolutely continuous df V_1, pdf v_1 and cf ω_1. Assume that $EX_1=0$. If* (i) *V_1 belongs* *to the domain of normal attraction of a stable df* V_0 *of index* α, $1<\alpha<2$, (ii) ω_1 *is absolutely integrable in r-th power for some* $r\geqq 1$ *and* (iii) *the relation* (1.2) *holds, then for all large n, the df* V_n *of* Z_n *is absolutely continuous with pdf* v_n *such that*

$$\sup_x (1+|x|^{\alpha})|v_n(x)-v_0(x)|=O(n^{1-2\gamma}) \tag{1.4}$$

as $n\to\infty$, *where* v_0 *is the pdf of* V_0 *and* $\gamma=\alpha^{-1}$.

This theorem also helps us to prove the following theorem concerning probabilities of moderate deviations in situations with non-normal limit distributions:

Theorem 2. *Under the conditions of Theorem* 1, *as* $n\to\infty$,

$$\frac{1-V_n(L_n)+V_n(-L_n)}{1-V_0(L_n)+V_0(-L_n)}=1+O(n^{1-2\gamma}L_n) \tag{1.5}$$

for any sequence $\{L_n\}$ *of reals such that* $L_n\uparrow\infty$.

2. Notations and preliminary lemmas. With no loss of generality, the constant a in the previous section may be taken to be equal to 1.

Let X_0 denote a stable random variable having the df V_0 and let ω_0 denote its cf. We write $\omega_n(t)=E\exp(itZ_n)$ so that $\omega_n(t)=\{\omega_1(tn^{-\gamma})\}^n$. For any function $g(t)$ and a positive integer k, we will write $g^{(k)}(t)$ to denote $(d/dt)^k g(t)$, whenever such a derivative exists.

Then since V_1 belongs to the domain of normal attraction of V_0,

$$\lim_{n\to\infty}\omega_n(t)=\omega_0(t) \tag{2.1}$$

for all t. Also as in Basu [2],

$$\lim_{n\to\infty}\omega_n^{(1)}(t)=\omega_0^{(1)}(t) \tag{2.2}$$

for all t. This implies, in particular, that $EX_0=0$, so that from the cannonical form of $\omega_0(t)$, we have

$$\omega_0(t)=\{\omega_0(tn^{-\gamma})\}^n \tag{2.3}$$

for all t.

For each integer n and real x, we define for $k=0, 1$,

$$\alpha_{kn}(t, x)=\int_{|u|\leq|x|n^{\gamma}}\exp(itu)dV_k(u)\ , \tag{2.4}$$

$$\beta_{kn}(t, x)=\omega_k(t)-\alpha_{kn}(t, x)\ , \tag{2.5}$$

$$A_{kn}(t, x)=\{\alpha_{kn}(tn^{-\gamma}, x)\}^n\ , \tag{2.6}$$

$$B_{1n}(t, x)=\omega_n(t)-A_{1n}(t, x)=\sum_{j=1}^{n}\binom{n}{j}\{\alpha_{1n}(tn^{-\gamma}, x)\}^{n-j}\{\beta_{1n}(tn^{-\gamma}, x)\}^j \tag{2.7}$$

and

$$B_{0n}(t, x)=\omega_0(t)-A_{0n}(t, x)=\sum_{j=1}^{n}\binom{n}{j}\{\alpha_{0n}(tn^{-\gamma}, x)\}^{n-j}\{\beta_{0n}(tn^{-\gamma}, x)\}^j . \tag{2.8}$$

The last equality in (2.8) follows from (2.3) and (2.5).

Note that for each fixed n and x, $\alpha_{kn}(t, x)$, $k=0, 1$, are differentiable any number of times under the integral sign. Whenever the following inversion integrals are absolutely convergent, we set

$$v_n(u)=(2\pi)^{-1}\int_{-\infty}^{\infty}\omega_n(t)\exp(-itu)dt , \tag{2.9}$$

$$a_{kn}(u, x)=(2\pi)^{-1}\int_{-\infty}^{\infty}A_{kn}(t, x)\exp(-itu)dt , \qquad k=0, 1 , \tag{2.10}$$

$$b_{kn}(u, x)=(2\pi)^{-1}\int_{-\infty}^{\infty}B_{kn}(t, x)\exp(-itu)dt , \qquad k=0, 1 , \tag{2.11}$$

for each integer n and reals u and x. In fact, the absolutely convergent integrals (2.9) provide the continuous pdfs that we shall use and are subjects of our theorem.

