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A GENERALIZATION OF REIDEMEISTER-SINGER THEOREM ON HEEGAARD SPLITTINGS

By

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Introduction. The Reidemeister-Singer theorem [Rd], [Sg] states that any two Heegaard splittings of a 3-manifold are stably equivalent. If one deals with periodic homeomorphism on 3-manifolds, it is natural to ask whether there is an equivariant version of the Reidemeister-Singer theorem. In this paper we establish such a theorem in the case of involutions, i.e. periodic homeomorphisms of order 2.

Let f: $M^m \to M^m$ be an involution on an m-manifold M. In [BL] Browder and Livesay define a submanifold N of codimension one in M to be *characteristic* if there are submanifolds U and V such that $M=U\cup V$, $U\cap V=\partial U=\partial V=N$ and $f(U)=V$. If $m=3$, the submanifold N is a surface. Moreover if f is an orientation preserving involution, then there exists a characteristic submaniforld N of M such that (M, U, V) is a Heegaard splitting (see Proposition 2.4): We call such a splitting a *characteristic Heegaard splitting* and use the notation $(M; U, V, f)$ to emphasize f. Since we are dealing with involutions, a stable equivalence relation for characteristic Heegaard splittings must involve attaching pairs of handles. We define an equivalence relation, called *ss*-equivalence, on the set of characteristic Heegaard splittings. This corresponds to stable equivalence in the usual Heegaard theory. Roughly, the *ss*-equivalence relation is defined by two operations: (1) An equivalence operation which says that two characteristic Heegaard splittings $(M; U_1, V_1, f_1)$ and $(M; U_2, V_2, f_2)$ are equivalent if there is an equivalence of Heegaard splittings h: $(M, U_1, V_1)\to(M, U_2, V_2)$ such that $f_2=h\circ f_1\circ h^{-1}$. (2) A stabilization operation that is a 2-fold stabilization in the usual Heegaard theory and requires that whenever a stabilizing handle is added to one side of a splitting, the image of the handle be added to the other.

Our main result is the following theorem:

Theorem. *Let M be an orientable closed 3-manifold. Then two characteristic Heegaard splittings $(M; U_1, V_1, f_1)$ and $(M; U_2, V_2, f_2)$ are ss-equivalent if and only*

1) The results in this article are contained in the auther's Ph.D. thesis written at University of Illinois under the direction of Professor R. Craggs.

if there is a homeomorphism $h: M \to M$ *such that* $f_2 = h \circ f_1 \circ h^{-1}$ *and* $h(\partial U_1) \cap \partial U_2$ *contains a neighborhood of the fixed point set for* f_2 *in* U_2. *In particular if* f_1 *and* f_2 *are conjugate, fixed point free involutions, then* $(M; U_1, V_1, f_1)$ *and* $(M; U_2, V_2, f_2)$ *are ss-equivalent.*

In § 2 we give a construction for producing characteristic Heegaard splittings (CHS-construction) starting with a fixed involution f on M and a fixed triangulation K on M invariant under f. Such splittings are said to be *standard*. In § 3 we define the *ss*-equivalence relation and show that any two standard Heegaard splittings, starting with the same involution and triangulation, are *ss*-equivalent provided that the Heegaard surfaces coincide in a neighborhood of the fixed point set. In § 4 we prove our Reidemeister-Singer theorem. The proof is suggested in part by Craggs's proof in [Cr].

We wish to thank Professor R. Craggs for helpful discussion and encouragement.

§ 1. Notation and definitions

We work in the piecewise linear category.

Maps are all piecewise linear maps. The interior, closure, and boundary of $(\cdots)$ are denoted by $\mathrm{Int}(\cdots)$, $\mathrm{Cl}(\cdots)$, and $\partial(\cdots)$ respectively.

For a set Z of simplexes, $|Z|$ denotes the union of simplexes in Z. Hence if Z is a complex than $|Z|$ is the *carrier* or underlying space of Z. For a complex K, K_i denotes the i-skeleton of K, i.e. the set of simplexes in K whose dimension are not bigger than i. Hence K_i is a subcomplex of K. For a subset X of $|K|$, $\bar{N}(X; K)$ denotes the smallest subcomplex of K whose carrier contains a neighborhood of X in $|K|$, and $N(X; K)$ denotes the carrier of the complex $\bar{N}(X; K)$. We denote by $K(X)$ the smallest subcomplex of K whose carrier contains X. Note that for a subcomplex J of K we have $K(|J|)=J$. For a simplex b in K, $\hat{b}$ denotes the barycenter of the simplex b, and $\hat{K}$ denotes the barycentric subdivision of K. For two joinable subcomplexes B and C of K, $B*C$ denotes the join of B and C; hence $B*C$ is a polyhedron in K. If B and C are simplexes in K, then we assume that $B*C$ is also a simplex in K (see [Zm] for the definition of join).

A 1-dimensional complex T is called a *tree* if it is simplicially collapsible.

Let M be a manifold and X a polyhedron in M. In this paper, for a regular neighborhood N of X in M, we always assume that N is small with respect to things previously defined and that N meets the boundary of M regularly. That is to say, we construct N as follows: Choose a triangulation K in which all subspaces, previously mentioned in the argument, are subcomplexes, let K'' be a second derived subdivision of K, and let $N=N(X; K'')$.

For a monifold M and a submanifold N, we say that N is ***proper*** in M provided that $\partial N\subset\partial M$ and $\text{Int}\, N\subset\text{Int}\, M$.

For a closed 3-manifold M, a *Heegaard splitting* of M is a triple $(M; U, V)$ such that each of U and V is a cube-with-handles, $U\cup V=M$, and $U\cap V=\partial U=\partial V$.

A map f: $M\to M$ is an *involution* provided that $f\circ f$ is the identity map. We denote by $F(f)$ the ***fixed point*** set of f.

A triangulation K of a 3-manifold M is *invariant* with respect to an involution f: $M\to M$ provided that f: $K\to K$ is simplicial and $F(f)$ is triangulated by K.

A Heegaard splitting $(M; U, V)$ is *characteristic* with respect to f if $f(U)=V$. To emphasize f we use the notion $(M; U, V, f)$ instead of $(M; U, V)$. Two characteristic Heegaard splittings $(M; U, V, f)$ and $(M; X, Y, f)$ are equivalent if there is an orientation preserving homeomorphism g: $(M, U, V)\to(M, X, Y)$ such that $f\circ g=g\circ f$. Note that if f is an orientation preserving involution, then $(M; U, V, f)$ and $(M; V, U, f)$ are equivalent. The equivalence is induced by f: $(M, U, V)\to (M, V, U)$.

For a map f: $M\to M$, a subset Z of M is an *invariant* set for f if $f(Z)=Z$. Note that for a characteristic Heegaard splitting $(M; U, V, f)$, ∂U is an invariant set for f and $F(f)\subset\partial U$.

Let K be an invariant triangulation of M with respect to f such that $N(F(f); K)$ is a regular neighborhood of $F(f)$. A proper annulus in $N(F(f); K)$ is ***admissible*** provided that the annulus satisfies the following four conditions (see Fig. 1):

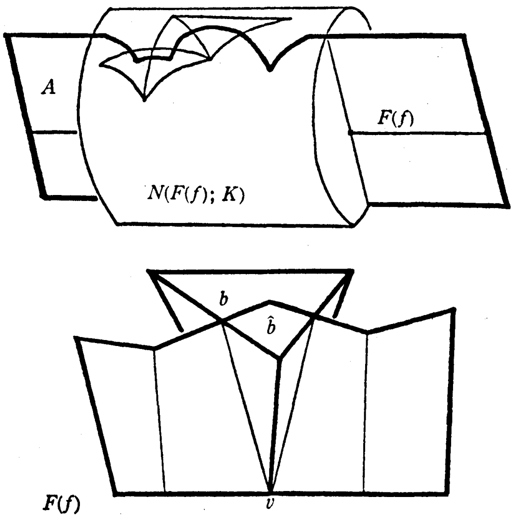
(1) The annulus is an invariant set for f.

(2) If J is a triangulation of $\partial N(F(f); K)$ induced by K, then the boundary of the annulus consists of dual 1-cells of 1-simplexes of J with respect to the barycentric subdivision $\hat{J}$.

(3) For each 1-simplex b of K, $\text{Int}\, b$ meets the interior of the annulus if and only if b lies in $F(f)$.

(4) The annulus contains a component of $F(f)$.

Throughout this paper we assume that f: $M\to M$ is a fixed orientation preserving involution and that $F(f)$ is a disjoint union of 1-spheres $S(1), \cdots, S(n)$. Furthermore, for any triangulation K of M that we deal with, we assume that K is invariant with respect to f and that K is sufficiently fine so that $N(F(f); K)$ is a regular neighborhood of $F(f)$. The existence of such a triangulation can be shown as follows: Since the orbit space of f is a manifold, let J be a triangu-



b: a simplex on $\partial N(F(f); K)$

Fig. 1.

lation of the orbit space. Then J induces a triangulation of M by publing back. Let K be the second barycentric subdivision of the triangulation. Then K is a desired invariant triangulation of M.

We use the sign $\square$ to indicate the end of proofs.

§ 2. Existence of characteristic Heegaard splittings

The main result in this section is Proposition 2.4, a construction for producing characteristic Heegaard splittings. This construction will provide a very specific model handling the difficult tasks in proving the Reidemeister-Singer theorem. The notation in the sequence of propositions is cumulative; once introduced, something will be proper subject matter for future propositions.

