YOKOHAMA MATHEMATICAL
JOURNAL VOL. 27, 1979

SOME ESTIMATES OF THE CONCENTRATION FUNCTION OF SUMS OF ABSOLUTELY REGULAR SEQUENCES

By

KEN-ICHI YOSHIHARA

(Received December 19, 1978)

1. Introduction. The concentration function of a real-valued random variable X is defined to be

(1.1) $$Q(X;\lambda)=\sup_x P(x\leqq X\leqq x+\lambda)\ .$$

Some properties of $Q(X;\lambda)$ are shown in Hengartner and Theodorescu [3] and Petrov [5].

Let $\{\xi_i\}$ be a strictly stationary, absolutely regular sequence of random variables, i.e., $\{\xi_i\}$ satisfies the absolute regularity condition

(1.2) $$\beta(n)=E\{\sup_{A\in\mathscr{M}_n^\infty}|P(A|\mathscr{M}_{-\infty}^0)-P(A)|\}\downarrow 0 \qquad (n\to\infty)$$

where $\mathscr{M}_a^b$ denotes the σ-algebra of events generated by $\xi_a,\dots,\xi_b$ $(a\leqq b)$.

Next, let

(1.3) $$S_n=\sum_{j=1}^n \xi_j \qquad \text{and} \qquad S_0=0\ .$$

In this paper, we shall estimate $Q(S_n;\lambda n^{1/\alpha})$ $(\alpha>0,\ \lambda>0)$.

2. Main result. Let $\{\xi_j\}$ be a strictly stationary, absolutely regular process with coefficient $\beta(n)$.

(2.1) $$E|\xi_1|^\alpha<\infty\ .$$

Let

(2.2) $$\gamma=\begin{cases}\alpha & \text{if}\ \ 0<\alpha\leqq 1\ ,\\ \alpha' & \text{if}\ \ 1<\alpha'<\alpha\leqq 2\ ,\\ 2 & \text{if}\ \ \alpha>2\ ,\end{cases}$$

and put

(2.3) $$B_n(\gamma)=cn^{1/\gamma}$$

where c is a positive constant.

We shall consider the following conditions:

Condition A. (AI). $\{\xi_j\}$ is absolutely regular, (2.1) holds with α $(0<\alpha\leqq 1)$ and $\beta(n)$ is arbitrary;

(AII). $\{\xi_j\}$ is absolutely regular, (2.1) holds with α $(1<\alpha\leqq 2)$, $E\xi_1=0$ and $\sum_1^\infty \{\beta(n)\}^{1-(\alpha'/\alpha)}<\infty$ for some α' $(0<\alpha'<\alpha)$;

(AIII). $\{\xi_j\}$ is absolutely regular, (2.1) holds with α $(\alpha>2)$, $E\xi_1=0$ and $\sum_1^\infty \{\beta(n)\}^{(\alpha-2)/\alpha}<\infty$.

Condition B. There exist functions $p=p(n)$ and $q=q(n)$ for which the following relations hold:

(i) $p\to\infty$, $q\to\infty$, $k\to\infty$, and $p/q\to\infty$ as $n\to\infty$;

(ii) $\max\{kq/n,\ k\beta(q)\}=O(k^{-1/\gamma})$

where $k=[n/(p+q)]$ and $[s]$ denotes the largest integer m such that $m\leqq s$.

We shall prove the following theorem.

Theorem. *Let $\{\xi_j\}$ be a strictly stationary absolutely regular process with coefficient $\beta(n)$. Let Condition B and one of Conditions* (AI)-(AIII) *be satisfied. Suppose that the distribution of $S_n/B_n(\gamma)$ converges weakly to a stable distribution with exponent γ then the inequalities*

$$K(\lambda)k^{-1/\gamma}\leqq Q(S_n:\ \lambda B_n(\gamma))\leqq M([\lambda]+1)k^{-1/\gamma} \tag{2.4}$$

hold, where λ is a positive number, $K(\lambda)$ is some positive constant depending on λ and M is an absolute constant.