In what follows, $c, c_0, c_1, \cdots$ or $C, C_0, C_1, \cdots$ etc. will denote some positive constants independent of n and x and their meanings are not of much importance and may change from one step to another.

We need the following lemmas.

Lemma 1. *For $k=0, 1$,*

$$z^{\alpha}R_k(z)\equiv z^{\alpha}P(|X_k|>z)\to c_k>0 , \tag{2.12}$$

$$\int_{|u|>z}|u|dV_k(u)=O(z^{-\alpha+1}) \tag{2.13}$$

and

$$\int_{|u|\le z}u^2dV_k(u)=O(z^{-\alpha+2}) \tag{2.14}$$

as $z\to\infty$.

Proof. While (2.12) is well-known (see e.g. Gnedenko and Kolmogorov [5]), the proofs of (2.13) and (2.14) can be found in Basu and Maejima [4].

Lemma 2. *There exist positive constants ε, sufficiently small, c and C such that for $k=0, 1$,*

$$|A_{kn}(t, x)| \leqq C \exp(-c|t|^{\alpha})$$

for all t in the range $|t| \leqq \varepsilon n^{\gamma}$, *all* x *with* $|x| \geqq 1$ *and all large* n.

Proof. See Basu and Maejima [4].

Lemma 3. *Let* $\varepsilon > 0$ *and* $c > 0$ *be as in Lemma 2. Then under* (1.2) *there exists a constant* $C > 0$ *such that*

$$|\omega_n(t) - \omega_0(t)| \leqq C n^{1-2\gamma} t^2 \exp(-c|t|^{\alpha})$$

for all t in the range $|t| \leqq \varepsilon n^{\gamma}$, *all* x *with* $|x| \geqq 1$ *and all large* n.

Proof. This is Lemma 1 in Banys [1].

Lemma 4. *Let the positive constants* ε *and* c *be as in Lemma 2. Then under* (1.2), *there exist polynomials* $P_1(\cdot)$, $P_2(\cdot)$ *and* $P_3(\cdot)$ *with non-negative coefficients independent of* n *and* x *such that for all large* n, *the relations*

$$|\alpha_{1n}^{n-j}(tn^{-\gamma}, x) - \alpha_{0n}^{n-j}(tn^{-\gamma}, x)| \leqq n^{1-2\gamma} P_1(|t|) \exp\{-c|t|^{\alpha}(n-j)/n\}, \qquad 1 \leqq j \leqq n$$

and

$$|A_{1n}^{(2)}(t, x) - A_{0n}^{(2)}(t, x)| \leqq n^{1-2\gamma}\{P_2(|t|) + |x|^{2-\alpha} P_3(|t|)\} \exp(-c|t|^{\alpha})$$

hold for all t in the range $|t| \leqq \varepsilon n^{\gamma}$ *and all* x *with* $|x| \geqq 1$.

Proof. The proof is similar to that of Lemma 2.3 of Basu and Maejima [4]. In fact, the adjustments necessary are rather easy in view of the condition (1.2).

The follwing lemma is being quoted here from Smith and Basu [11] for ready reference.

Lemma 5. *Let* $\varepsilon > 0$ *and integer* n_0 *be fixed. Then if* Θ *is the set where* $|x| \geqq 1$, $|t| \geqq \varepsilon$, $n \geqq n_0$ *and*

$$\lambda_k \equiv \sup_{\Theta} |\alpha_{kn}(t, x)|,$$

it follows that $0 \leqq \lambda_k < 1$, $k = 0, 1$.

3. Proof of Theorem 1. Observe that because of assumption (ii) in Theorem 1, the integral (2.9) is absolutely convergent for all large n. Further a direct application of Parseval's identity reveals that under the assumptions of the theorem, the integrals in (2.10) and (2.11) are absolutely convergent for all large n and $x \neq 0$ (see e.g. Smith and Basu [11]) so that for all large n,

$$v_n(x) = a_{1n}(x, x) + b_{1n}(x, x) \tag{3.1}$$

and

$$v_0(x) = a_{0n}(x, x) + b_{0n}(x, x). \tag{3.2}$$

We shall first prove the relation

$$\sup_x |v_n(x)-v_0(x)|=O(n^{1-2\gamma}) \tag{3.3}$$

as $n\to\infty$. In view of (3.3), then it is enough to prove that

$$\sup_{|x|\geq 1} |x|^\alpha |b_{1n}(x, x)-b_{0n}(x, x)|=O(n^{1-2\gamma}) \tag{3.4}$$

and

$$\sup_{|x|\geq 1} |x|^\alpha |a_{1n}(x, x)-a_{0n}(x, x)|=O(n^{1-2\gamma}) \tag{3.5}$$

as $n\to\infty$. The relations (3.3), (3.4) and (3.5) together imply (1.4).