Definition. A set of annuli $A(1), \cdots, A(n)$ in $N(F(f); K)$ is called a *constructive A-set* provided that for each $i=1, \cdots, n$, the annulus $A(i)$ is an admissible annulus in $N(S(i); K)$.

Definition. A set ν of vertices in K is called a *constructive* V-set provided that the set ν satisfies the following three conditions:

(1) $\nu \cap f(\nu)=\phi$.

(2) The union of ν and $f(\nu)$ is the set of all vertices in K which do not lie in the fixed point set $F(f)$.

(3) There is a constructive A-set $\{A(1), \cdots, A(n)\}$ such that for each $i=1, \cdots, n$, $\partial A(i)$ separates the set $\nu \cap N(S(i); K)$ and $f(\nu) \cap N(S(i); K)$ on $\partial N(S(i); K)$.

The A-set is called a *companion* A-set to ν.

Proposition 2.1. *There are an invariant triangulation K of M and a constructive A-set.*

Proof. Any triangulation of the orbit space of M which has the image of $F(f)$ (under the identification map) in the 1-skeleton pulls back to an invariant triangulation of M. Let K be the second barycentric subdivision of such an invariant triangulation of M. Then by an argument similar to the one in Lemma 1 of [MS] there is, for each $i=1, \cdots, n$, an admissilble annulus $A(i)$ in $N(S(i); K)$. Thus the set $\{A(1), \cdots, A(n)\}$ is a constructive A-set. $\square$

Proposition 2.2. *For each constructive A-set α (constructive with respect to an invariant triangulation K of M) there is a constructive V-set ν such that α is a companion A-set to ν.*

Proof. Let α consist of annuli $A(1), \cdots, A(n)$. For each $i=1, \cdots, n$, let $Q(i)$ be a connected component of $N(S(i); K)-A(i)$. Then there is a tree $T(i)$ in K_1 such that $|T(i)|\subset Q(i)$ and $T(i)_0=K_0\cap Q(i)$. Let J be the complex obtained from K by identifying each simplex b in K and its image $f(b)$. Let g: $K\to J$ be the identification map. Let $M'=\mathrm{Cl}\,(M-N(F(f); K))$. Then $g|M'$: $M'\to g(M')$ is a covering map. Now $g(\bigcup_{i=1}^{n} T(i))$ is a disjoint union of n trees in $g(M')$. Since M' is connected, so is $g(M')$. Hence there is a maximal tree G in $J(g(M'))$ which contains each $g(T(i))$ as a subtree. Since $g|M'$ is a double covering, $g^{-1}(G)$ consists of two trees. Let T be one of these. Then T_0 is the desired constructive V-set. $\square$

It is easy to check the following proposition.

Proposition 2.3. *Suppose that ν is a constructive V-set in K and α is a companion A-set to ν. Let $a(1), \cdots, a(m)$ denote the 1-simplexes of K that intersect both ν and $f(\nu)$. For each $j=1, \cdots, m$, let $a^*(j)$ be the dual 2-cell of the simplex $a(j)$ with respect to $K(M')$. Then $\bigcup_{j=1}^{m} a^*(j)$ is a proper surface in M' whose boundary is $\bigcup_{i=1}^{n} \partial A(i)$.* $\square$

Definition. Let H be the union of $\bigcup_{i=1}^{n} A(i)$ and the surface $\bigcup_{j=1}^{m} a^*(j)$. We call H the (α, ν)-*surface.*

A surface in M is called a *standard surface* provided that there are a constructive V-set ν and a companion A-set α such that the surface is the (α, ν)-surface.

Let $U(H)$ be the closure of the union of the connected components of $M-H$

which intersect ν. And let $V(H)=\mathrm{Cl}\,(M-U(H))$. Then $f(U(H))=V(H)$ and H is invariant under f. But generally it is not true that $U(H)$ is a cube-with-handles. Furthermore $U(H)$ may be disconnected. We shall modify H so that H becomes a Heegaard surface. The following proposition shows a canonical methods for constructing a characteristic Heegaard surface from a standard surface.

Proposition 2.4 (*CHS-construction*). *There exists a characteristic Heegaard splitting.*

Proof. Let H be an (α, ν)-surface. Let $U(H)$ be the closure of the union of the connected components of $M-H$ that intersect the constructive V-set ν. Let $Z(H)$ be the union of 1-simplexes in K whose interiors intersect $\mathrm{Int}\,U(H)$. Note that $Z(H)$ does not contain any 1-simplexes in the fixed point set $F(f)$. Let J be the complex obtained from K by identifying each simplex b in K and its image $f(b)$, and let g: $K\to J$ be the identification map. Then there is a regular neighborhood N of $Z(H)\cap U(H)$ in $U(H)$ such that $N\cap H$ is invariant under f, and the image of each connected component of $N\cap H$ under g is a disk.

We modify N to N^* as follows; the modification will be called a *wedge-modification* (see Fig. 2): First of all, note that for each connected component C of $g(N\cap H)$, $g^{-1}(C)$ is connected if and only if C intersects $g(F(f))$. This follows from the fact that $N\cap H$ is also a regular neighborhood of $Z(H)\cap H$ in H. We choose disks in each connected component C of $g(N\cap H)$ as follows. If C intersects $g(F(f))$, then we choose a disk B in C such that $B\cap g(F(f))=\phi$, and $\mathrm{Cl}\,(\partial B-\partial C)$ is a proper arc in C. If C misses $g(F(f))$, then we choose two disjoint disks $B(1)$ and $B(2)$ in C such that for $i=1, 2$, $\mathrm{Cl}\,(\partial B(i)-\partial C)$ is a proper arc in C. Let E be the union of disks B, $B(1)$, $B(2)$ described for the various components. Now $N\cap H$ consists of mutually disjoint disks $D(1), \cdots, D(s)$. For each $i=1, \cdots, s$, $D(i)\cap g^{-1}(E)$ consists of two disjoint disks $D(i1)$ and $D(i2)$. By re-indexing, we may assume that $f(\bigcup_{i=1}^{s} D(i1))=\bigcup_{i=1}^{s} D(i2)$. Let $P(i)=\mathrm{Cl}\,(\partial D(i1)-\partial D(i))$.

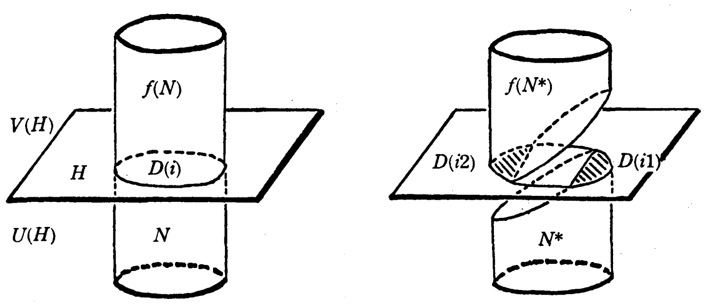


Fig. 2.

Then $P(i)$ is a proper arc in $D(i)$. Let $R(i)$ be a regular neighborhood of $\mathrm{Cl}(D(i)-D(i1))$ in N relative to $P(i)$. Let $N^*=\mathrm{Cl}(N-\bigcup_{i=1}^{s} R(i))$. Now $f(N^*)\cap N^*=\phi$. Let $U=\mathrm{Cl}(U(H)-N^*)\cup f(N^*)$ and $V=\mathrm{Cl}(M-U)$. Then $(M; U, V, f)$ is a characteristic Heegaard splitting, because (i) $\mathrm{Cl}(U(H)-N^*)$ is a regular neighborhood of the union of dual 1-cells of 2-simplexes in K whose interiors meet $\mathrm{Int}\, U(H)$, (ii) $f(N^*)$ is cubes-with-handles, and (iii) $\mathrm{Cl}(U(H)-N^*)\cap f(N^*)$ consists of disks. $\square$

By the process described in the above proposition, each (α, ν)-surface H induces a characteristic Heegaard splitting. The characteristic Heegaard splitting is determined up to equivalence by the constructive V-set ν. Any characteristic Heegaard splitting induced from the (α, ν)-surface by the CHS-construction will be called an *(α, ν)-Heegaard splitting.*

A characteristic Heegaard splitting will be called *standard* Heegaard splitting provided that there are constructive V-set ν and a companion A-set α such that the splitting is an (α, ν)-Heegaard splitting.

§ 3. An equivalence relation on the set of characteristic Heegaard splittings

In this section we define an equivalence relation on the set of characteristic Heegaard splittings which corresponds to the stable equivalence relation in the Heegaard splitting theory. We will consider as fixed a 3-manifold M, an involution f on M, and an invariant triangulation K of M.

First we define an equivalence relation on the set of constructive V-sets in K: Let $\nu=\{v(1), \cdots, v(t)\}$ and ν' be constructive V-sets. Suppose that there are integers h and j such that

$$\nu'=\{v(1), \cdots, v(j-1), f(v(j)), v(j+1), \cdots, v(t)\}$$

and $f(v(j))$ and $v(h)$ are the vertices of a 1-simplex in K. Then we say that ν' is obtained from ν by an *elementary trade* of $v(j)$ for $f(v(j))$, and the elementary trade is denoted by $\nu\rightarrow\nu'$. Let ν and ν' be constructive V-set that have a common A-set α. We say that ν and ν' are A-equivalent provided that there is a sequence of elementary trades

$$\nu\rightarrow\nu(1)\rightarrow \cdots \rightarrow\nu(k)\rightarrow\nu'$$

such that the intervening V-sets have the same companion A-set α.