3. Auxiliary results. In what follows, by the letter M, we shall denote any quantity (not always the same) which is bounded in absolute value. Let $\{\xi_j\}$ be a strictly stationary, absolutely regular process with coefficient $\beta(n)$. The following lemma is easily proved by the method of the proof of Theorem 1 in Yoshihara [6].

Lemma 1. *Let p, q and k be arbitrary positive integers. Let $\{\eta_j\}$ be a family of random variables such that for each j η_j is measurable with respect to $\mathscr{M}_{(j-1)(p+q)+1}^{(j-1)(p+q)+p}$. Then, for any k*

$$P(|\sum_{j=1}^k Y_j|\in A)-2k\beta(p)\leqq P(|\sum_{j=1}^k \eta_j|\in A)\leqq P(|\sum_{j=1}^k Y_j|\in A)+2k\beta(p) \tag{3.1}$$

where A is a Borel set of the real line and $\{Y_j\}$ is a family of independent random variables such that for each j Y_j has the same distribution as that of η_j.

Next, we shall prove the following

Lemma 2. *Let $\{\xi_i\}$ be absolutely regular. Suppose that one of Conditions* (AI)-

(AIII) *is satisfied. Then, for any positive integer* m

$$E|\sum_{i=1}^{m} \xi_i|^{r} \leqq Mm \tag{3.2}$$

where γ *is the number defined in* (2.2).

Proof. (3.2) holds obviously under Condition (AI) or (AIII).

Now, we shall prove (3.2) under Condition (AII). Let

$$\zeta_j = \begin{cases} \xi_j & \text{if } |\xi_j| < m^{1/\alpha'}, \\ 0 & \text{otherwise}, \end{cases}$$

$\bar{\zeta}_j = \zeta_j - E\zeta_j$ and $\bar{\bar{\zeta}}_j = \xi_j - \zeta_j$. Since, by assumption, $E\xi_j = 0$, so $E\bar{\bar{\zeta}}_j = 0$. We note that

$$E(\sum_{j=1}^{m} \bar{\zeta}_j)^2 \leqq MmE\zeta_1^2 .$$

So

$$\begin{aligned} E|\sum_{j=1}^{m} \bar{\zeta}_j|^{\alpha'} &\leqq \{E|\sum_{j=1}^{m} \bar{\zeta}_j|^2\}^{\alpha'/2} \leqq Mm^{\alpha'/2}\{E\zeta_0^2\}^{\alpha'/2} \\ &\leqq Mm^{\alpha'/2}[\{m^{1/\alpha'}\}^{2-\alpha'} E|\xi_1|^{\alpha'}]^{\alpha'/2} \leqq MmE|\xi_1|^{\alpha'} . \end{aligned} \tag{3.3}$$

Since the inequality $|a+b|^r \leqq |a|^r + |b|^r$ holds for any two numbers a and b and for any $0 < r \leqq 1$, so we have $|\sum_{j=1}^{m} \bar{\bar{\zeta}}_j|^{\alpha'-1} \leqq \sum_{j=1}^{m} |\bar{\bar{\zeta}}_j|^{\alpha'-1}$. Hence, by Lemma 2.1 in Davydov [1] and Condition (AII) we have

