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CONVERGENCE RATES OF THE INVARIANCE PRINCIPLE FOR ABSOLUTELY REGULAR SEQUENCES

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1. Introduction. Let $\{\xi_j\}$ be a strictly stationary sequence of random variables which are defined on a probability space $(\Omega, \mathscr{A}, P)$. For $a\leqq b$, let $\mathscr{M}_a^b$ denote the σ-algebra of events generated by $\xi_a, \ldots, \xi_b$. We shall say that the sequence satisfies the absolute regularity (a.r.) condition if

$$\beta(n)=E\{\sup_{A\in\mathscr{M}_n^\infty}|P(A|\mathscr{M}_{-\infty}^0)-P(A)|\}\downarrow 0 \qquad (n\to\infty)\,. \tag{1.1}$$

For any $T(0<T\leqq\infty)$, let $C_T=C[0, T]$ be the space of all continuous functions on $[0, T]$. We give the uniform topology by defining the distance between two points x and y in C_T as

$$\rho_T(x, y)=\sup_{0\leq t\leq T}|y(t)-x(t)|\,. \tag{1.2}$$

Let $S_k=\sum_{j=1}^k \xi_j$ and $S_0=0$. Define a random element $\bar{S}_T=\{\bar{S}_T(t)\colon 0\leqq t\leqq T\}$ in C_T by

$$\bar{S}_T(t)=\begin{cases} S_k & \text{for} \quad t=k,\ k=0, 1, \ldots, [T]\,, \\ \text{linearly interpolated for } t\in[k-1, k]\,, & k=1, \ldots, [T]\,, \\ S_{[T]} & \text{for} \quad [T]\leqq t\leqq T\,. \end{cases} \tag{1.3}$$

where $[s]$ denotes the largest integer m such that $m\leqq s$. Further, let $X_n=\{X_n(t)\colon 0\leqq t\leqq 1\}$ be the random elements in C_1 defined by

$$X_n(t)=(n^{1/2}\sigma)^{-1}\sum_{j=1}^{[nt]} S_j \qquad (0\leqq t\leqq 1) \tag{1.4}$$

where

$$\sigma^2=E\xi_0^2+2\sum_{j=1}^\infty E\xi_0\xi_j>0\,. \tag{1.5}$$

In what follows, we assume that $\sigma=1$.

It is known that if $E\xi_0=0$ and for some $\delta>0$, $E|\xi_0|^{2+\delta}<\infty$ and $\sum\{\beta(n)\}^{\delta/(2+\delta)}<\infty$, then X_n converges weakly to a standard Wiener process $w=\{w(t)\colon 0\leqq t\leqq 1\}$. So, if we put

(1.6) $$F_n(z)=P(\sup_{0\le t\le 1}|X_n(t)|\leqq z)$$

and

(1.7) $$F(z)=P(\sup_{0\le t\le 1}|w(t)|\leqq z)$$
$$=\frac{4}{\pi}\sum_{j=0}^{\infty}(2j+1)^{-1}\exp\left\{-\frac{(2j+1)^2\pi^2}{8z^2}\right\},$$

then

(1.8) $$\lim_{n\to\infty}F_n(z)=F(z).$$

In this paper, we shall prove the following theorems using Rosenkranz's method in [3].

Theorem 1. *Let $\{\xi_j\}$ be a strictly stationary, a.r. sequence of random variables such that $E\xi_0=0$ and for some $\delta>0$ $E|\xi_0|^{4+\delta}<\infty$ and $\sum n\{\beta(n)\}^{\delta/(4+\delta)}<\infty$. Then*

(1.9) $$\sup_z|F_n(z)-F(z)|=O(n^{-1/8}(\log n)^{1/2}).$$

Next, we denote by P_W the Wiener measure on C_1. As in [3], by a "functional" F, we mean a real-valued function F with domain C_1. We define

(1.10) $$\Psi_n(z)=P(-\infty<F(X_n)\leqq z)$$

and

(1.11) $$\Psi(z)=P_W(-\infty<F(w)\leqq z).$$

Then, we can extend Theorem 1 as follows.