Proposition 3.1. *Let α be a constructive A-set. Let $\nu=\{v(1), \cdots, v(t)\}$ and ν' be constructive V-sets that have the common companion A-set α. Suppose that for each $i=1, \cdots, n$,*

$$\nu \cap N(S(i); K) = \nu' \cap N(S(i); K) .$$

Then ν *and* ν' *are A-equivalent.*

Proof. We use downward induction on the number of common vertices of ν and ν'. By reindexing vertices we may assume that

$$\nu' = \{v(1), \cdots, v(k), f(v(k+1)), \cdots, f(v(t))\} .$$

Let H and H' be the (α, ν)- and (α, ν')-surfaces respectively. Let $J(H)$ be the union of 1-simplexes in K that pierce H and similarly let $J(H')$ be the union of k-simplexes in K which pierce H'. Note that H determines both the constructive V-set ν and the set $J(H)$. Thus $J(H)=J(H')$ if and only if $H=H'$. Hence $J(H)=J(H')$ if and only if $\nu=\nu'$. Without loss of generality we may assume that there is a simplex b in $J(H)$ that is not in $J(H')$. Then, since b does not pierce H', either the two vertices of b lie in ν' or the two vertices of $f(b)$ lie in ν'. Since $f(b)$ is also an element in $J(H)-J(H')$, we may assume that the two vertices of b lie in ν'. Then a vertex of the simplex b lies in the set $\{v(1), \cdots, v(k)\}$ and the other vertex of b lies in the set $\{f(v(k+1)), \cdots, f(v(t))\}$, say it is $f(v(j))$. Thus we can make an elementary trade of $v(j)$ for $f(v(j))$ to get a new V-set ν'' from ν. Now

$$\nu'' = \{v(1), \cdots, v(k), v(k+1), \cdots, v(j-1), f(v(j)), v(j+1), \cdots, v(t)\} .$$

Hence ν'' and ν' have $k+1$ common vertices $v(1), \cdots, v(k), f(v(j))$, and the number of common vertices increases. □

We define as follows an equivalence relation on the set of characteristic Heegaard splittings of M. First we define, for an arbitrary splitting $(M; U, V, f)$, three elementary modifications of $(M; U, V, f)$ to a new splitting $(M; X, Y, g)$.

Type 1 (equivalence): There is an equivalence of Heegaard splittings h: $(M, U, V) \to (M, X, Y)$ (note that $h(U)=X$) such that $g=hfh^{-1}$.

Type 2 (attaching a pair of handles of index 1): Here $f=g$ and there is a disk D in U such that $\partial U \cap \operatorname{Int} D=\phi$, $\operatorname{Cl}(\partial D-\partial U)$ is a proper arc in U, say P, and $P \cap f(P)=\phi$. There is a regular neighborhood N of P in U such that $f(N) \cap N=\phi$. Finally $X=\operatorname{Cl}(U-N) \cup f(N)$ and $Y=\operatorname{Cl}(M-X)$.

Type 3 (attaching a pair of handles of index 2): Here $f=g$ and there are proper disks $D(1)$ and $D(2)$ in V and U respectively such that $D(1) \cap F(f)=\phi$, and $\partial D(1) \cap \partial D(2)$ consists of one crossing point on ∂U. There is a regular neighborhood N of $D(1)$ in V such that $f(N) \cap N=\phi$. Finally $X=\operatorname{Cl}(U-f(N)) \cup N$ and $Y=\operatorname{Cl}(M-X)$.

Notes. (1) Modifications of Type 2 increase the genus of Heegaard surfaces, and modifications of Type 3 decrease the genus of Heegaard surfaces. (2) In the modification of Type 3, the only purpose $D(2)$ serves is to guarantee that we are destabilizing the splittings.

We define two characteristic Heegaard splittings $(M; U, V, f)$ and $(M; X, Y, g)$ to be *ss-equivalent* provided that there is a sequence of elementary modifications and their inverses transforming $(M; U, V, f)$ to $(M; X, Y, g)$.

The notion of *ss*-equivalence will provide the basis of our formulation, in the next section, of a Reidemeister-Singer theorem for characteristic Heegaard splittings. Our plan for obtaining this theorem is as follows: First obtain a Reidemeister-Singer theorem for a restricted class of splittings—standard characteristic Heegaard splittings with respect to an involution f and an invariant triangulation K of M under f where the splittings coincide on the set $N(F(f); K)$. Second show that every characteristic Heegaard splitting is *ss*-equivalent to a standard Heegaard splitting. Finally show that *ss*-equivalence does not change twisting numbers which are defined later. The first two steps in this approach are suggested by Craggs's proof of the Reidemeister-Singer theorem [Cr].

The following proposition handles an important special case for the first step in our plan.

Proposition 3.2. *Let α be a constructive A-set. Let ν and ν' be constructive V-sets which have the common companion A-set α. Let $(M; U, V, f)$ and $(M; X, Y, f)$ be an (α, ν)- and an (α, ν')-Heegaard splitting respectively. Then the two splittings are ss-equivalent provided that for each $i=1, \cdots, n$,*

$$\nu \cap N(S(i); K) = \nu' \cap N(S(i); K) .$$

Proof. By Proposition 3.1, the constructive V-set ν is A-equivalent to ν'. We may assume that ν' is obtained by an elementary trade of a vertex v in ν for $f(v)$.

Let L be the link of $f(v)$ in K. Then L is a subcomplex of K. Let T be a maximal tree in L. Let $b(1), \cdots, b(q)$ be an indexing of the 1-simplexes of T so that T collapses to a point $w(0)$ by the collapsing of the successive simplexes $b(q), b(q-1), \cdots, b(1)$. For each $k=0, 1, \cdots, q$, let $T(k)$ be the subcomplex of T whose carrier is the union of $w(0)$ and $\bigcup_{j=1}^{k} b(j)$. Then $T(q)=T$ and $T(k)$ collapses to $T(k-1)$. Let $w(k)$ be the vertex in $T(k)-T(k-1)$. And let $a(k)$ be the 1-simplex of K whose vertices are $f(v)$ and $w(k)$.

Let H be the (α, ν)-surface. Let $U(H)$ be the closure of the union of the connected components of $M-H$ which intersect ν. Let $V(H)=\mathrm{Cl}\,(M-U(H))$. Let $Z(H)$ be the union of 1-simplexes of K whose interiors intersect $\mathrm{Int}\,U(H)$. Let N^* be the wedge-modification of a regular neighborhood N of $|Z(H)|\cap U(H)$ in $U(H)$ used to get the (α,ν)-Heegaard splitting $(M; U, V, f)$ as described in the CHS-construction. To prove the proposition we consider two cases:

(1) $L\cap f(\nu)=\phi$.

(2) $L\cap f(\nu)\neq\phi$.

Case (1): Suppose that $L\cap f(\nu)=\phi$. Let B be the connected component of $V(H)$ which contains $f(v)$. Since $L\cap f(\nu)=\phi$, every 1-simplex of K containing $f(v)$ runs from $f(v)$ to a vertex in $U(H)$; thus B is the 3-ball whose boundary is made up of the dual cells to the 1-simplexes mentioned above. For each $i=1,\cdots,q$, we choose a subset $M(i)$ of $f(N^*)$ as follows: Let $D(i)$ be the connected component $N\cap H$ which the 1-simplex $a(i)$ pierces. Let $D(i2)=D(i)\cap f(N^*)$ (see Fig. 2). Let $M(i)$ be a regular neighborhood of $D(i2)$ in $f(N^*)$. Remember that each regular neighborhood is small with respect to all things previously defined. Thus we may assume that the $M(i)$'s are mutually disjoint. Let

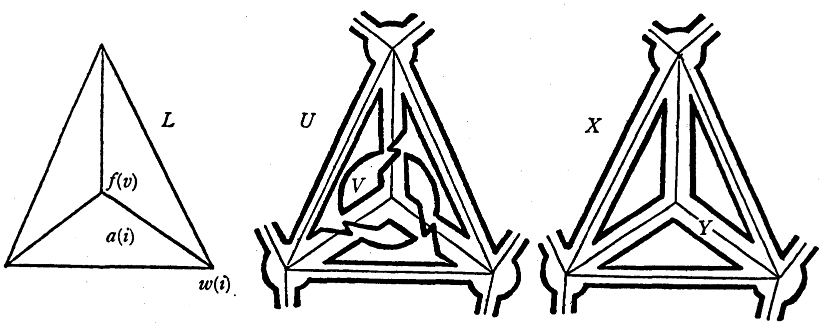
$$U(1)=\mathrm{Cl}\,(U-\bigcup_{i=1}^{q}M(i))\cup\bigcup_{i=1}^{q}f(M(i)) \quad \text{and}$$
$$V(1)=\mathrm{Cl}\,(M-U(1)) \quad \text{(see Fig. 3A).}$$

Then $(M; U(1), V(1), f)$ is equivalent to $(M; X, Y, f)$ by expanding $\mathrm{Cl}\,(B-((N^*\cap B)-\bigcup_{i=1}^{q}M(i))$ to B.

Claim 1. *$(M; U, V, f)$ is ss-equivalent to $(M; U(1), V(1), f)$.*

Proof. For each $k=1,\cdots,q$, let

$$U(k)=\mathrm{Cl}\,(U-\bigcup_{i=k}^{q}M(i))\cup\bigcup_{i=k}^{q}f(M(i)) \quad \text{and}$$
$$V(k)=\mathrm{Cl}\,(M-U(k))\,.$$



Dimension is reduced to two.

