YOKOHAMA MATHEMATICAL
JOURNAL VOL. 27, 1979

MODIFIED CHAIN CONDITIONS FOR RINGS WITHOUT IDENTITY

By

F. S. CATER

(Received September 20, 1977)

Introduction.

Charles Hopkins (Annals of Math., 1939, pp. 712-730) has shown that a unital Artinian left R-module over a left Artinian ring R with identity, must be Noetherian. Moreover, any left Artinian ring with identity must be left Noetherian. These results are known to be false for rings without identity.

I. S. Cohen (Duke Math. Journal, 1950, pp. 27-42) has shown that a commutative Noetherian ring with identity is Artinian if and only if every proper prime ideal of the ring is maximal. Again, this is false for rings without identity.

In this paper we will invent generalizations of the definitions "Artinian" and "Noetherian" to obtain analogues of Hopkins' and Cohen's theorems that do not require an identity. For rings with identity "almost left Noetherian (Artinian)" will be equivalent to "left Noetherian (Artinian)". We will discover that many of the well known properties of left Artinian (Noetherian) rings are also properties of almost left Artinian (Noetherian) rings. In general, every almost left Artinian ring must be almost left Noetherian. In our analogue of Cohen's theorem we will eliminate both the commutativity and the identity. To this end, we invent "almost prime and almost maximal" ideals. Indeed an almost left Noetherian ring is almost left Artinian if and only if every proper almost prime ideal is an almost maximal left ideal.

Our work will incidentally prove the following results.

1. In a left Artinian ring R, any expanding sequence of left ideals $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$ satisfying $\bigcup_n J_n = R(\bigcup_n J_n)$, must terminate.

2. Any left Artinian ring R with nil radical W satisfying $W=RW$, must be left Noetherian.

3. A commutative Noetherian ring R is the direct sum of an Artinian ring with a nilpotent ring if and only if every proper prime ideal of R is a maximal ideal of R. If R has an identity, then the nilpotent summand vanishes, and this becomes Cohen's theorem.

Essay.

Let $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots \subseteq J_n \subseteq \cdots$ be an expanding sequence of (left) ideals in a ring R. We say that the sequence *terminates* if for some $n \in Z_+$, $J_n = J_{n+k}$ for all $k \in Z_+$. We say that the sequence *almost terminates* if for some n, $q \in Z_+$, $R^q(\bigcup_n J_n) \subseteq J_m$. (Equivalently, $R^p(\bigcup_n J_n) \subseteq J_p$ for some $p \in Z_+$; just let p be the maximum of q and m.) Of course the sequence almost terminates if it terminates.

Now let $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq \cdots$ be a contracting sequence of (left) ideals of a ring R. We say that the sequence *terminates* if for some $n \in Z_+$, $I_n = I_{n+k}$ for all $k \in Z_+$. We say that the sequence *almost terminates* if for some m, $q \in Z_+$, $R^q I_m \subseteq \bigcap_n I_n$. (Equivalently, $R^p I_p \subseteq \bigcap_n I_n$ for some $p \in Z_+$; just let p be the maximum of q and m.) Of course the sequence almost terminates if it terminates.

We say that the ring R is *left Artinian* if every contracting sequence of left ideals of R terminates. We say that R is *almost left Artinian* if every contracting sequence of left ideals almost terminates. We say that R is *left Noetherian* if every expanding sequence of left ideals of R terminates. We say that R is *almost left Noetherian* if every expanding sequence of left ideals almost terminates.

Of course in a ring R satisfying $r \in Rr$ for each $r \in R$ (this happens, for example, if R has an identity), "left Artinian" is equivalent to "almost left Artinian" and "left Noetherian" is equivalent to "almost left Noetherian". On the other hand every nilpotent ring is both almost left Artinian and almost left Noetherian. The reader can easily construct a nilpotent ring that is neither left Artinian nor left Noetherian.

We show that these properties are invariant under ring homomorphisms.

Proposition 1. *Any homomorphic image of an almost left Artinian (Noetherian) ring is almost left Artinian (Noetherian).*

Proof. Let f be a homomorphism of the almost left Artinian ring R. Let $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_n \supseteq \cdots$ be a contracting sequence of left ideals of the ring fR. Then $f^{-1}J_1 \supseteq f^{-1}J_2 \supseteq \cdots \supseteq f^{-1}J_n \supseteq \cdots$ is a contracting sequence of left ideals of R. By hypothesis, there is an index q such that $R^q f^{-1} J_q \subseteq \bigcap_n f^{-1} J_n$ and

$$(fR)^q J_q = f[R^q f^{-1} J_q] \subseteq f(\bigcap_n f^{-1} J_n) = \bigcap_n J_n \,.$$

Thus fR is also almost left Artinian.