To prove (3.3) we observe that

$$2\pi|v_n(x)-v_0(x)|\leqq I_{1n}+I_{2n}+I_{3n} \tag{3.6}$$

where

$$I_{1n}=\int_{|t|\leqq \varepsilon n^\gamma} |\omega_n(t)-\omega_0(t)|dt ,$$

$$I_{2n}=\int_{|t|>\varepsilon n^\gamma} |\omega_n(t)|dt$$

and

$$I_{3n}=\int_{|t|>\varepsilon n^\gamma} |\omega_0(t)|dt ,$$

$\varepsilon>0$ being as in Lemma 3.

By Lemma 3 it now follows that

$$I_{1n}=O(n^{1-2\gamma}) \tag{3.7}$$

as $n\to\infty$. That

$$I_{2n}=O(n^{1-2\gamma}) \tag{3.8}$$

and

$$I_{3n}=O(n^{1-2\gamma}) \tag{3.9}$$

as $n\to\infty$, follow as consequences of the fact that both V_1 and V_0 are absolutely continuous and also the relation (2.3) holds. Thus (3.3) follows from relations (3.6) through (3.9).

We now turn to the proof of (3.4). Towards that we first note that as in Smith and Basu [11], for all $x\neq 0$ and $k=0, 1$,

$$\int_{-\infty}^{\infty} |\alpha_{kn}(t, x)|^n dt=O(n^{-\gamma}) , \tag{3.10}$$

$$\int_{-\infty}^{\infty} |\beta_{kn}(t, x)|^s dt=O(1) , \tag{3.11}$$

where s is a very large number.

Now

$$2\pi\{b_{1n}(x,x)-b_{0n}(x,x)\}=\{\Sigma'+\Sigma''+\Sigma'''\}\binom{n}{j}\int_{-\infty}^{\infty}\{\alpha_{1n}^{n-j}(tn^{-\gamma},x)\beta_{1n}^{j}(tn^{-\gamma},x) \tag{3.12}$$
$$-\alpha_{0n}^{n-j}(tn^{-\gamma},x)\beta_{0n}^{j}(tn^{-\gamma},x)\}\exp(-itx)dt$$
$$\equiv J_{1n}(x)+J_{2n}(x)+J_{3n}(x)\ ,$$

say, where Σ', Σ'', Σ''' denote summation over ranges $1\leqq j\leqq[n/2]$, $[n/2]+1\leqq j\leqq n-2s$ and $n-2s+1\leqq j\leqq n$ respectively, s being some fixed integer with $n>2s$. Here $[y]$ means 'the integer part of y'.

Next

$$J_{1n}(x)=\Sigma'\binom{n}{j}\int_{-\infty}^{\infty}\alpha_{1n}^{n-j}(tn^{-\gamma},x)\{\beta_{1n}^{j}(tn^{-\gamma},x)-\beta_{0n}^{j}(tn^{-\gamma},x)\}\exp(-itx)dt \tag{3.13}$$
$$+\Sigma'\binom{n}{j}\int_{-\infty}^{\infty}\beta_{0n}^{j}(tn^{-\gamma},x)\{\alpha_{1n}^{n-j}(tn^{-\gamma},x)-\alpha_{0n}^{n-j}(tn^{-\gamma},x)\}\exp(-itx)dt$$
$$\equiv J_{11n}(x)+J_{12n}(x)\ ,$$

say.