Fig. 3A.

Let $U(q+1)=U$ and $V(q+1)=V$. It is sufficient to show that for each $k=1, \cdots, q$, $(M; U(k+1), V(k+1), f)$ is ss-equivalent to $(M; U(k), V(k), f)$. Now we can assume that $(f(v)*b(k))\cap V$ ($*$ denotes join) is a proper disk in V and also in $V(k+1)$. Since the proper disk intersects the disk $D(k2)$ at one crossing point on $\partial U(k+1)$, $(M; U(k+1), V(k+1), f)$ is ss-equivalent to $(M; U(k), V(k), f)$ by attaching a pair of handles of index 2. This completes the proof of the proposition for Case (1).

Case (2): Suppose that $L\cap f(\nu)\neq\phi$. We shall establish our claim in three steps.

First step: Since $L\cap f(\nu)\neq\phi$, we can choose $w(0)$ in $f(\nu)$. For each $i=1, \cdots, q$, we choose a subset $N(i)$ of N^* as follows. If $a(i)\notin Z(H)$, let $N(i)=\phi$. If $a(i)\in Z(H)$, then $a(i)$ pierces H. Let $D(i)$ be the connected component of $N\cap H$ which the 1-simplex $a(i)$ pierces. Let $D(i1)=D(i)\cap N^*$ (see Fig. 2). Let $N(i)$ be a regular neighborhood of $D(i1)$ in N^*. Again we assume that the $N(i)$'s are mutually disjoint. Let

$$U(1)=\mathrm{Cl}\,(U-\bigcup_{i=1}^{q} f(N(i)))\cup\bigcup_{i=1}^{q} N(i) \quad \text{and}$$
$$V(1)=\mathrm{Cl}\,(M-U(1)) \quad \text{(see Fig. 3B).}$$

Claim 2. *$(M; U, V, f)$ is ss-equivalent to $(M; U(1), V(1), f)$.*

Proof. For each $k=1, \cdots, q$, let

$$U(k)=\mathrm{Cl}\,(U-\bigcup_{i=k}^{q} f(N(i)))\cup\bigcup_{i=k}^{q} N(i) \quad \text{and}$$
$$V(k)=\mathrm{Cl}\,(M-U(k))\,.$$

Let $U(q+1)=U$ and $V(q+1)=V$. It is sufficient to show that for each $k=1, \cdots, q$, $(M; U(k+1), V(k+1), f)$ is ss-equivalent to $(M; U(k), V(k), f)$. There are three cases:

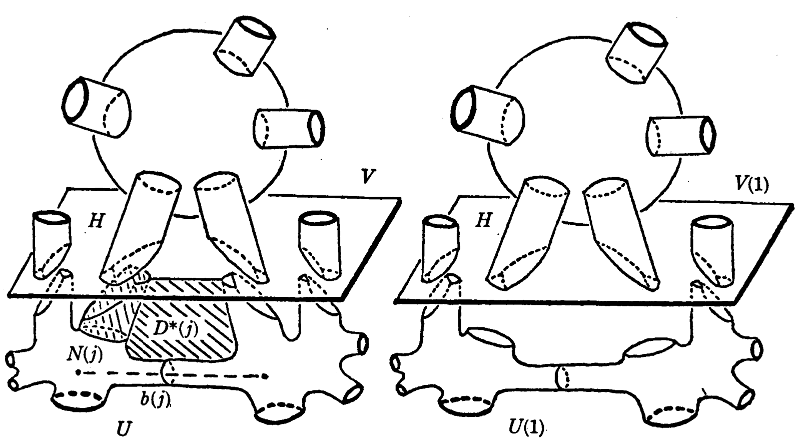


Fig. 3B.

(1) $a(k) \notin Z(H)$.

(2) $a(k) \in Z(H)$ and $b(k)$ pierces H.

(3) $a(k) \in Z(H)$ and $b(k) \subset U(H)$.

Case (1): If $a(k) \notin Z(H)$ then $N(k)=\phi$. Hence $U(k+1)=U(k)$ and $V(k+1)=V(k)$. Thus $(M; U(k+1), V(k+1), f)$ is ss-equivalent to $(M; U(k), V(k), f)$.

Case (2): If $a(k) \in Z(H)$ and $b(k)$ pierces H, we can assume that $(f(v)*b(k)) \cap U$ is a proper disk in U and also in $U(k+1)$. Since the proper disk intersects the proper disk $D(i1)$ at one crossing point on $\partial U(k+1)$, $(M; U(k+1), V(k+1), f)$ is ss-equivalent to $(M; U(k), V(k), f)$ by attaching a pair of handles of index 2.

Case (3): This case is similar to the second case. This establishes Claim 2.

Second step: Let $L(U)=|L| \cap U(H)$ and $L(V)=|L| \cap V(H)$. Let B be a 1-dimensional subcomplex of L such that $|G|$ is a spine of $L(U)$, and G does not collapse to any proper 1-dimensional subcomplex of G (see Fig. 3C). Let $C(1), \cdots, C(t)$ be the connected components of $L(U)-|G|$. Then for each $i=1, \cdots, t$, $C(i)$ is homeomorphic to the half open annulus $S^1 \times [0, 1]$ and the boundary of $C(i)$ lies in $L(V)$. For each $i=1, \cdots, t$, let $L(i)$ be a 1-dimensional subcomplex of L such that

(1) the carrier of $L(i)$ is a simple arc,

(2) the carrier of $L(i)$ meets $|G|$ and the boundary of $C(i)$ at one point respectively,

(3) the intersection of $|L(i)|$ and $L(U)$ lies in the closure of $C(i)$, and

(4) $L(i)-J$ consists of a 1-simplex and a vertex. (see Fig. 3D).

Let $e(1), \cdots, e(h)$ be 1-simplexes in G such that $G-\bigcup_{i=1}^{h} e(i)$ is a union of trees, and h is the rank of $H_1(G)$. Let T' be a union of trees in J such that each tree is a maximal tree in a connected component of J, $T'_0=J_0$, $J \cap L(i) \subset T'$ for each $i=1, \cdots, t$, and $G-T'=\bigcup_{i=1}^{h} e(i)$ (see Fig. 3D). Let $d(1), \cdots, d(p)$ be an indexing

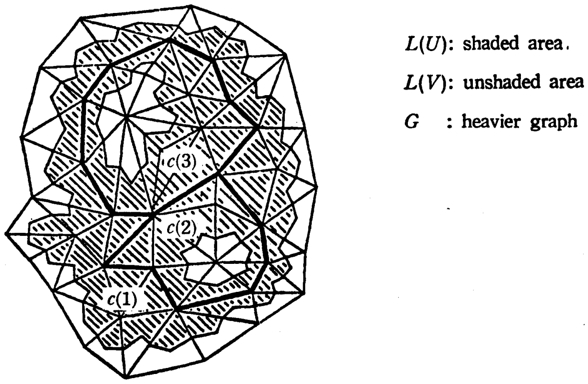
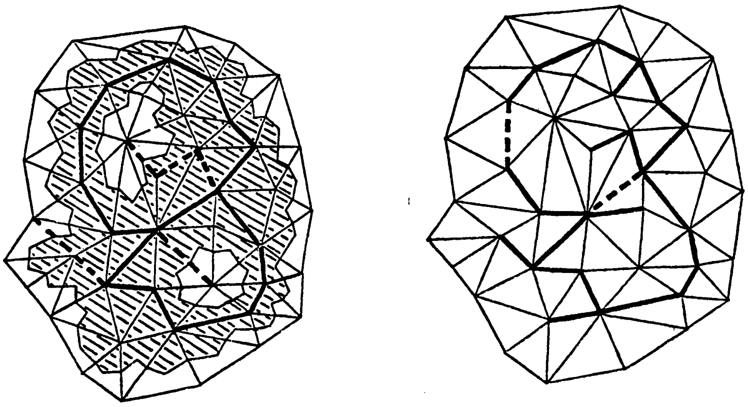


Fig. 3C.



$L(i)$: broken graph

$e(i)$: broken graph
T' : heavier graph

Fig. 3D.

of 1-simplexes of T' so that T' collapses to a union P of points by the collapsing of successive simplexes $d(p)$, $d(p-1), \cdots, d(1)$. For each $k=1, \cdots, p$, let $T'(k)=P \cup \bigcup_{i=1}^{k} d(i)$. Then $T'(p)=T'$ and $T'(k)$ collapses to $T'(k-1)$. For each $k=1, \cdots, p$, let $r(k)$ be the 2-simplex $(f(v)*d(k))$. We can assume that for each $k=1, \cdots, p$, $r(k) \cap V(1) \cap V(H)$ is a proper disk in $V(1)$. Let $R(k)$ be a regular neighborhood of the disk in $V(1)$. Let

$$U(2, p)=\mathrm{Cl}\,(U(1)-\bigcup_{i=1}^{p} f(R(i))) \cup \bigcup_{i=1}^{p} R(i) \quad \text{and}$$
$$V(2, p)=\mathrm{Cl}\,(M-U(2, p))\,.$$

Claim 3. *$(M; U(1), V(1), f)$ is ss-equivalent to $(M; U(2, p), V(2, p), f)$.*

Proof. For each $k=1, \cdots, p$, let

$$U(2, k)=\mathrm{Cl}\,(U(1)-\bigcup_{i=1}^{k} f(R(i))) \cup \bigcup_{i=1}^{k} R(i) \quad \text{and}$$
$$V(2, k)=\mathrm{Cl}\,(M-U(2, k))\,.$$

Let $U(2, 0)=U(1)$ and $V(2, 0)=V(1)$. It is sufficient to show that for each $k=0, 1, \cdots, p-1$, $(M; U(2, k), V(2, k), f)$ is *ss*-equivalent to $(M; U(k+1), V(k+1), f)$. Let $w(j(k))$ be the vertex in $T'(k+1)-T'(k)$. Then one of the two disks $D(j(k)1)$ and $D(j(k)2)$ is a proper disk in $U(2, k)$ and meets $r(k) \cap V(1) \cap V(H)$ at one crossing point on $\partial U(2, k)$. Thus $(M; U(2, k), V(2, k), f)$ is *ss*-equivalent to $(M; U(2, k+1), V(2, k+1), f)$ by attaching a pair of handles of index 2. This establishes Claim 3.