Now suppose R is almost left Noetherian. Let $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \cdots$ be any expanding sequence of left ideals of fR. Then $f^{-1}J_1 \subseteq f^{-1}J_2 \subseteq \cdots \subseteq f^{-1}J_n \subseteq \cdots$ is an expanding sequence of left ideals of R. By hypothesis, there is an index q such that $R^q(\bigcup_n f^{-1}J_n)\subseteq f^{-1}J_q$ and

$$(fR)^q[\bigcup_n J_n]=f[R^q \bigcup_n f^{-1}J_n]\subseteq f[f^{-1}J_q]=J_q .$$

Thus fR is also almost left Noetherian.

Analogous arguments prove that left Artinian and left Noetherian are also invariant under homomorphisms. We proceed to the direct sum of two rings.

Proposition 2. *Let R be the direct sum of the rings R_1 and R_2; $R=R_1\oplus R_2$. Then R is almost left Artinian (Noetherian) if and only if R_1 and R_2 are almost left Artinian (Noetherian).*

Proof. The only if part follows from Proposition 1. So suppose R_1 and R_2 are almost left Artinian. Let $J_1\supseteq J_2\supseteq\cdots\supseteq J_n\supseteq\cdots$ be a contracting sequence of left ideals of R. Then $R_1J_1\supseteq R_1J_2\supseteq\cdots\supseteq R_1J_n\supseteq\cdots$ is a contracting sequence of left ideals of R_1, and $R_2J_1\supseteq R_2J_2\supseteq\cdots\supseteq R_2J_n\supseteq\cdots$ is a contracting sequence of left ideals of R_2. Hence there is an index q such that $R_i^qR_iJ_q\subseteq\bigcap_n R_iJ_n\subseteq\bigcap_n J_n$ $(i=1, 2)$; just let q be the maximum of the indices obtained for R_1 and R_2. Then

$$R^{q+1}J_q=(R_1^{q+1}J_q)+(R_2^{q+1}J_q)\subseteq\bigcap_n J_n .$$

Now suppose R_1 and R_2 are almost left Noetherian. Let $J_1\subseteq J_2\subseteq\cdots\subseteq J_n\subseteq\cdots$ be an expanding sequence of left ideals in R. As before, there is an index q such that $R_i^q(\bigcup_n R_iJ_n)\subseteq R_iJ_q\subseteq J_q$ $(i=1, 2)$, and $R^{q+1}(\bigcup_n J_n)\subseteq R_1^q(\bigcup_n R_1J_n)+R_2^q(\bigcup_n R_2J_n)\subseteq J_q$.

The corresponding argument for left Artinian (Noetherian) is trickier. Let (J_n) be a contracting sequence of left ideals of R where R_1 and R_2 are left Artinian. Put $I_n=\{x\in R_1\colon x+y\in J_n$ for some $y\in R_2\}$. Evidently I_n is a left ideal of R_1, and $I_1\supseteq I_2\supseteq\cdots\supseteq I_n\supseteq\cdots$. Also $(R_2\cap J_n)_n$ is a contracting sequence of left ideals of R_2. So there is an index q such that $I_q=I_{q+k}$ and $R_2\cap J_q=R_2\cap J_{q+k}$ for all $k\in Z_+$. Now take any $z\in J_q$. Say $z=x+y$ $(x\in R_1, y\in R_2)$. Then $x\in I_q=I_{q+k}$ $(k\in Z_+)$. But for any $k\in Z_+$, $x+w\in J_{q+k}$ for some $w\in R_2$, $x+y\in J_q$, $y-w=(x+y)-(x+w)\in R_2\cap J_q=R_2\cap J_{q+k}$, and $z=x+y=(x+w)+(y-w)\in J_{q+k}$. Thus $J_q=J_{q+k}$ for all $k\in Z_+$, and R is left Artinian.

Now suppose R_1 and R_2 are left Noetherian and (J_n) is an expanding sequence of left ideals of R. Define the left ideals I_n of R_1 as before. This time (I_n) is expanding and $(R_2\cap J_n)_n$ is an expanding sequence of left ideals of R_2. Say $q\in Z_+$ such that $I_q=I_{q+k}$ and $R_2\cap J_q=R_2\cap J_{q+k}$ for all $k\in Z_+$. Fix $k\in Z_+$ and take any $z\in J_{q+k}$. Say $z=x+y$ $(x\in R_1, y\in R_2)$. Then $x\in I_{q+k}=I_q$ and so there is a $w\in R_2$ such that $x+w\in J_q\subseteq J_{q+k}$. Whence

$$(x+y)-(x+w)=y-w \in R_2 \cap J_{q+k}=R_2 \cap J_q ,$$

and so $z=x+y=(x+w)+(y-w) \in J_q$. Thus $J_q=J_{q+k}$ for all $k \in Z_+$, and R is left Noetherian.