$$\begin{aligned} E|\sum_{j=1}^{m} \bar{\bar{\zeta}}_j|^{\alpha'} &\leqq E(\sum_{i=1}^{m} |\bar{\bar{\zeta}}_i|)|\sum_{j=1}^{m} \bar{\bar{\zeta}}_j|^{\alpha'-1} \leqq E(\sum_{i=1}^{m} |\bar{\bar{\zeta}}_i|)(\sum_{j=1}^{m} |\bar{\bar{\zeta}}_j|^{\alpha'-1}) \\ &= E(\sum_{i=1}^{m} |\bar{\bar{\zeta}}_i|^{\alpha'} + \sum_{i \neq j} |\bar{\bar{\zeta}}_i||\bar{\bar{\zeta}}_j|^{\alpha'-1}) \\ &\leqq M[\sum_{i=1}^{m} E|\bar{\bar{\zeta}}_i|^{\alpha'} + \sum_{i \neq j} \{E|\bar{\bar{\zeta}}_0|E|\bar{\bar{\zeta}}_0|^{\alpha'-1} + 10(E|\bar{\bar{\zeta}}_0|^{\alpha})^{\alpha'/\alpha}(\beta(|j-i|))^{1-(\alpha'/\alpha)}\}] \\ &\leqq M[mE|\bar{\bar{\zeta}}_0|^{\alpha'} + m^2 m^{-(2\alpha-\alpha')/\alpha'}(E|\bar{\bar{\zeta}}_0|^{\alpha})^2 \\ &\quad + m(E|\bar{\bar{\zeta}}_0|^{\alpha})^{\alpha'/\alpha} \sum \{\beta(j)\}^{1-(\alpha'/\alpha)}] \leqq Mm . \end{aligned} \tag{3.4}$$

Hence, from (3.3) and (3.4)

$$E|\sum_{j=1}^{m} \xi_j|^{\alpha'} \leqq M[E|\sum_{j=1}^{m} \bar{\zeta}_j|^{\alpha'} + E|\sum_{j=1}^{m} \bar{\bar{\zeta}}_j|^{\alpha'}] \leqq Mm ,$$

which implies (3.2). Thus, the proof is completed.

4. Proof of Theorem. Let $\delta = \delta_n = (\log n)^{-1/2} (n \geqq 2)$. Let x be arbitrary. By Lemma 1

$$
\begin{aligned}
(4.1)\qquad P_0 &= P(x-B_n(\gamma)/2 \leqq S_n \leqq x+B_n(\gamma)/2) \\
&\leqq P(x-B_n(\gamma)(1+\delta)/2 \leqq \sum_{j=1}^{k}\sum_{i=1}^{p} \xi_{(j-1)(p+q)+i} \leqq x+B_n(\gamma)(1+\delta)/2) \\
&\quad + P(|\sum_{j=1}^{k}\sum_{i=1}^{q} \xi_{(j-1)(p+q)+p+i}| \geqq \delta B_n(\gamma)/4) + P(|S_n - S_{k(p+q)}| \geqq \delta B_n(\gamma)/4) \\
&\leqq P(x-B_n(\gamma)(1+\delta)/2 \leqq \sum_{j=1}^{k} Y_j \leqq x+B_n(\gamma)(1+\delta)/2) \\
&\quad + P(|\sum_{j=1}^{k} Z_j| \geqq \delta B_n(\gamma)/4) + P(|S_n - S_{k(p+q)}| \geqq \delta B_n(\gamma/4)) + 4k\beta(q) \\
&= P_1 + P_2 + P_3 + O(k^{-1/\gamma}) \qquad \text{(say)}
\end{aligned}
$$

where $\{Y_1, \ldots, Y_k\}$ ($\{Z_1, \ldots, Z_k\}$) is a family of i.i.d. random variables such that the distribution of $Y_1(Z_1)$ has the same distribution as that of $\sum_1^p \xi_i(\sum_1^q \xi_j)$.