Now after applying an equivalence isotopy if necessary, we may assume that for each $i=1, \cdots, h$, where h is the rank of $H_1(G)$, $(f(v)*e(i)) \cap V(2, p) \cap V(H)$ is a proper disk in $V(2, p)$. Let $E(i)$ be a regular neighborhood of this proper disk

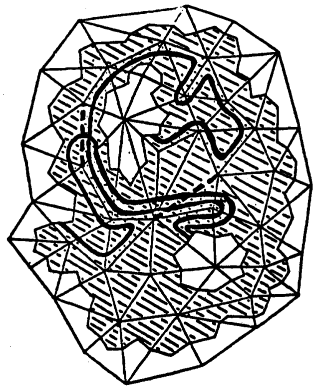


Fig. 3E.

in $V(2, p)$. Let

$$U(3, 1)=\mathrm{Cl}\,(U(2, p)-\bigcup_{i=1}^{h} f(E(i)))\cup\bigcup_{i=1}^{h} E(i) \qquad \text{and}$$
$$V(3, 1)=\mathrm{Cl}\,(M-U(3, 1))\,.$$

Claim 4. *$(M; U(2, p), V(2, p), f)$ is ss-equivalent to $(M; U(3, 1), V(3, 1), f)$.*

Proof. For each $k=1, \cdots, h$, let

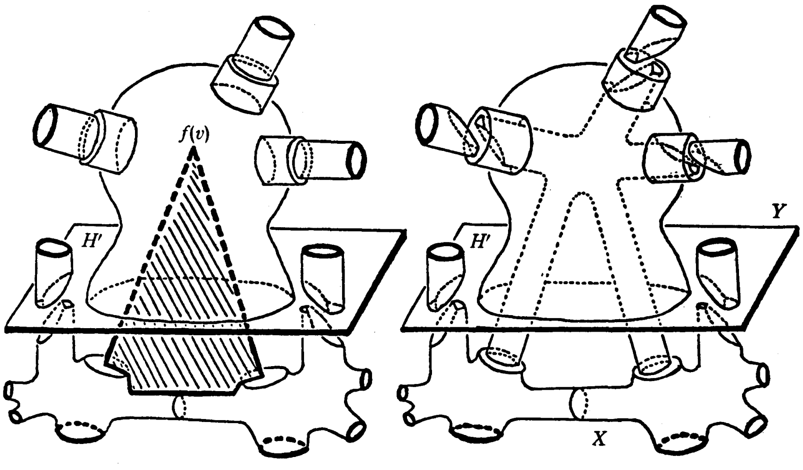
$$U(3, k)=\mathrm{Cl}\,(U(2, p)-\bigcup_{i=k}^{h} f(E(i)))\cup\bigcup_{i=k}^{h} E(i) \qquad \text{and}$$
$$V(3, k)=\mathrm{Cl}\,(M-U(3, k))\,.$$

Let $U(3, h+1)=U(2, p)$ and $V(3, h+1)=V(2, p)$. It is sufficient to show that for each $k=1, \cdots, h$, $(M; U(3, k+1), V(3, k+1), f)$ is ss-equivalent to $(M; U(3, k), V(3, k), f)$. Let

$$\tilde{Z}(h)=(|T'|\cup\bigcup_{j=1}^{t}|L(j)|)\cap L(U) \qquad \text{and}$$
$$\tilde{Z}(k)=Z(h)\bigcup_{i=k+1}^{h} e(i)\,, \quad 1\leq k\leq h\,.$$

For each $k=1, \cdots, h$, let $M(k)$ be a regular neighborhood of $\tilde{Z}(k)$ in $L(U)$. Then there are two disjoint simple arcs on $\partial M(k)$ which are proper arcs in $L(U)$ and cross $e(k)$ once each. Let $P(k)$ be one of these (see Fig. 3E). Let $Q(k)$ be a simple arcs on $|L|$ such that $P(k)\subset Q(k)$, $\partial Q(k)\subset L(v)\cap L_0$, and each interior of a connected component of $Q(k)-P(k)$ lies in the interior or a 2-simplex of L. Let $Q'(k)$ be the cone of $Q(k)$ with cone point $f(v)$. After applying an equivalnece isotopy if necessary, we may assume that $Q'(k)\cap U(3, k+1)$ is a proper disk in $U(3, k+1)$ which meets the proper disk $(f(v)*e(k))\cap V(2, p)\cap V(H)$ at one crossing point on $\partial U(3, k+1)$. Hence $(M; U(3, k+1), V(k+1), f)$ is ss-equivalent to $(M; U(3, k), V(3, k), f)$ by attaching a pair of handles of index 2. This establishes Claim 4.

Third step: We shall prove that $(M; U(3, 1), V(3, 1), f)$ is ss-equivalent to



$(f(v)*b(i)) \cap U(H')$: shaded area

Fig. 3F.

$(M; X, Y, f)$ by a sequence of attaching pairs of handles of index 1. This will be done by the CHS-construction around the vertex $f(v)$.

Let B^* be a regular neighborhood of $f(v)$ in $U(3, 1)$. Let

$$U^*(3, 1) = \mathrm{Cl}\,(U(3, 1)B^*) \cup f(B^*) .$$

For each $k=1, \cdots, q$, and $i=0, 1, \cdots, q$, let

$$B^*(k) = (f(v)*b(k)) \cap U^*(3, 1) \quad \text{and}$$
$$L^*(i) = a(i) \cap U^*(3, 1) .$$

After applying an equivalence isotopy if necessary, we may assume that for each $i=0, 1, \cdots, q$, $L^*(i)$ is a proper arc in $U^*(3, 1)$ (see Fig. 3F). Furthermore we may assume that for each $k=1, \cdots, q$, $B^*(k)$ is a disk, and $\mathrm{Cl}\,(\partial B^*(k) - \partial U^*(3, 1))$ consists of two proper arcs in $U^*(3, 1)$. For each $i=0, 1, \cdots, q$, let $N^*(i)$ be a regular neighborhood of $L^*(i)$ in $U^*(3, 1)$. For each $k=0, 1, \cdots, q$, let

$$U(4, k) = \mathrm{Cl}\,(U^*(3, 1) - \bigcup_{i=0}^{k} N^*(i)) \cup \bigcup_{i=0}^{k} f(N^*(i)) \quad \text{and}$$
$$V(4, k) = \mathrm{Cl}\,(M - U(4, k)) .$$

Then $(M; U(3, 1), V(3, 1), f)$ is equivalent to $(M; U(4, 0), V(4, 0), f)$ by pushing the ball $B^* \cup L^*(0)$ off from $U(3, 1)$. And $(M; U(4, q), V(4, q), f)$ is equivalent to $(M; X, Y, f)$. We shall show that for each $k=1, \cdots, q$, $(M; U(4, k-1), V(k-1), f)$ is *ss*-equivalent to $(M; U(4, k), V(4, k), f)$. Since for each $k=1, \cdots, q$, $B^*(k)$ is a disk in $U(4, k-1)$, and $\mathrm{Cl}\,(\partial B^*(k) - \partial U(4, k-1))$ is a proper arc in $U(4, k-1)$, we conclude that $(M; U(4, k-1), V(4, k-1), f)$ is *ss*-equivalent to $(M; U(4, k), V(4, k), f)$ by attaching a pair of handles of index 1. Thus $(M; U(4, 0), V(4, 0), f)$ is *ss*-equivalent to $(M; U(4,q), V(4,q), f)$. Hence $(M; U(3,1), V(3,1), f)$ is *ss*-equivalent to $(M; X, Y, f)$.

By Claim 2, 3, and 4, $(M; U, V, f)$ is *ss*-equivalent to $(M; U(3,1), V(3,1), f)$. Therefore (M, U, V, f) is *ss*-equivalent to $(M; X, Y, f)$. This completes the proof of the proposition for Case (2). □

§ 4. A Reidemeister-Singer theorem for characteristic Heegaard splittings

In this section we establish the promised Reidemeister-Singer theorem for characteristic Heegaard splittings. Since characteristic Heegaard splittings are more special than the usual ones, we should expect the Reidemeister-Singer theorem to be more delicate. More precisely, it will be seen that characteristic Heegaard splittings are determined, up to *ss*-equivalence, by the nature of the Heegaard surface near $F(f)$. But *ss*-equivalence *can not* modify the embedding type of the neighborhood of $F(f)$ in M. Thus to get the full picture we will have to look at twisting of the annuli $A(i)$ about the components $S(i)$ of the fixed point set $F(f)$.