Of course Proposition 2 can be extended to the direct sum of finitely many rings. We now come to a result for our "almost" properties that does not carry over to "left Artinian" or "left Noetherian."

Proposition 3. *Let I be a left ideal in an almost left Artinian (Noetherian) ring R. Then I, as a ring, is almost left Artinian (Noetherian).*

Proof. Suppose R is almost left Artinian and (J_n) is a contracting sequence of left ideals of the ring I. Then (IJ_n) is a contracting sequence of left ideals of R. So $R^q(IJ_q) \subseteq \bigcap_n IJ_n$ for some $q \in Z_+$. We have

$$I^{q+1}J_q \subseteq R^q(IJ_q) \subseteq \bigcap_n IJ_n \subseteq \bigcap_n J_n .$$

Thus I is almost left Artinian. The argument for "almost left Noetherian" is analogous and is left to the reader.

Consider the ring of 2 by 2 matrices of the form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ where b and c are real numbers and a is a rational number. Let I be the left ideal composed of those matrices with $a=c=0$. The reader can show that R is left Artinian and left Noetherian, but the ring I is neither.

Now we find alternative ways of defining our properties.

Proposition 4. *Let R be a ring. Then R is almost left Artinian (Noetherian) if and only if for each nonvoid family $\mathscr{I}$ of left ideals, there is a member I of $\mathscr{I}$ and a $q \in Z_+$ such that $R^qI \subseteq J$ ($R^qJ \subseteq I$) for any J in $\mathscr{I}$ satisfying $J \subseteq I$ ($I \subseteq J$).*

Proof. The condition implies almost left Artinian (Noetherian); just let $\mathscr{I}$ be the set of left ideals occurring in the sequence. Suppose R is almost left Artinian, and let $\mathscr{I}$ be a nonvoid family of left ideals. Suppose further that the condition does not hold for $\mathscr{I}$. Choose any $I_1 \in \mathscr{I}$. Then there is an $I_2 \in \mathscr{I}$ such that $I_1 \supseteq I_2$ but $RI_1 \not\subseteq I_2$. There is an $I_3 \in \mathscr{I}$ such that $I_2 \supseteq I_3$ but $R^2I_2 \not\subseteq I_3 \cdots$. There is an $I_n \in \mathscr{I}$ such that $I_{n-1} \supseteq I_n$ but $R^{n-1}I_{n-1} \not\subseteq I_n$, and so forth. The contracting sequence $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ violates the hypothesis that R is almost left Artinian.

Suppose that R is almost left Noetherian and $\mathscr{I}$ is a nonvoid family of left ideals for which the condition does not hold. Pick any $I_1 \in \mathscr{I}$. Then there is an $I_2 \in \mathscr{I}$ such that $I_1 \subseteq I_2$ but $R^2I_2 \not\subseteq I_1$. There is an $I_3 \in \mathscr{I}$ such that $I_2 \subseteq I_3$ but $R^3I_3 \not\subseteq I_2 \cdots$. There is an $I_n \in \mathscr{I}$ such that $I_{n-1} \subseteq I_n$ but $R^nI_n \not\subseteq I_{n-1}$, and so forth.

The expanding sequence $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ violates the hypothesis the R is almost left Noetherian.

Likewise R is left Artinian if and only if every nonvoid family of left ideals has a minimal member, and R is left Noetherian if and only if every nonvoid family of left ideals has a maximal member.

Next we come to several results on our "almost" properties, culminating in Theorem 1, that are usually associated with "left Artinian" and "left Noetherian".

By the *nil radical* W of a ring R, we mean the sum of all the nilpotent ideals of R. Of course, W is a nil ideal. It is also known that W equals the sum of the nilpotent left ideals of R, and W equals the sum of the nilpotent right ideals of R. Consult M. Gray, *A radical Approach to Algebra*, p. 28.

Lemma 1. *Let W be the nil radical of an almost left Noetherian ring R. Then W is nilpotent.*

Proof. Let $\mathscr{I}$ denote the family of all nilpotent left ideals. For example, $(0) \in \mathscr{I}$, so $\mathscr{I}$ is nonvoid. By Proposition 4, there is a nilpotent left ideal I and a $q \in Z_+$ such that $R^q J \subseteq I$ for any nilpotent left ideal $J \supseteq I$. Say $I^n = (0)$. Then

$$J^{nq+n} \subseteq (R^q J)^n \subseteq I^n = (0) ,$$

and $J^{nq+n} = (0)$ for any nilpotent left ideal $J \supseteq I$.