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Let F be a uniformly continuous functional, i.e.,*

(1.12) $$|F(f)-F(g)|\leqq K\rho_1(f,g)\quad(f,g\in C_1)$$

and suppose that for some positive constant L

(1.13) $$|\Psi(z+h)-\Psi(z)|\leqq L|h|.$$

Then

(1.14) $$|\Psi_n(z)-\Psi(z)|=O(n^{-1/8}(\log n)^{1/2}).$$

2. Auxiliary results. In this and following sections, by the letter K, with or without subscript, we shall denote various positive constants.

We shall consider some lemmas.

Lemma 2.1. *If the conditions of Theorem 1 are statisfied, then for any posi-*

tive integer m

(2.1) $$E\{\max_{1\leq j\leq m}|S_j|^4\}\leqq Km^2 .$$

Hence, for any $\gamma>0$

(2.2) $$P(\max_{1\leq j\leq m}|S_j|>\gamma)\leqq Km^2/\gamma^4 .$$

((2.1) is the special case of Theorem 3 in [5]).

Next, let $\{\eta_j\}$ be a strictly stationary, a.r. process such that $E\eta_1=0$, $\operatorname{Var}\eta_1=c^2>0$, and $E|\eta_1|^{2+\delta}<\infty$ for some $\delta>0$. Let $\bar{w}=\{w(t): 0\leqq t<\infty\}$ be a standard Wiener process defined on the probability space $(\Omega, \mathscr{A}, P)$.

Lemma 2.2. *Let* $\{\eta_j\}$ *be the sequence defined above. Let* $k>0$ *and* $i_1<i_2<\cdots<i_k$ *be arbitrary integers. Let* $g(y_1, \ldots, y_k)$ *be any bounded Borel function on the k-dimensional Euclidean space* R^k, *say,* $|g(y_1, \ldots, y_k)|\leqq K_0$. *Then, there exists a sequence of nonnegative i.i.d. random variables* $\tau_1, \ldots, \tau_k$ *with the following properties*:

(2.3) $$\begin{aligned}|Eg(\eta_{i_1}, \eta_{i_2}, \ldots, \eta_{i_k})-Eg(w(T_1), w(T_2)-w(T_1), \ldots, w(T_k)-w(T_{k-1}))|\\ \leqq 2K_0k\beta(d)\end{aligned}$$

where $T_j=\tau_1+\cdots+\tau_j$ $(j=1, \ldots, k)$, $T_0=0$ *and* $d=\min_{1\leq j\leq k-1}(i_{j+1}-i_j)$. *Moreover, we have*

$$E\tau_i=c^2$$

and

$$E\tau_i^j\leqq K_jE|\eta_i|^{2j} \qquad (j=1, 2, \ldots) .$$

Proof. Let $\{Y_i\}$ be the sequence of i.i.d. random variables, each Y_i having the same distribution function as that of η_i. Then, from the proof of Lemma 1 in [4] it follows that

$$|Eg(\eta_{i_1}, \eta_{i_2}, \ldots, \eta_{i_k})-Eg(Y_{i_1}, Y_{i_2}, \ldots, Y_{i_k})|\leqq 2K_0k\beta(d) .$$

So, applying Rosenkranz's general version of the Skorokhod representation theorem (Theorem 4 in [3]) to $\{Y_{i_1}, Y_{i_2}, \ldots, Y_{i_k}\}$, we have the lemma.

Remark. We can easily show that Lemma 2.2 remains true, if we use a function g of the type in Lemma 1 in [4] instead of the bounded function.

3. Proof of Theorems. For any positive integer n, we define the following numbers:

$$m=[n^{5/13}]\,,\qquad r=[nm^{-1}]\,,$$
$$\varepsilon=\varepsilon_n=n^{-1/8}(\log n)^{1/2}$$
$$\gamma=\gamma_n=(1/4)\varepsilon_n n^{1/2}$$

To prove Theorem 1, we need some lemmas.