As a corollary to our Reidemeister-Singer theorem for Heegaard splittings. A sketch of the proof for our main theorem is given in Diagram 4.

In order to apply Chillingworth's theorem, as Craggs does in [Cr], we have change the definition of standard surfaces. Let H be the (α, ν)-surface associated with an invariant triangulation K. Now H satisfies the following four conditions:

(1) There is a polyhedron $U(H)$ in M such that $U(H) \cup f(U(H)) = M$ and $U(H) \cap f(U(H)) = H$.

(2) The fixed point set $F(f)$ is the union of 1-simplexes in K which lie in H.

(3) Any 1-simplex b in K which misses $F(f)$ either misses H or pierces H.

$$\begin{array}{ccc} (M; U, V, f) & & (M; X, Y, f) \\ \cong\big\downarrow & \begin{array}{c}\text{Standardizing}\\ \text{(Prop. 4.5)}\end{array} & \cong\big\downarrow \\ (M; U', V', f)_{K_1} & & (M; X', Y', f)_{K_2} \\ \cong\big\downarrow & \begin{array}{c}\text{Subdivision}\\ \text{(Prop. 4.4)}\end{array} & \cong\big\downarrow \\ (M; U'', V'', f)_K & \xrightarrow[\cong]{\text{Prop. 3.2}} & (M; X'', Y'', f)_K \end{array}$$

$\cong$ denotes *ss*-equivalence,
K_1 and K_2 are triangulations of M,
K is a common subdivision of K_1 and K_2,
$(;,,)_J$ denotes a standard Heegaard splitting with respect to the triangulation J.

Diagram. 4.

(4) For any 3-simplex c in K which meets H at an interior point, $H\cap c$ is a proper disk in c.

Suppose that $\tilde{H}$ is an invariant surface in M which satisfies the above four conditions and the following condition:

A 1-simplex b in K pierces $\tilde{H}$ if and only if b pierces H.

Then $\tilde{H}$ separates the constructive V-set ν from its image $f(\nu)$. Let $U(\tilde{H})$ be the closure of the union of connected components of $M-\tilde{H}$ which intersect ν. Let $Z(H)$ be the union of 1-simplexes in K whose interiors meet $\operatorname{Int} U(H)$ and similarly $Z(\tilde{H})$ the union of 1-simplexes in K whose interiors meet $\operatorname{Int} U(\tilde{H})$. Then we have $Z(H)=Z(\tilde{H})$. Let $\tilde{N}^*$ be a wedge-modification of a regular neighborhood of $Z(\tilde{H})\cap U(\tilde{H})$ in $U(H)$ as in the CHS-construction. Let $\tilde{U}=\operatorname{Cl}(U(\tilde{H})-\tilde{N}^*)\cup f(\tilde{N}^*)$ and $\tilde{V}=\operatorname{Cl}(M-\tilde{U})$. Then the standard Heegaard splitting $(M; U, V, f)$, formed from $Z(H)$ and $U(H)$, is equivalent to $(M; \tilde{U}, \tilde{V}, f)$ since, in the orbit space, the images of H and $\tilde{H}$ are ambient isotopic fixing $F(f)$. Hence from now on a standard surface with respect to an invariant triangulation K means an invariant surface H which satisfies the following four conditions:

(1) There is a polyhedron $U(H)$ in M such that $U(H)\cup f(U(H))=M$ and $U(H)\cap f(U(H))=H$.

(2) The fixed point set $F(f)$ is the union of 1-simplexes in K which lie in H.

(3) Any 1-simplex b in K which misses $F(f)$ either misses H or pierces H.

(4) For any 3-simplex c in K which meets H at an interior point, H separates c into two convex sets.

For any surface H which is standard in the old sense, there is a new standard surface $\tilde{H}$ such that a 1-simplex b in K pierces $\tilde{H}$ if and only if b pierces H. There are infinitely many such new standard surfaces for each standard surfaces in the old sense. For example, we can construct as follows a new standard surface $\tilde{H}$: Suppose that H is the (α, ν)-surface in the old sense. Let h: $K\to[0, 1]$ be the map such that h is affine on each simplex of K and for each vertex v in K

$$h(v)=\begin{cases}1 & \text{if} \quad v\in\nu\\ 1/2 & \text{if} \quad v\in F(f)\\ 0 & \text{if} \quad v\in f(\nu)\,.\end{cases}$$

Then for each 3-simplex c in K, the intersection of $h^{-1}(1/2)$ and c is a proper convex disk in c provided that c meets H at an interior point of c. Hence $h^{-1}(1/2)$ separates c into two convex sets. Let $\tilde{H}=h^{-1}(1/2)$.

For any new standard surface H, there is a standard surface H in the old sense such that a 1-simplex b in K pierces H if and only if b pierces $\tilde{H}$. Hence all propositions in Sections 2 and 3 hold for new standard surfaces. We need Condition (4) to apply Chillingworth's theorem.

In [Ch], Chillingworth proves the following theorem:

Theorem. *Let K be a finite simplicial complex which is topologically a closed ball of dimension $m \leq 3$, embedded rectilinearly as a convex subset of Euclidean n space ($m \leq n$). Let L be a subcomplex of the boundary of K which is connected and simply connected and not the whole of the boundary of K. Then K simplicially collapses to L.*

The following proposition, which is derived from Chillingworth's theorem, is a special case of Lemma 1 in [Cr].

Proposition 4.1. *Let a be a 3-simplex. Suppose that K is a subdivision of a. Let L be the union of all 1-simplexes in K whose interiors lie in* Int a. *Then there is a 2-dimensional subcomplex J of K containing L such that J simplicially collapses to $J \cap K(\partial a)$.*

Proposition 4.2. *Suppose that $(M; U, V, f)$ is a characteristic Heegaard splitting of M. Let D be a proper disk in U. Suppose that L is a connected 1-complex in D such that $|L| \cap \partial D \neq \phi$. Let N be a regular neighborhood of $|L|$ in U such that $f(N) \cap N = \phi$. Let $X = \mathrm{Cl}(U-N) \cup f(N)$ and $Y = \mathrm{Cl}(M-X)$. Then $(M; U, V, f)$ is ss-equivalent to $(M; X, Y, f)$.*

Proof. There is a subcomplex J of L such that $|L| \cup \partial D$ collapses to $|J| \cup \partial D$, and $|J| \cup \partial D$ has no free face. Now J induces an open cell complex structure on D. Then the cell complex collapses cellwise to a point in ∂D. Equivalently a point in ∂D expands cellwise to the cell complex. Let W be a regular neighborhood of $|J|$ in U such that $f(W) \cap W = \phi$. Let $U(1) = \mathrm{Cl}(U-W) \cup f(W)$ and $V(1) = \mathrm{Cl}(M-U(1))$. The existence of the cellwise expansions allows us to conclude that $(M; U, V, f)$ is *ss*-equivalent to $(M; U(1), V(1), f)$. The argument is similar in style to the argument in Proposition 3.2. In the sequence of expansions, each expansion along a 1-cell corresponds to a Type 1 modification and each expansion along 2-cell corresponds to a Type 2 modification in the definition of *ss*-equivalence. It is easy to check that $(M; U(1), V(1), f)$ is equivalent to $(M; X, Y, f)$. □

The following proposition follows from the usual argument in Heegaard theory.

Proposition 4.3. *Let $(M; U, V, f)$ be a characteristic Heegaard splitting.*

Suppose that K is a 2-dimensional complex in U and L is a subcomplex of K such that K collapses to L and $|K|\cap\partial U=|L|$. Let N^* be a wedge-modification of a regular neighborhood of $\mathrm{Cl}(|K_1|-|L|)$ *in U. Let $U'=\mathrm{Cl}(U-N^*)\cup f(N^*)$ and $V'=\mathrm{Cl}(M-U)$. Then $(M;U,V,f)$ is ss-equivalent to $(M;U',V',f)$.* □

Proposition 4.4 (*Subdivision process*). *Let K be an invariant triangulation of M under f. Let ν be a constructive V-set and α a companion A-set. Let $(M;U,V,f)$ be a standard Heegaard splitting* (*with respect to* K) *defined by an (α,ν)-surface H. Let K' be an invariant subdivision of K.*

Define as follows a new partition $(M;U',V')$ of M. *Let $U(H)$ be the closure of the union of the connected components of $M-H$ which meet ν. Let Z' be the union of* 1*-simplexes in K' whose interiors meet* $\mathrm{Int}\,U(H)$. Let N'^* *be a wedge-modification of a regular neighborhood N' of $Z'\cap U(H)$ in $U(H)$ such that $N'\cap H$ is invariant under f. Let $U'=\mathrm{Cl}(U(H)-N'^*)\cup f(N'^*)$ and $V'=\mathrm{Cl}(M-U')$. Then $(M;U',V',f)$ is a characteristic Heegaard splitting and is ss-equivalent to $(M;U,V,f)$.*

Proof. Let $N^*=\mathrm{Cl}(U(H)-U)$ and let $Z(H)$ be the union of 1-simplexes in K whose interiors meet $\mathrm{Int}\,U(H)$. Let

$$Z(2)=\bigcup\{b\in K':\dim b=1,\ b\subset Z',\ b\not\subset|K_1|,\text{ and } b\subset|K_2|\}\quad\text{and}$$
$$Z(3)=\bigcup\{b\in K':\dim b=1,\ b\subset Z',\text{ and } b\not\subset|K_2|\}\,.$$

Let $N'(2)$ be a regular neighborhood of $Z(2)\cap U$ in U, and $N'^*(2)$ a wedge-modification of $N'(2)$. Let $U(0)=\mathrm{Cl}(U-N'^*(2))$ and $V(0)=\mathrm{Cl}(M-U(0))$. Proposition 4.2 shows that $(M;U,V,f)$ is ss-equivalent to $(M;U(0),V(0),f)$ by attaching pairs of handles of index 2.