Now let $x_1, \cdots, x_{nq+n} \in W$. Since the sum of two nilpotent left ideals is nilpotent, it follows that $x_i \in J_i$ for some nilpotent left ideals J_i $(i=1, \cdots, nq+n)$. Put $J = J_1 + \cdots + J_{nq+n} + I$. Then J is nilpotent and $J \supseteq I$. By the preceding paragraph, $J^{nq+n} = (0)$ so $x_1 \cdots x_{nq+n} = 0$. It follows that $W^{nq+n} = (0)$.

Lemma 2. (*Hopkins*) *Let I be any nil left ideal of an almost left Artinian ring R. Then I is nilpotent. In particular, the nil radical of R is nilpotent.*

Proof. Assume I is not nilpotent. We first claim that there is a minimal nonnilpotent left ideal contained in I. Suppose there is not. Put $I_1 = I$. Then $(0) \neq I_1^2 \subseteq RI_1$ and RI_1 is not nilpotent since I_1 is not. So there is a nonnilpotent left ideal $I_2 \subsetneqq RI_1 \subseteq I_1$. Then $(0) \neq I_2^3 \subseteq R^2 I_2$ and $R^2 I_2$ is not nilpotent since I_2 is not. ...So there is a nonnilpotent left ideal $I_n \subsetneqq R^{n-1} I_{n-1} \subseteq I_{n-1}$. Then $(0) \neq I_n^{n+1} \subseteq R^n I_n$ and $R^n I_n$ is not nilpotent since I_n is not. And so forth. Finally, the contracting sequence $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ conflicts with the hypothesis that R is almost left Artinian. Thus there is such a minimal left ideal; call it K. But $K^2 \subseteq K$ and K^2 is not nilpotent since K is not. Then $K^2 = K$ by the minimality of K.

Now let $\mathscr{S}$ denote the family of all left ideals $J \subseteq K$ such that $KJ \neq (0)$. For example, $K \in \mathscr{S}$. We claim that $\mathscr{S}$ has a minimal element. Suppose not. Put $K_1 = K$. Then

$$(0) \neq KK_1 = K^2K_1 \subseteq K(RK_1),$$

so $RK_1 \in \mathscr{S}$. Choose $K_2 \in \mathscr{S}$ such that $K_2 \subsetneqq RK_1 \subseteq K_1$. Then

$$(0) \neq KK_2 = K^3K_2 \subseteq K(R^2K_2),$$

so $R^2K_2 \in \mathscr{S}$.... Choose $K_n \in \mathscr{S}$ such that $K_n \subsetneqq R^{n-1}K_{n-1} \subseteq K_{n-1}$. Then

$$(0) \neq KK_n = K^{n+1}K_n \subseteq K(R^nK_n),$$

so $R^nK_n \in \mathscr{S}$. Choose $K_{n+1} \in \mathscr{S}$ such that $K_{n+1} \subsetneqq R^nK_n \subseteq K_n$. And so forth. The contracting sequence of left ideals $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$ conflicts with the hypothesis that R is almost left Artinian. Thus $\mathscr{S}$ has a minimal element; call it J.

Now $KJ \neq (0)$, so choose an element $x \in J$ such that $Kx \neq (0)$. Then Kx is a left ideal of R, and $K(Kx) = K^2x = Kx \neq (0)$. But $Kx \subseteq K$ since $x \in J \subseteq K$, so $Kx \in \mathscr{S}$. Also $Kx \subseteq J$ since $x \in J$. But J is minimal in $\mathscr{S}$, so $Kx = J$. Thus there is an $a \in K$ such that $ax = x$. It follows that

$$0 \neq x = ax = a^2x = \cdots = a^nx = \cdots$$

for all $n \in Z_+$. Hence $a^n \neq 0$ for all $n \in Z_+$ and a is not nilpotent. But $a \in K \subseteq I$, contrary to the hypothesis that I is nil.

Proposition 5. *Let R be an almost left Noetherian ring and let I be a nonzero nil right ideal of R. Then R has a nonzero nilpotent left ideal.*

Proof. For each $x \in R$, let $f(x)$ denote the left ideal $\{y \in R: yx = 0\}$. We may assume, without loss of generality, that $f(x) \neq R$ if $x \neq 0$; for otherwise the right annihlator of R is the desired left ideal. Let $\mathscr{S}$ be the family of left ideals $\{f(x)\}$ where x runs over the nonzero elements of I. By Proposition 4, there exists a nonzero $x_0 \in I$ and a $q \in Z_+$ such that $R^qf(x) \subseteq f(x_0)$ for any nonzero $x \in I$ satisfying $f(x_0) \subseteq f(x)$.