Firstly, we note that the following lemma is easily proved by Lemma 2.1.

Lemma 3.1. *If the conditions of Theorem* 1 *are satisfied, then*

$$mP(\max_{1\le j\le r}|S_j|\geqq\gamma)=o(n^{-1/8}(\log n)^{1/2})\,. \tag{3.1}$$

Lemma 3.2. *Assume that the conditions of Theorem* 1 *are satisfied. Let* $\tilde{S}_n=\{\tilde{S}_n(t)\colon 0\leqq t\leqq n\}$ *be the random element* C_n *defined by*

$$\tilde{S}_n(t)=\begin{cases}S_{rj} & \textit{for}\quad t=rj,\quad j=1,\dots,m\,,\\ S_{mr} & \textit{for}\quad mr\leqq t\leqq n\\ \textit{linearly interpolated for} & t\in[(j-1)r, jr)\,,\quad j=0,1,\dots,m\,.\end{cases} \tag{3.2}$$

Then

$$P(\rho_n(\bar{S}_n,\tilde{S}_n)\geqq\varepsilon_n n^{1/2})=o(n^{-1/8}(\log n)^{1/2})\,. \tag{3.3}$$

Proof. It follows from the method used in the proof of (10) in [1] that

$$\begin{aligned}\rho_n(\bar{S}_n,\tilde{S}_n)&=\sup_{0\leq t\leq n}|\bar{S}_n(t)-\tilde{S}_n(t)|\\ &\leqq 2\max_{1\leq i\leq m}\{\sup_{(i-1)r\leq j\leq ir}|S_j-S_{(i-1)r}|\}+\sup_{mr\leq j\leq n}|S_j-S_{mr}|\,.\end{aligned} \tag{3.4}$$

So

$$\begin{aligned}&P(\rho_n(\bar{S}_n,\tilde{S}_n)\geqq 4\gamma)\\ &\leqq\sum_{i=1}^{m}P(\max_{(i-1)r\leq j\leq ir}|S_j-S_{(i-1)r}|\geqq\gamma)+P(\max_{mr\leq j\leq n}|S_j-S_{mr}|\geqq 2\gamma)\\ &\leqq(m+1)P(\max_{0\leq j\leq r}|S_j|\geqq\gamma)\,.\end{aligned} \tag{3.5}$$

Hence, (3.3) follows from Lemma 3.1 and (3.5), and the proof is completed.

Lemma 3.3. *Let* $q=[n^{1/4}]$ *and* $p=r-q$. *Put*

$$\begin{aligned}\eta_i&=\sum_{j=1}^{p}\xi_{(i-1)r+j}\qquad(i=1,\dots,m)\\ \zeta_i&=\sum_{j=1}^{q}\xi_{(i-1)r+p+j}\qquad(i=1,\dots,m)\\ \zeta_{m+1}&=\sum_{i=1}^{n-mr}\xi_{mr+i}\,.\end{aligned} \tag{3.6}$$

Further, let $\hat{S}_n=\{\hat{S}_n(t): 0\leqq t\leqq n\}$ *be the random element in* C_n *defined by*

$$(3.7)\qquad \hat{S}_n(t)=\begin{cases}\sum_{i=1}^{j}\eta_i & \text{for } t=rj,\ j=0,1,\dots,m\,,\\ \sum_{i=1}^{m}\eta_i & \text{for } mr\leqq t\leqq n\,,\\ \text{linearly interpolated for} & t\in[(j-1)r, jr)\,,\quad j=1,\dots,m\,.\end{cases}$$

Then

$$(3.8)\qquad P(\rho_n(\tilde{S}_n, \hat{S}_n)\geqq 4\gamma)=o(n^{-1/8}\log n)^{1/2})\,.$$