Let $e(1),\cdots,e(p)$ be the 3-simplexes of K whose interiors meet $\mathrm{Int}\,U(H)$. Let $N(1),\cdots,N(p)$ be regular neighborboods of $e(1)\cap Z(3)\cap U(0),\cdots,e(p)\cap Z(3)\cap U(0)$ in $U(0)$ respectively which are mutually disjoint. Let N be a regular neighborhood of $Z(H)\cap U$ in U such that N^* is a wedge-modification of N. If necessary, modify $N(1),\cdots,N(p)$ so that $N\cup N'(2)\cup\bigcup_{i=1}^{p}N(i)=N'$. For each $i=1,\cdots,p$, let $N^*(i)=N(i)\cap N'^*$ and let

$$U(i)=\mathrm{Cl}(U(i-1)-N^*(i))\cup f(N^*(i))\qquad\text{and}$$
$$V(i)=\mathrm{Cl}(M-U(i))\,.$$

Of course $U(p)=U'$ and $V(p)=V'$.

Claim. *For each $i=1,\cdots,p$, $(M;U(i-1),V(i-1),f)$ is ss-equivalent to $(M;U(i),V(i),f)$.*

Proof. There are two cases:

(1) $e(i)\subset U(H)$.

(2) $e(i)-U(H)\neq\phi$.

Case (1): Suppose that $e(i)\subset U(H)$. Then by Proposition 4.1 there is a 2-dimensional subcomplex C of $K'(e(i))$ such that $e(i)\cap Z(3)\subset|C|$, and C collapses to $C\cap K'(\partial e(i))$. Hence by Proposition 4.3 $(M;\ U(i-1),\ V(i-1), f)$ is *ss*-equivalent to $(M;\ U(i),\ V(i), f)$ by attaching pairs of handles of index 1.

Case (2): Suppose that $e(i)-U(H)\neq\phi$. Note that $e(i)\cap U(H)$ is a convex linear cell. Thus by intersecting the simplexes of K' with this cell we can exhibit $e(i)\cap U(H)$ as the carrier of a convex linear cell complex. Now use Lemma 1 in Chapter 1 in [Zm] to subdivide this cell complex into a simplicial complex J without adding vertices. Let

$$W=\bigcup\{b\in J_1\colon (\operatorname{Int} b)\cap(\operatorname{Int} U(H))\neq\phi,\ \text{and}\ b\subset Z(2)\cup Z(3)\}\,.$$

Let P^* be a wedge-modification of a regular neighborhood of $W\cap U(i)$ in $U(i)$. Let $U^*=\operatorname{Cl}(U(i)-P^*)\cup f(P^*)$ and $V^*=\operatorname{Cl}(M-U^*)$. Then by the same argument as the first case, $(M;\ U(i-1),\ V(i-1), f)$ is *ss*-equivalent to $(M;\ U^*,\ V^*, f)$. Similarly $(M;\ U(i),\ V(i), f)$ is *ss*-equivalent to $(M;\ U^*,\ V^*, f)$. Hence $(M;\ U(i-1),\ V(i-1), f)$ is *ss*-equivalent to $(M;\ U(i),\ V(i), f)$. This establishes the claim.

Therefore $(M;\ U,\ V, f)$ is *ss*-equivalent to $(M;\ U',\ V', f)$ and $(M,\ U',\ V', f)$ is a characteristic Heegaard splitting. This completes the proof of Proposition 4.4. □

Proposition 4.5 (*Standardizing process*). *Suppose that $(M;\ X,\ Y, f)$ is a characteristic Heegaard splitting of M and that ∂X is a standard surface with respect to an invariant triangulation K of M. Then there are an invariant subdivision K'' of K and a standard Heegaard splitting $(M;\ U'',\ V'', f)$ (standard with respect to K'' and ∂X) so that ∂X is a standard surface with respect to K''* and *$(M;\ X,\ Y, f)$ is ss-equivalent to $(M;\ U'',\ V'', f)$.*

Proof. Let K' be an invariant subdivision of K such that ∂X is a standard surface with respect to K', and $N(\partial X;\ K')$ is a bicollar neighborhood of ∂X. It is possible to find such a subdivision by Condition (4) in the definition of standard surface. Let $X'=\operatorname{Cl}(X-N(\partial X;\ K'))$ and $Y'=\operatorname{Cl}(Y-N(\partial X;\ K'))$. Let $J=K'(X')$. Of course $|J|=X'$. Since X' is a cube-with-handles, there are a subdivision J' of J and a 2-dimensional subcomplex L of J' such that $L_1=(J')_1$, and L simplicially collapses to $L\cap J(\partial X')$ (see the argument in the proof of Lemma 1 in [Cr]). Since the map $f\colon K'(X')\to K'(Y')$ is an isomorphism, J' induces a subdivision of $K'(Y')$ which maps simplicially onto J' under f. These subdivisions induce an invariant subdivision K' of K' such that

$$(K'')_0 \cap U = (J')_0 \cup ((K')_0 \cap F(f)) \quad \text{and}$$
$$K''(X') = J'.$$

We construct as follows a new partition $(M; X(1), Y(1))$ of M. Let

$$G(1) = \bigcup \{b \in K'' : \dim b = 1, \operatorname{Int} b \subset N(\partial X; K') - F(f)\} \quad \text{and}$$
$$G(2) = \bigcup \{b \in J' : \dim b = 1, b \not\subset G(1)\}.$$

Let $N(1)$ be a regular neighborhood of $G(1) \cap X$ in X. Let $N^*(1)$ be a wedge-modification of $N(1)$. Let

$$X(1) = \mathrm{Cl}\,(X - N^*(1)) \cup f(N^*(1)) \quad \text{and}$$
$$Y(1) = \mathrm{Cl}\,(M - X(1)).$$

Then by the same argument as in the second case for the claim in Proposition 4.4, $(M; X, Y, f)$ is *ss*-equivalent to $(M; X(1), Y(1), f)$.

Let $N(2)$ be a regular neighborhood of $G(2)$ in X'. Let $N^*(2)$ be a wedge-modification of $N(2)$. We can assume that $N(1) \cup N(2)$ is a regular neighborhood of $(G(1) \cup G(2)) \cap X$ in X. Let

$$U'' = \mathrm{Cl}\,(X(1) - N^*(2)) \cup f(N^*(2)) \quad \text{and}$$
$$V'' = \mathrm{Cl}\,(M - U'').$$

Then $(M; U'', V'', f)$ is a standard Heegaard splitting of M with respect to ∂X and K''. By Proposition 4.3 and the existence of the complex L, we can conclude that $(M; X(1), Y(1), f)$ is *ss*-equivalent to $(M; U'', V'', f)$. Therefore (M, X, Y, f) is *ss*-equivalent to $(M; U'', V'', f)$. □

Proposition 4.6. *Let $(M; X, Y, f)$, K, K'', and $(M; U'', V'', f)$ be as above. Let $(M; U, V, f)$ be a standard Heegaard splitting of M with respect to the standard surface ∂X and the triangulation K. Then $(M; X, Y, f)$ is ss-equivalent to $(M; U, V, f)$.*

Proof. By Proposition 4.4, $(M; U, V, f)$ is *ss*-equivalent to $(M; U'', V'', f)$. By Proposition 4.5, $(M; X, Y, f)$ is *ss*-equivalent to $(M; U'', V'', f)$. Therefore $(M; X, Y, f)$ is *ss*-equivalent to $(M; U, V, f)$. □

Proposition 4.7. *Let $(M; U, V, f)$ and $(M; X, Y, f)$ be characteristic Heegaard splittings of M. Suppose that there is an invariant triangulation K of M such that both ∂U and ∂X are standard surfaces with respect to K, and $U \cap N(F(f); K) = X \cap N(F(f); K)$. Then $(M; U, V, f)$ is ss-equivalent to $(M; X, Y, f)$.*

Proof. Let $(M; U', V', f)$ and $(M; X', Y', f)$ be standard Heegaard splittings of M with respect to the standard surfaces ∂U and ∂X, and the triangulation K respectively. By Proposition 4.6, $(M; U, V, f)$ is *ss*-equivalent to $(M; U', V', f)$, and $(M; X, Y, f)$ is *ss*-equivalent to $(M; X', Y', f)$. By Proposition 3.2, $(M; U', V', f)$ is *ss*-equivalent to $(M; X', Y', f)$. Therefore $(M; U, V, f)$ is *ss*-equivalent to $(M; X, Y, f)$. □

Let h: $M\times I\rightarrow M$ be a map. For each $t\in I$, let h_t: $M\rightarrow M$ denote the map $h_t(x)=h(x, t)$ for all $x\in M$.

Proposition 4.8. *Let $(M; U(1), V(1), f)$ and $(M; U(2), V(2), f)$ be characteristic Heegaard splittings of M. Suppose that, for some neighborhood N of $F(f)$ in M, we have $U(1)\cap N=U(2)\cap N$. Then there are an invariant triangulation K of M and ambient isotopies $h(1), h(2)$: $M\times I\rightarrow M$ fixing $F(f)$ such that for each $x\in M$, $t\in I$, and $i=1, 2$:*

(1) $h(i)_t\circ f=f\circ h(i)_t$.