Suppose now $v \in R$ and $x_0v \neq 0$. Then x_0v is a nonzero element of I since $x_0 \in I$ and I is a right ideal. So x_0v is nilpotent. Say $0 = (x_0v)^k \neq (x_0v)^{k-1}$. Of course $k \geq 2$ and $x_0v \in f((x_0v)^{k-1}) = f(x_0r)$, so $R^qx_0v \subseteq R^qf(x_0r) \subseteq f(x_0)$ since $f(x_0) \subseteq f(x_0r)$. Hence $(R^qx_0v)x_0 = R^qx_0vx_0 = (0)$.

Even if $x_0v = 0$, we still have $R^qx_0vx_0 = (0)$. In other words, $R^qx_0Rx_0 = (0)$. Whence $(Rx_0 + Zx_0)^{q+3} = (0)$ and $Rx_0 + Zx_0$ is a nilpotent left ideal containing $x_0 \neq 0$.

Before our next result we make an observation. If I is a nonzero nil left ideal of R, then R has a nonzero nil right ideal. To see this we can suppose, without loss of generality, that $IR\neq(0)$; for otherwise I is a right ideal. Choose some $x\in I$ such that $xR\neq(0)$. For any $r\in R$, rx is nilpotent, $(xr)^k=x(rx)^{k-1}r$, and clearly xr is also nilpotent. Thus xR is a nonzero nil right ideal. The converse is proved analogously. If R has a nonzero nil right ideal, R must also contain a nonzero nil left ideal. We employ this immediately.

Proposition 6. *Let the ring R be almost left Artinian or almost left Noetherian. Then any nil one sided ideal of R is nilpotent.*

Proof. The nil radical W of R is nilpotent by Lemmas 1 and 2. Also R/W is either almost left Artinian or almost left Noetherian. Now suppose I is a nil one sided ideal that is not nilpotent. Then $J=(I+W)/W$ is a nil one sided ideal of R/W that is nonzero. By the remark preceding this proposition, by Proposition 5, and by Lemma 2, R/W has a nonzero nilpotent left ideal. But this is impossible.

Next we generalize a result of Brauer concerning Artinian rings and idempotent elements.

Lemma 3. (*Brauer*) *Let I be a nonnilpotent left ideal in an almost left Artinian ring R. Then I contains a nonzero idempotent element.*

Proof. In view of Lemma 2, I is nonnil. Just as in the proof of Lemma 2, there is a minimal nonnilpotent left ideal $K\subseteq I$. For each $x\in K$, let $g(x)$ denote the left ideal $\{y\in K\colon yx=0\}$. Of course K is nonnil by Lemma 2.

Suppose a is any nonnilpotent element of K. Then $a^2\in Ra$ and a^2, and hence Ra, is not nilpotent. But $Ra\subseteq K$, and by minimality, $Ra=K$. Likewise $a^3\in Ra^2\subseteq K$ and $Ra^2=K$. There is a $b\in Ra=K$ such that $ba=a$. Then $b^2a=ba=a$ so $(b^2-b)a=0$, and $b^2-b\in g(a)$. Put $c=a+b-ab\in K$; note that

$$ca=(a+b-ab)a=a^2+ba-aba=a^2+a-a^2=a\ .$$

Since $ca=a\neq0$, c is not nilpotent. Hence $Ra=K$ and $g(c)\subseteq g(a)$.

In the last paragraph, we showed that for any nonnilpotent $a\in K$, there exist $b,\ c\in K$ such that $ba=ca=a$, $c=a+b-ab$, $b-b^2\in g(a)$, and $g(c)\subseteq g(a)$. Of course c is not nilpotent since $ca=a\neq0$.

Put $a_1=a$, $b_1=b$, $a_2=c$ and proceed by induction to produce sequences (a_n), $(b_n)\subseteq K$ such that each a_n is nonnilpotent, $b_na_n=a_{n+1}a_n=a_n$, $a_{n+1}=a_n+b_n-a_nb_n$, $b_n-b_n^2\in g(a_n)$, $g(a_{n+1})\subseteq g(a_n)$ for all $n\in Z_+$. The contracting sequence of left ideals $g(a_1)\supseteq g(a_2)\supseteq\cdots\supseteq g(a_n)\supseteq\cdots$ must almost terminate. So there is a $q\in Z_+$ such that $R^qg(a_q)\subseteq g(a_{q+1})$. Thus $R^q(b_q-b_q^2)\subseteq R^qg(a_q)\subseteq g(a_{q+1})$ and $b_q^{q+1}-b_q^{q+2}\in g(a_{q+1})$. But