Proof. Since

$$\rho_n(\tilde{S}_n, \hat{S}_n)\leqq \max_{1\leqq j\leqq m}|\sum_{i=1}^{j}\zeta_i|+|\zeta_{m+1}|\,,$$

so from the method of the proof of Theorem 3 in [4] (cf. the proof of Lemma 3.4 (below)) and Lemma 2.1

$$\begin{aligned}P(\rho_n(\tilde{S}_n, \hat{S}_n)\geqq 4\gamma_n)&\leqq P(\max_{1\leqq j\leqq m}|\sum_{i=1}^{j}\zeta_i|\geqq 2\gamma_n)+P(|\zeta_{m+1}|\geqq 2\gamma_n)\\ &\leqq K\{(m^2E\zeta_0^4/\gamma_n^4+m\beta(p))+E\zeta_{m+1}^4/\gamma_n^4\}\\ &\leqq K(m^2q^2/\gamma_n^4+r^2/\gamma_n^4+m\beta(p))\\ &=o(n^{-1/8}(\log n)^{1/2})\,.\end{aligned}$$

Lemma 3.4. *Let* $\{\eta_j\}$ *be the random variables defined by* (3.6). *Then, there exists a sequence of non-negative i.i.d. random variables* $\tau_1^{(\nu)}, \dots, \tau_m^{(\nu)}$ $(\nu=1, 2)$ *with the following properties*:

$$(3.9)\qquad\begin{aligned}&P(\max_{1\leqq j\leqq m}|w(T_j^{(1)})|\leqq z-\varepsilon)+o(n^{-1/8}(\log n)^{1/2})\\ &\leqq P(\sup_{0\leqq t\leqq 1}|\tilde{S}_n(nt)|\leqq z)\\ &\leqq P(\max_{1\leqq j\leqq m}|w(T_j^{(2)})|\leqq z+\varepsilon)+o(n^{-1/8}(\log n)^{1/2})\end{aligned}$$

where $T_j^{(\nu)}=\tau_1^{(\nu)}+\cdots+\tau_j^{(\nu)}$ $(j=1, \dots, m)$, $T_0^{(\nu)}=0$ *and*

$$(3.10)\qquad\begin{aligned}&E\tau_1^{(\nu)}=n^{-1}E\eta_1^2\\ &E(\tau_1^{(\nu)})^j\leqq Kn^{-j}E\eta_1^{2j}\qquad (j=1, 2, \dots)\,,\end{aligned}$$

$(\nu=1, 2)$.

Proof. We note that $\{\eta_1, \dots, \eta_m\}$ satisfies the a.r. condition and from Lemma 3.1 $E(p^{-1/2}\eta_i)^4\leqq K$. So putting

$$g_\nu(y_1, \ldots, y_m)=\begin{cases}1 & \text{if } \max_{1\le j\le m} (n^{-1/2}\sum_{i=1}^{j} y_i)\le z+(-1)^\nu\varepsilon\,, \\ 0 & \text{otherwise,}\end{cases}$$

$$(\nu=1, 2)$$

by Lemmas 3.2 and 2.2 we can conclude that there exists a sequence of i.i.d. random variables $\tau_1, \ldots, \tau_m$ satisfying (3.10) such that

$$\begin{aligned}P(\sup_{0\le t\le 1}|\tilde{S}_n(nt)|\le zn^{1/2}) &\ge P(\sup_{0\le t\le 1}|\hat{S}_n(nt)|\le(z-\varepsilon)n^{1/2})-P(\rho_n(\tilde{S}_n, \hat{S}_n)\ge\varepsilon n^{1/2}) \\ &=P(\max_{1\le j\le m}|\sum_{i=1}^{j}\eta_i|\le(z-\varepsilon)n^{1/2})+o(n^{-1/8}(\log n)^{1/2}) \\ &\ge P(\max_{1\le j\le m}|w(T_j^{(1)})|\le z-\varepsilon)-Km\beta(q)+o(n^{-1/8}(\log n)^{1/2})\end{aligned}$$