In view of the condition (1.2) and Lemma 1 for $j\geqq1$ and $|x|\geqq1$,

$$\begin{aligned}&|\beta_{1n}^{j}(tn^{-\gamma},x)-\beta_{0n}^{j}(tn^{-\gamma},x)|\\&=|\{\beta_{1n}(tn^{-\gamma},x)-\beta_{0n}(tn^{-\gamma},x)\}\\&\quad\times\{\beta_{1n}^{j-1}(tn^{-\gamma},x)+\beta_{1n}^{j-2}(tn^{-\gamma},x)\beta_{0n}(tn^{-\gamma},x)+\cdots+\beta_{0n}^{j-1}(tn^{-\gamma},x)\}|\\&\leqq\{C_1x^{-2}n^{-2\gamma}\}\{R_1^{j-1}(|x|n^{\gamma})+R_1^{j-2}(|x|n^{\gamma})R_0(|x|n^{\gamma})+\cdots+R_0^{j-1}(|x|n^{\gamma})\}\\&\leqq\{C_1x^{-2}n^{-2\gamma}\}\{j(C_2|x|^{-\alpha}n^{-1})^{j-1}\}\\&=Cn^{-2\gamma}j(C_2/n)^{j-1}x^{-2}\ .\end{aligned}\tag{3.14}$$

Using (3.10) and (3.14), we then have, as $n\to\infty$,

$$\begin{aligned}|J_{11n}(x)|&\leqq Cn^{-2\gamma}x^{-2}\Sigma'\binom{n}{j}j(C_2/n)^{j-1}\\&\leqq Cx^{-2}n^{1-2\gamma}\ .\end{aligned}\tag{3.15}$$

Now, by Lemma 4, for all x in the range $|x|\geqq1$, as $n\to\infty$,

$$\begin{aligned}&\int_{|t|\leqq\epsilon n^{\gamma}}\{\alpha_{1n}^{n-j}(tn^{-\gamma},x)-\alpha_{0n}^{n-j}(tn^{-\gamma},x)\}\exp(-itx)dt\\&=O(n^{1-2\gamma})\ ,\end{aligned}\tag{3.16}$$

provided $1\leqq j\leqq[n/2]$. Further for each j, $1\leqq j\leqq[n/2]$,

$$|\alpha_{1n}(tn^{-\gamma},x)|^{n-j-m-1}|\alpha_{0n}(tn^{-\gamma},x)|^{m}\leqq|\alpha_{kn}(tn^{-\gamma},x)|^{[n/4]}\ , \tag{3.17}$$

where $k=1$ or 0 according as $1\leqq m\leqq[n/4]$ or $[n/4]+1\leqq m\leqq n-j-1$. Also from the integrability of $|\omega_k(t)|^r$, $r\geqq 1$, $k=0, 1$ and the relation (3.11) it follows that for all large n and some fixed $h>0$,

$$\int_{|t|>\varepsilon n^\gamma}|\alpha_{kn}(tn^{-\gamma}, x)|^{[n/4]}dt\leqq C\lambda^{[n/4]-h}, \tag{3.18}$$

where $\lambda=\max(\lambda_0, \lambda_1)$, λ_0 and λ_1 being defined as in Lemma 5. In view of Lemma 5, then (3.17) and (3.18) imply that

$$\begin{aligned}&\int_{|t|>\varepsilon n^\gamma}\{\alpha_{1n}^{n-j}(tn^{-\gamma}, x)-\alpha_{0n}^{n-j}(tn^{-\gamma}, x)\}\exp(-itx)dt\\ &\quad=O(n^{1-2\gamma})\end{aligned} \tag{3.19}$$

as $n\to\infty$.

Using (3.16), (3.19) and Lemma 1, we now observe that for all $|x|\geqq 1$,

$$\begin{aligned}|J_{12n}(x)|&\leqq Cn^{1-2\gamma}\Sigma'\binom{n}{j}R_0^j(|x|n^\gamma)\\ &\leqq C|x|^{-\alpha}n^{1-2\gamma},\end{aligned} \tag{3.20}$$

for all large n.

Similarly, again because of (3.10) and Lemma 1,

$$\begin{aligned}|J_{2n}(x)|&\leqq C\Sigma''\binom{n}{j}\{R_1^j(|x|n^\gamma)+R_0^j(|x|n^\gamma)\}n^\gamma\\ &\leqq C|x|^{-\alpha}n^{1-2\gamma},\end{aligned} \tag{3.21}$$

for all large n and all x in the range $|x|\geqq 1$.