$$\begin{aligned}0&=(b_q^{q+1}-b_q^{q+2})a_{q+1}=(b_q^{q+1}-b_q^{q+2})(a_q+b_q-a_qb_q)\\&=b_q^{q+1}a_q+b_q^{q+2}-b_q^{q+1}a_qb_q-b_q^{q+2}a_q-b_q^{q+3}+b_q^{q+2}a_qb_q\\&=a_q+b_q^{q+2}-a_qb_q-a_q-b_q^{q+3}+a_qb_q\\&=b_q^{q+2}-b_q^{q+3}\ .\end{aligned}$$

From $b_q^{q+2}=b_q^{q+3}$ obtains $(b_q^{q+2})^2=b_q^{q+2}$. Moreover, $b_q\in K\subseteq I$, so $b_q^{q+2}\in I$. Finally, $b_q^{q+2}a_q=a_q\neq 0$, so $b_q^{q+2}\neq 0$.

Lemma 4. *Let W be the nil radical of an almost left Artinian ring R. Let $\mathscr{J}$ be a left ideal of R/W. Then $\mathscr{J}$ has an idempotent generator E such that $E=e+W$ for some idempotent element $e\in R$. Moreover, if $\mathscr{J}$ is an ideal of R/W, E is the identity of the ring $\mathscr{J}$.*

Proof. Say $\mathscr{J}=I/W$ for an appropriate left ideal $I\supseteq W$ of R. We may suppose, without loss of generality, that $\mathscr{J}\neq(0)$; otherwise just let $e=0$, $E=W$. So I is not nilpotent. For each $x\in I$, let $f(x)$ denote the left ideal $\{y\in I;\ yx=0\}$. We claim that there exists a nonzero idempotent element $e\in I$ such that $f(e)\subseteq W$. Suppose not. By Lemma 3, there is a nonzero idempotent element $a\in I$. Then $f(a)\nsubseteq W$. By Lemma 3, there is a nonzero idempotent element $b\in f(a)$. Put $c=a+b-ab\in I$. Then $bc=b(a+b-ab)=ba+b^2-bab=b\neq 0$, so $b\notin f(c)$. Also

$$\begin{aligned}c^2=(a+b-ab)^2&=a^2+b^2+abab+ab+ba-a^2b-aba-bab-ab^2\\&=a+b+ab-ab-ab\\&=a+b-ab=c\end{aligned}$$

and c is a nonzero idempotent element of I. But

$$ca=(a+b-ab)a=a^2+ba-aba=a$$

so $f(c)\subseteq f(a)$.

In the preceding paragraph we have shown that for any nonzero idempotent element $a\in I$, there exist nonzero idempotent elements b, $c\in I$ such that $b\in f(a)$, $b\notin f(c)$, $f(c)\subseteq f(a)$.

Let a_1 be any nonzero idempotent element of I and by induction define sequences (a_n), $(b_n)\subseteq I$ of idempotent elements such that $b_n\in f(a_n)$, $b_n\notin f(a_{n+1})$, $f(a_{n+1})\subseteq f(a_n)$. Then the contracting sequence $f(a_1)\supseteq f(a_2)\supseteq\cdots\supseteq f(a_n)\supseteq\cdots$ of left ideals must almost terminate. Say $q\in Z_+$ such that $R^qf(a_q)\subseteq f(a_{q+1})$. Whence $b_q=b_q^q\cdot b_q\in R^qf(a_q)\subseteq f(a_{q+1})$, which is impossible. Thus there is a nonzero idempotent element $e\in I$ such that $f(e)\subseteq W$.

For any $x\in I$, $xe-x\in f(e)$ evidently, so $xe-x\in W$. Thus $(x+W)(e+W)=x+W$ for all $x\in I$. Put $E=e+W$ in I/W. Then $XE=X$ for any $X\in\mathscr{J}$. Thus E is an idempotent generator of the left ideal $\mathscr{J}$ of R/W.

Finally, let $\mathscr{J}$ be an ideal of R/W. Let $\mathscr{I}$ denote the right ideal of R/W composed of all elements of the form $EY-Y$ for some $Y \in \mathscr{J}$. For any $X \in \mathscr{J}$ and any $EY-Y \in \mathscr{I}$, we have

$$X(EY-Y)=XE(EY-Y)=X(EY-EY)=0$$

in R/W. But $\mathscr{I} \subseteq \mathscr{J}$, so $\mathscr{I}^2=(0)$. Since R/W has no nonzero nilpotent right ideal, $\mathscr{I}=(0)$. Thus $EY=Y=YE$ for each $Y \in \mathscr{J}$. This completes the proof.