Firstly, we shall estimate P_2. As $Z_1, \ldots, Z_k$ are i.i.d. random variables, so from Marcinkiewicz-Zygmund inequalities [4] and Lemma 2

$$(4.2)\qquad E|\sum_{j=1}^{k} Z_j|^\gamma \leqq Mk^{\gamma/2}E|Z_1|^\gamma \leqq Mk^{\gamma/2}q$$

and so

$$(4.3)\qquad P_2 \leqq M\delta^{-\gamma}B_n^{-\gamma}(\gamma)k^{\gamma/2}q \leqq Mk^{-1/\gamma}$$

Next, from Lemma 2

$$(4.4)\qquad P_3 \leqq M\delta^{-\gamma}B_n^{-\gamma}(\gamma)(n-k(p+q)) \leqq Mk^{-1/\gamma}\,.$$

Finally, since from Lemma 2 $E|Y_{n1}/B_n(\gamma)|^\gamma$ is bounded and $Y_{nj}/B_n(\gamma)$'s are i.i.d., so from Theorems 4.3 and 5.1 in Esseen [2] we have

$$(4.5)\qquad P_1 = P\left(B_n^{-1}(\gamma)x - \frac{1+\delta}{2} \leqq \sum_{j=1}^{k}(Y_{nj}/B_n(\gamma)) \leqq B_n^{-1}(\gamma)x + \frac{1+\delta}{2}\right) \leqq Mk^{-1/\gamma}\,.$$

Thus, from (4.1) and (4.3)-(4.5)

$$(4.6)\qquad P(x-B_n(\gamma)/2 \leqq S_n \leqq x+B_n(\gamma)/2) \leqq Mk^{-1/\gamma}\,,$$

which implies

$$(4.7)\qquad Q(S_n:\ B_n(\gamma)) \leqq Mk^{-1/\gamma}\,.$$

In general, for arbitrary positive numbers c and λ

$$Q(X;\ \lambda c) \leqq M([\lambda]+1)Q(X:\ c)$$

holds. So, from (4.7) we have the right-hand side inequality of (2.4).

Similarly, we have

(4.8) $$P_0 \geqq P_4 - P(|\sum_{j=1}^{k} Z_{nj}| \geqq B_n(\gamma)\delta/4) - P(|S_n - S_{k(p+q)}| \geqq B_n(\gamma)\delta/4) = P_4 + O(k^{-1/\gamma})$$

where

$$P_4 = P(x - B_n(\gamma)(1-\delta)/2 \leqq \sum_{j=1}^{k} Y_{nj} \leqq x + B_n(\gamma)(1-\delta)/2) .$$

As from Lemma 2 $E|Y_{n1}|^{\gamma} \leqq MB_n^{\gamma}(\gamma)$ and Y_{nj}'s are i.i.d., so from Theorems 4.3 and 5.1 in Esseen [2] we have

(4.9) $$P_4 \geqq Mk^{-1/\gamma} .$$

Hence, from (4.8) and (4.9), we have the left-hand inequality in (2.4). Thus the proof is completed.

References

[1] Yu. A. Davydov: *Convergence of distributions generated by stationary processes.* Theory Probab. Appl., **13**, 691–696 (1968).

[2] C. G. Esséen: *On the concentration function of a sum of independent random variables.* Z. Wahrscheinlichkeitstheorie verw. Gebiete **9**, 290–308 (1968).

[3] W. Hengartner and R. Theodorescu: *Concentration functions.* Academic Press. New York (1973).

[4] J. Marcinkiewicz and A. Zygmund: *Sur les fonctions independentes.* Fund. Math., **29**, 60–90 (1937).

[5] V. V. Petrov: *Sums of independent random variables.* Springer-Verlag. Berlin Heidelberg (1975).

[6] M. Udagawa and K. Yoshihara: *Weak convergence of random elements in $D[0, \infty)$ defined by the partial sums of stationary sequences.* Sci. Rep. Tokyo Kyoiku Daigaku Sec. A, **11**, No. 294 (1972).

[7] K. Yoshihara: *Probability inequalities for sums of absolutely regular processes and their applications.* Z. Wahrscheinlichkeitstheorie verw. Gebiete., **43**, 319–330 (1978).

Yokohama National University
Department of Mathematics,
Faculty of Engineering,
156, Tokiwadai, Hodogaya,
Yokohama, Japan