Finally using (3.11), we find that for $n>6s$, $|x|\geqq 1$,

$$\begin{aligned}|J_{3n}(x)|&\leqq C\Sigma'''n^{2s-2}\{R_1^{j-2s}(|x|n^\gamma)+R_0^{j-2s}(|x|n^\gamma)\}\\ &\leqq Cn^{2s-2}(|x|^{-\alpha}n^{-1})^{n-4s+1}\\ &\leqq C|x|^{-\alpha}n^{1-2\gamma}.\end{aligned} \tag{3.22}$$

The relation (3.4) now follows from relations (3.12) through (3.22).

It now remains to prove (3.5). For each $k=0, 1$, simple calculation together with the integrability of $|\omega_k(t)|^r$, $r\geqq 1$, informs us that $A_{kn}^{(2)}(t, x)$ is also absolutely integrable for all large n and $x\neq 0$. Hence

$$\begin{aligned}&2\pi|x|^\alpha|a_{1n}(x, x)-a_{0n}(x, x)|\\ &\quad=|x|^{\alpha-2}\left|\int_{-\infty}^{\infty}\{A_{1n}^{(2)}(t, x)-A_{0n}^{(2)}(t, x)\}\exp(-itx)dt\right|\\ &\quad\leqq|x|^{\alpha-2}\{I_{1n}^*(x)+I_{2n}^*(x)\},\end{aligned} \tag{3.23}$$

where

$$I_{1n}^{*}(x)=\int_{|t|\leq \varepsilon n^{\gamma}}|A_{1n}^{(2)}(t,x)-A_{0n}^{(2)}(t,x)|dt$$

and

$$I_{2n}^{*}(x)=\int_{|t|>\varepsilon n^{\gamma}}|A_{1n}^{(2)}(t,x)-A_{0n}^{(2)}(t,x)|dt\ ,$$

$\varepsilon>0$ being as in Lemma 4.

Since $\alpha<2$, we observe that as $n\to\infty$,

$$\sup_{|x|\geq 1}|x|^{\alpha-2}I_{1n}^{*}(x)=O(n^{1-2\gamma}) \tag{3.24}$$

as a consequence of Lemma 4. That

$$\sup_{|x|\geq 1}|x|^{\alpha-2}I_{2n}^{*}(x)=O(n^{1-2\gamma})\ , \tag{3.25}$$

as $n\to\infty$, follows by exactly the same way as in Basu and Maejima [4]. Relations (3.23) through (3.25) imply (3.5).

This completes the proof of the theorem.

4. **Proof of Theorem 2.** In view of our result (1.4),

$$\sup_{x}|x|^{\alpha-1}|V_n(x)-V_0(x)|=O(n^{1-2\gamma})\ . \tag{4.1}$$

Theorem 2 now easily follows from (4.1) and (2.12).

5. **Remarks.** Given the above results, it seems natural to ask if similar results (stated in Theorem 3 below) can be obtained for the case $0<\alpha\leqq 1$ through similar calculations. Of course the lemmas necessary in this case have to be modified accordingly (see Basu and Maejima [4]). In this case, we need the following.

Definition. A df V_0 is called strictly stable if for any $a_1, a_2>0$, there exists a constant $a>0$ such that

$$V_0(a_1x)*V_0(a_2x)=V_0(ax)\ ,$$

where $*$ means the convolution of distributions.

We recall the representation of cf of a stable df:

$$\omega_0(t)=\exp\left\{it\xi-c|t|^{\alpha}\left(1-i\beta\frac{|t|}{t}\psi(t,\alpha)\right)\right\}\ , \tag{5.1}$$

where α, β, ξ, c are constants (ξ is any real number, $0<\alpha<2$, $|\beta|\leqq 1$, $c\geqq 0$) and $\psi(t,\alpha)=\tan(\pi\alpha/2)$ or $2\pi^{-1}\log|t|$ according as $\alpha\neq 1$ or $\alpha=1$. Then V_0 is strictly stable if and only if

$$\beta=0 \text{ in (5.1)} \quad \text{for} \quad \alpha=1\,,$$
$$\xi=0 \text{ in (5.1)} \quad \text{for} \quad \alpha\neq 1\,.$$

(See Lukacs [6].) It is easily seen that if V_0 is strictly stable then the corresponding cf ω_0 satisfies (2.3), on which our method of proof of Theorem 1 depended largely. Also, in the case $1<\alpha<2$, if $EX_1=0$, then V_0 is strictly stable and conversely.