We remark that if W is any nil ideal of R and if D is any idempotent element of R/W, then there is an idempotent element d of R such that $d+W=D$. Consult N. Jacobson, *The Structure of Rings*, pp. 53–4.

We reach our first major conclusion about almost left Artinian rings.

Theorem 1. *Let W be the nil radical of an almost left Artinian ring R that is not nilpotent. Then W is nilpotent and R/W is a left Artinian ring with identity.*

Proof. By Lemma 2, W is nilpotent. Let $\mathscr{J}=R/W$ in Lemma 4. Then R/W has an identity. But R/W is almost left Artinian by Proposition 1, so R/W must also be left Artinian.

Let W be the nil radical of a ring R which has some nonzero idempotent elements. We say that the idempotent element x is greater than the idempotent element y if $xy-y \in W$ and $yx-y \in W$. The reader can easily verify that this is a reflexive transitive ordering of the idempotent elements of R. The least idempotent element is 0. Lemma 4 assures us that if R is almost left Artinian, the set of idempotent elements has at least one upper bound, namely an idempotent element e for which $e+W$ is the identity of R/W.

The direct sum of any left Artinian ring with any nilpotent ring is necessarily almost left Artinian. This follows from Proposition 2. We have a partial converse.

Theorem 2. *Let R be an almost left Artinian ring. Then R is the direct sum of a left Artinian ring with identity and a nilpotent ring if and only if the center of R contains an upper bound for the set of idempotent elements of R.*

Proof. Suppose R is such a direct sum. Then every idempotent element lies in the left Artinian summand. The identity of the left Artinian summand is an upper bound of the set of idempotent elements, and it lies in the center of R.

Now suppose e is an upper bound of the set of idempotent elements and e lies in the center of R. Thus $ue-u=eu-u \in W$ for any idempotent element u of R.

Take any $x \in R$. By Lemma 4, there is an idempotent element $u \in R$ such that $u+W$ is the identity of the ring R/W. Thus $(x+W)(u+W)=x+W$ and $xu-x \in W$. So

$$\begin{aligned} xe-x &= (xe-xue)+(xue-xu)+(xu-x) \\ &= (x-xu)e+x(ue-u)+(xu-x) \in W\,. \end{aligned}$$

Let $R_1=Re$ and $R_2=\{xe-x\colon x \in R\}$. Since e is in the center of R, both R_1 and R_2 are ideals of R. Indeed if $y \in R_1 \cap R_2$, then $y=xe-x$ for some $x \in R$ and $y=ye=xe-xe=0$. Thus $R_1 \cap R_2=(0)$. For any $r \in R$, $r=(r-re)+re \in R_1+R_2$. Finally $R=R_1 \oplus R_2$. By the preceding paragraph, $R_2 \subseteq W$, so R_2 is nilpotent. Moreover, e is evidently the identity of the ring R_1. By Proposition 1, R_1 is almost left Artinian, but R_1 has an identity. So R_1 is left Artinian.

Next we characterize commutative almost Artinian rings.

Theorem 3. *A commutative ring R is almost Artinian if and only if R is the direct sum of an Artinian ring with identity and a nilpotent ring.*

The proof follows from Proposition 2 and Theorem 2. We leave the details to the reader.

Unfortunately the analogue of Theorem 3 for Noetherian rings is false. Let F be the real field. Then $xF[x]$ is almost Noetherian with (0) nil radical (note that $xF[x]$ is an ideal of $F[x]$ and $F[x]$ is Noetherian), but $xF[x]$ is not Noetherian or the direct sum of two nonzero rings. And $xF[x]$ is also an example of an ideal of a Noetherian ring that is not the direct sum of a Noetherian ring with a nilpotent ring.

Now let R be the ring of all 2 by 2 matrices over F whose second column entries are 0. Then R is almost left Artinian (R is a left ideal of a left Artinian ring) but R is not the direct sum of a left Artinian ring with a nilpotent ring. Thus commutativity is essential in Theorem 3.

These two examples were supplied by Prof. Lawrence Levy of the University of Wisconsin.

Corollary 1. *Let I be a left ideal of an almost left Artinian ring R such that the ring I is commutative. Then I is the direct sum of an Artinian ring with a nilpotent ring.*

The proof is left to the reader.

Corollary 2. *Let I be a left ideal of an almost left Artinian ring R such that the ring I has an identity. Then I is a left Artinian ring.*

The proof is left to the reader.

Before we obtain the major results of this paper, we must discuss left R-modules.