and

$$\begin{aligned}P(\sup_{0\le t\le 1}|\tilde{S}_n(t)|\le zn^{1/2}) &\le P(\sup_{0\le t\le 1}|\hat{S}_n(nt)|\le(z+\varepsilon)n^{1/2})+P(\rho_n(\tilde{S}_n, \hat{S}_n)\ge\varepsilon n^{1/2}) \\ &=P(\max_{1\le j\le m}|\sum_{i=1}^{j}\eta_i|\le(z+\varepsilon)n^{1/2})+o(n^{-1/8}(\log n)^{1/2}) \\ &\le P(\max_{1\le j\le m}|w(T_j^{(2)})|\le z+\varepsilon)+Km\beta(q)+o(n^{-1/8}(\log n)^{1/2})\,.\end{aligned}$$

Since $m\beta(q)=O(n^{-8/13})$, (3.9) follows from the above inequalities.

Lemma 3.5. *Let $T_0, T_1, \ldots, T_m$ be one of two families $\{T_0^{(\nu)}, T_1^{(\nu)}, \ldots, T_m^{(\nu)}\}$ $(\nu=1, 2)$ obtained in Lemma 3.4. Then*

$$P\left(\max_{1\le j\le m}\left|w(T_j)-w\left(\frac{j}{m}\right)\right|\ge\varepsilon\right)=O(n^{-1/8}(\log n)^{1/2})\,. \tag{3.11}$$

Proof. We define random variables Z_j $(j=1, \ldots, m)$ by

$$Z_j=\sum_{i=1}^{j}\left(\tau_i-\frac{1}{m}\right)=T_j-\frac{j}{m}\,.$$

Then, we can rewrite $w(T_j)$ as

$$w(T_j)=w\left(\frac{j}{m}+Z_j\right) \qquad (j=1, \ldots, m)\,.$$

We note here that

$$\begin{aligned}|E\eta_1^2-p| &\le 2p[\sum_{j=p}^{\infty}E\xi_0\xi_j+p^{-1}\sum_{j=1}^{p}j|E\xi_0\xi_j|] \\ &\le 2p\{E|\xi_0|^{4+\delta}\}^{2/(4+\delta)}[\sum_{j=p}^{\infty}\{\beta(j)\}^{(2+\delta)/(4+\delta)}+p^{-1}\sum_{j=1}^{p}j\{\beta(j)\}^{(2+\delta)/(4+\delta)}] \\ &=O(1)\end{aligned}$$

and so

$$|E\tau_1-m^{-1}|=|n^{-1}E\eta_1^2-m^{-1}|=|n^{-1}p\{1+O(p^{-1})\}-m^{-1}|\leqq Kn^{-1}(q+O(1))$$
$$=O(n^{-3/4})\ .$$

On the other hand

$$\mathrm{Var}\,(\tau_1)\leqq E\tau_1^2\leqq Kn^{-2}E\eta_1^2\leqq Kn^{-2}p^2\ .$$

Hence, if we put $\lambda=n^{-1/4}$, then by Kolmogorov's inequality

$$\begin{aligned}P(\max_{1\leqq j\leqq m}|Z_j|>\lambda)&\leqq P(\max_{1\leqq j\leqq m}|\sum_{i=1}^{j}(\tau_i-E\tau_i)|>\lambda-\sum_{j=1}^{m}|m^{-1}-E\tau_j|)\\&\leqq P(\max_{1\leqq j\leqq m}|\sum_{i=1}^{j}(\tau_i-E\tau_i)|>\lambda/2)\\&\leqq\lambda^{-2}\sum_{j=1}^{m}\mathrm{Var}\,\tau_j\leqq K\lambda^{-2}mn^{-2}p^2=o(n^{-1/8}(\log n)^{1/2})\ .\end{aligned}$$

The rest of the proof is the same as the proof of Lemma 6 in [3] and so is omitted.

Proof of Theorem 1. The proof is easily followed from Lemmas 3.2, 3.4 and 3.5.

Proof of Theorem 2. The proof is the same as that of Theorem 5 in [3] and so is omitted.

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