The assumption that $EX_1=0$ in Theorem 1 was made merely to give some simplifications in its proof and it does not in any way restrict the scope of our result. In fact, at the expense of a little more algebra (see Section 2 in Basu and Maejima [4]), a similar approach leads to the following theorem. This time, it is assumed that $Z_n=(an^{1/\alpha})^{-1}\Sigma_{j=1}^{n}X_j-M_n$ converges in law to a stable df V_0 for some sequence of reals $\{M_n\}$, although $M_n=0$ in Theorem 1.

Theorem 3. *Let $\{X_n\}$ be a sequence of iid random variables each with an absolutely continuous df V_1, pdf v_1 and cf ω_1. Suppose that* (i) *V_1 belongs to the domain of normal attraction of a stable df V_0 of index α or of a strictly stable df V_0 of index α, according as $1<\alpha<2$ or $0<\alpha\leqq 1$,* (ii) *ω_1 is absolutely integrable in r-th power for some $r\geqq 1$ and* (iii)

$$\int_{-\infty}^{\infty}|x|^{[\alpha]+1}|v_1(x)-v_0(x)|dx<\infty\,,$$

where v_0 is the pdf of V_0. Then for large n, the df V_n of Z_n is absolutely continuous with pdf v_n such that

$$\sup_x\,(1+|x|^{\alpha})|v_n(x)-v_0(x)|=O(n^{1-([\alpha]+1)/\alpha})\,.$$

Finally, it may be mentioned here that in the case $\alpha=2$ (i.e. equivalently when $EX_1^2<\infty$ and V_0 is the standard normal df) much stronger results concerning the non-uniform rate of convergence in the local limit theorem are available. For example, Maejima [7] gave a necessary and sufficient condition for

$$\text{(5.2)} \qquad \begin{aligned}&\sup_x\,\{1+x^2M(|x|n^{1/2})/M(n^{1/2})\}|v_n(x)-v_0(x)|\\ &\quad=O(M^{-1}(n^{1/2}))\end{aligned}$$

to hold as $n\to\infty$, when $M(x)=|x|^{\delta}$, $0<\delta\leqq 1$, and further Basu [3] has generalized it for a more general class of $M(x)$.

Also, results concerning probabilities of moderate deviations are well-known (see e.g. Rudin and Sethuraman [10], Michel [8], Patra and Basu [9]). Our

Theorem 2 above seeks to provide counterparts of such results in the set up discribed above.

References

[1] Banys, J.: *An estimation of the convergence rate of the limiting stable law in metric* L_1. Litovsk. Mat. Sb., **18**, no. 1 13–18 (1977). (In Russian.)

[2] Basu, Sujit K.: *On a local limit theorem concerning variables in the domain of normal attraction of a stable law of index* α, $1<\alpha<2$. Ann. Probability, **4**, 486–489 (1976).

[3] Basu, Sujit K.: *A non-uniform rate of convergence in a local limit theorem.* Math. Proc. Camb. Phil. Soc., **84**, 351–359 (1978).

[4] Basu, Sujit K. and Maejima, M.: *A local limit theorem for attractions under a stable law.* To appear in Math. Proc. Camb. Phil. Soc.

[5] Gnedenko, B. V. and Kolmogorov, A. N.: *Limit Distributions for Sums of Independent Random Variables.* (Revised ed.) Addison-Wesley, Reading, Mass., 1968.

[6] Lukacs, E.: *Characteristic Functions.* (2nd ed.) Griffin, London, 1970.

[7] Maejima, M.: *A non-uniform estimate in the local limit theorem for densities*, II. Yokohama Math. J., **26**, 119–135 (1978).

[8] Michel, R.: *Results on probabilities of moderate deviations.* Ann. Probability, **2**, 349–353 (1974).

[9] Patra, N. K. and Basu, Sujit K.: *On certain generalizations of results on probabilities of moderate deviations.* Submitted.

[10] Rubin, H. and Sethuraman, J.: *Probabilities of moderate deviations.* Sankhyā, **A27**, 325–346 (1965).

[11] Smith, W. L. and Basu, Sujit K.: *General moment functions and a density version of the central limit theorem.* Proc. Camb. Phil. Soc., **75**, 365–381 (1974).

Indian Institute of Management Calcutta
Post Box No. 16757
Alipore Post Office
Calcutta-700027, India

Department of Mathematics
Faculty of Engineering
Keio University
3-14-1, Hiyoshi, Kohoku-ku
Yokohama 223, Japan

Department of Statistics
Asutosh College
92, S.P. Mukherjee Road
Calcutta-700026, India