(2) *$h(i)_1(\partial U(i))$ is a standard surface with respect to K.*

(3) $h(1)_1(U(1))\cap N(F(f); K)=h(2)_1(U(2))\cap N(F(f); K)$.

Proof. Let Z be the orbit space of f. Let g: $M\rightarrow Z$ by the identification map. Let $G(i)=g(\partial U(i))$, $F=g(F(f))$. Let J be a triangulation of Z such that:

(1) The set F is triangulated by J.

(2) The set $N(F; J)$ is a resular neighborhood of F.

(3) On the set $N(F; J)$ we have $G(1)\cap N(F; J)=G(2)\cap N(F; J)$ and this polyhedron is an annulus.

There are an ambient isotopy k: $Z\times I\rightarrow Z$ fixing F and a subdivision J' of J such that the following conditions hold:

(1) The set $N(F; J')$ is a regular neighborhood of F.

(2) The intersection $k_1(G(i))\cap(J')_0$ is contained in F.

(3) For each 3-simplex c of J' in $N(F; J')$, if $k_1(G(i))\cap(c-F)\neq\phi$, the intersection $k_1(G(i))\cap c$ is a proper disk and splits c into two convex cells.

There are ambient isotopies $k(1), k(2)$: $Z\times I\rightarrow Z$ fixing $N(F; J')$ such that (*) below holds:

(*) For each $i=1, 2$, and each 1-simplex b in $J'-J'(F)$, the intersection $k(i)_1\circ k_1(G(i))\cap b$ consists of at most finitely many points at which $\text{Int}\, b$ pierces the surface $k(i)_1\circ k_1(G(i))$.

If necessary, further apply ambient isotopies fixing $N(F; J')$ and replace J' by a further subdivisions so that for each 3-simplex c in J', the intersection

$k(i)_1 \circ k_1(G(i)) \cap c$ consists of at most one proper disk in c, and (*) still holds. Further, if necessary, apply other ambient isotopies fixing $(J')_2$ and subdivide J' so that (**) below holds:

(**) For each 3-simplex c in J', the intersection $k(i)_1 \circ k(G(i)) \cap c$ consists of at most one disk, and for each 1-simplex b in $J' - J'(F)$, the intersection $k(i)_1 \circ k_1(G(i)) \cap b$ consists of at most one point which lies in $\mathrm{Int}\, b$.

Applying further ambient isotopies fixing F, we may assume that for each 2-simplex b in J', the intersection $k(i)_1 \circ k_1(G(i)) \cap b$ consists of at most one proper line segment in b whose end points are barycenters of 1-simplexes in J' or vertices of J' in F, and (**) still holds. And finally applying further ambient isotopies fixing $(J')_2$, we may assume that for each 3-simplex c in J', if $k(i)_1 \circ k_1(G(i)) \cap c \neq \phi$, then the surface $k(i)_1 \circ k_1(G(i))$ separates c into two convex sets.

Let $\tilde{h}(1), \tilde{h}(2)$: $Z \times I \to Z$ be ambient isotopies defined by

$$\tilde{h}(i)(z, t) = \begin{cases} k(z, 2t) & \text{if } t \leq 1/2 \\ k(i)(k_1(z), 2t-1) & \text{if } 1/2 \leq t\,. \end{cases}$$

Pull back the triangulation J' and the ambient isotopies $\tilde{h}(1)$ and $\tilde{h}(2)$ to get the triangulation K of M and ambient isotopies $h(1)$ and $h(2)$. It is easy to check that K and $h(1)$ and $h(2)$ satisfy the conditions. □

Theorem. *Let M be an orientable closed 3-manifold. Let f_1, f_2: $M \to M$ be orientation preserving involutions. Then $(M; U_1, V_1, f_1)$ and $(M; U_2, V_2, f_2)$ are ss-equivalent if and only if there is a homeomorphism h: $M \to M$ such that $f_2 = h \circ f_1 \circ h^{-1}$ and the intersection $h(\partial U_1) \cap \partial U_2$ contains a neighborhood of the fixed point set for f_2 in ∂U_2.*

In particular if f_1 and f_2 are conjugate fixed point free involutions, then $(M; U_1, V_1, f_1)$ and $(M; U_2, V_2, f_2)$ are ss-equivalent.

Proof. Suppose that the two splittings are *ss*-equivalent. Then there is a homeomorphism h: $M \to M$ such that $f_2 = h \circ f_1 \circ h^{-1}$ and $(M; h(U_1), h(V_1), f_2)$ and $(M; U_2, V_2, f_2)$ are *ss*-equivalent by modifications of Type 2 or 3. Since modifications of Type 2 or 3 can not modify a neighborhood of the fixed point set $F(f_2)$, there is a regular neighborhood N of $F(f_2)$ such that $N \cap h(\partial U_1) = N \cap \partial U_2$. Therefore the intersection $h(\partial U_1) \cap \partial U_2$ contains a neighborhood of $F(f_2)$ in ∂U_2.

Suppose that there is a homeomorphism h: $M \to M$ such that $f_2 = h \circ f_1 \circ h^{-1}$ and $h(\partial U_1) \cap \partial U_2$ contains a neighborhood of $F(f_2)$ in ∂U_2. Let $U(1) = h(U_1)$, $V(1) = h(V_1)$, $U(2) = U_2$, $V(2) = V_2$, and $f' = f_2$. Then it is sufficient to show that $(M; U(1), V(1), f')$ and $(M; U(2), V(2), f')$ are *ss*-equivalent.

By Proposition 4.8, there are an invariant triangulation K and ambient isotopies $h(1), h(2)$: $M\times I\to M$ such that for each $x\in M$, $t\in I$, and $i=1, 2$, the following conditions hold:

(1) $h(i)_t\circ f'=f'\circ h(i)_t$.

(2) $h(i)_1(\partial U(i))$ is a standard surface with respect to K.

(3) $h(1)_1(U(1))\cap N(F(f'); K)=h(2)_1(U(2))\cap N(F(f'); K)$.

Since $h(i)_1\circ f'=f'\circ h(i)_1$, we have $(M; U(i), V(i), f')$ equivalent to $(M; h(i)_1(U(i)), h(i)_1(V(i)), f')$. By Proposition 4.7, $(M; h(1)_1(U(1)), h(1)_1(V(1)), f')$ is *ss*-equivalent to $(M; h(2)_1(U(2)), h(2)_1(V(2)), f')$. Therefore $(M; U(1), V(1), f')$ is *ss*-equivalent to $(M; U(2), V(2), f')$. □

Notes. (1) Theorem holds for disconnected closed 3-manifolds if we change, as follows, the definition of characteristic Heegaard splittings: A tuple $(M; U, V, f)$ is a characteristic Heegaard splitting provided that (i) $(M; U, V)$ is a partition of M into regular neighborhoods of graphs (possibly disconnected) so that $\partial U=\partial V$ and (ii) $f(U)=V$. (2) For the cases of fixed point free involutions, Theorem does not require orientability of M if we change the definition as in (1).

As a special case of Theorem we give a proof of the usual Reidemeister-Singer theorem for Heegaard splittings.

Theorem (*Reidemeister-Singer*). *Any two Heegaard splittings of a closed* 3-*manifold M are stably equivalent.*

Proof. Let M' be a copy of M. Let h: $M\to M'$ be a homeomorphism. Let N be the disjoint union of M'. Then N is a closed 3-manifold. Let f': $N\to N$ be the involution defined by

$$f'(x)=\begin{cases} h(x) & \text{if } x\in M \\ h^{-1}(x) & \text{if } x\in M' . \end{cases}$$

Let $(M; U, V)$ and $(M; X, Y)$ be Heegaard splittings of M. Then h induces characteristic Heegaard splittings $(N; U', V', f')$ and $(N; X', Y', f')$ of N (as indicated in the preceding notes) such that $U'\cap M=U$ and $X'\cap M=X$.

Since f' is a fixed point free involution on N, the splittings $(N; U', V', f')$ and $(N; X', Y', f')$ are *ss*-equivalent by Theorem. Hence there is a sequence of elementary modifications as in the definition of *ss*-equivalence which converts the splitting to the other. The restriction of the sequence to M exhibits the desired stable equivalence of $(M; U, V)$ and $(M; X, Y)$. □

References

[Ch] D. R. J. Chillingworth: *Collapsing three-dimensional convex polyhedra*, Proc. Cambridge Phil. Soc. **63**, 335–357 (1967).

[Cr] R. Craggs: *A new proof of the Reidemeister-Singer theorem on stable equivalence of Heegaard splittings*, Proc. Amer. Math. Soc. **57**, 143–147 (1976).

[MS] D. Montgomery and H. Samelson: *A theorem of fixed points of involutions in* S^3, Can. J. Math. **7**, 208–220 (1955).

[Rd] K. Reidemeister: *Zur dreidimensionalen Topologie*, Abh. math. Sem. Univ. Hamburg **9**, 189–194 (1933).

[Sg] J. Singer: *Three dimensional manifolds and their Heegaard diagrams*, Trans. Amer. Math. Soc. **35**, 88–111 (1933).

[Zm] E. C. Zeeman: *Seminar on combinatorial topology*, Publ. Inst. des Hautes Etudes Sci. Paris, 1963 (mimeographed notes).

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