Let M be a left R-module of the ring R. We make several definitions for submodules of M analogous to our definitions for left ideals of the ring R. We say that a contracting or expanding sequence of submodules $(N_n)_n$ of M terminates if $N_q=N_{q+k}$ for some $q \in Z_+$ and all $k \in Z_+$. We say that an expanding sequence of submodules $(N_n)_n$ almost terminates if $R^q(\bigcup_n N_n) \subseteq N_m$ for some $m, q \in Z_+$ (equivalently, $R^q(\bigcup_n N_n) \subseteq N_q$ for some $q \in Z_+$). We say that a contracting sequence of submodules $(N_n)_n$ almost terminates if $R^q N_m \subseteq \bigcap_n N_n$ for some $q, m \in Z_+$ (equivalently, $R^q N_q \subseteq \bigcap_n N_n$ for some $q \in Z_+$). We say that M is Artinian (almost Artinian) if every contracting sequence of submodules of M terminates (almost terminates). And we say that M is Noetherian (almost Noetherian) if every expanding sequence of submodules of M terminates (almost terminates).

Just as in Propositions 1, 2, 3 we can prove that any (module) homomorphic image of a left R-module with one of these 4 properties has the same property, and the direct sum of 2 R-modules with one of these 4 properties has the same property. Also a submodule of a module with one of these 4 properties has the same property. Proofs are quite analogous to those of Propositions 1, 2, 3, so the arguments are left to the reader. Note that if K is a submodule of N and N is a submodule of M, then K is a submodule of M. It is not in general true that if N is a left ideal of a ring M and if K is a left ideal of the ring N, that K is a left ideal of M.

Note also that R is a left R-module in a natural way, and the ring R has any one of these 4 properties if and only if the R-module R has the same property. If $R^q M=(0)$ for some $q \in Z_+$, then the R-module M is almost Artinian and almost Noetherian. If R has identity e and if M is a unital R-module (this means $em=m$ for all $m \in M$), then "almost Artinian (Noetherian)" is equivalent to "Artinian (Noetherian)".

We begin with

Proposition 7. *Let N be a submodule of a left R-module M and suppose the R-modules N and M/N are both almost Artinian (Noetherian). Then M is almost Artinian (Noetherian).*

Proof. Let N and M/N be almost Artinian and let $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$ be a contracting sequence of submodules of M. Then $K_1 \cap N \supseteq K_2 \cap N \supseteq \cdots \supseteq K_n \cap N \supseteq \cdots$ is a contracting sequence of submodules of N, and $(K_1+N)/N \supseteq (K_2+N)/N \supseteq \cdots \supseteq$

$(K_n+N)/N \supseteq \cdots$ is a contracting sequence of submodules of M/N. So there is an index $q \in Z_+$ such that $R^q[(K_q+N)/N] \subseteq \bigcap_n [(K_n+N)/N]$ and $R^q(K_q \cap N) \subseteq \bigcap_n (K_n \cap N)$. Take any $j \in Z_+$ and any $r_1, \cdots, r_q \in R$ and any $m \in K_q$. Then $r_1 \cdots r_q(m+N) \subseteq R^q(K_q+N) \subseteq K_{q+j}+N$. Say $k \in K_{q+j}$, $s \in N$, such that $r_1 \cdots r_q m = k+s$. But $s \in K_q$ and so $s \in K_q \cap N$. Thus $R^q s \subseteq R^q(K_q \cap N) \subseteq K_{q+j} \cap N$. Finally.

$$R^q r_1 \cdots r_q m \subseteq R^q k + R^q s \subseteq K_{q+j} .$$

So $R^{2q} m \subseteq K_{q+j}$. And $R^{2q} K_q \subseteq K_{q+j}$. Since $j \in Z_+$ was arbitrary, $R^{2q} K_q \subseteq \bigcap_n K_n$.

Let N and M/N be left Noetherian and let $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$ be an expanding sequence of submodules of M. Then $K_1 \cap N \subseteq K_2 \cap N \subseteq \cdots \subseteq K_n \cap N \subseteq \cdots$ is an expanding sequence of submodules of N, etc. There is a $q \in Z_+$ such that $R^q(\bigcup_n K_n \cap N) \subseteq K_q \cap N$ and $R^q(\bigcup_n [(K_n+N)/N]) \subseteq (K_q+N)/N$. Take any $j \in Z_+$ and any $r_1, \cdots, r_q \in R$ and any $m \in K_{q+j}$. Then $r_1 \cdots r_q(m+N) \subseteq R^q(K_{q+j}+N) \subseteq K_q+N$. Say $k \in K_q$, $s \in N$, such that $r_1 \cdots r_q m = k+s$. But $s \in K_{q+j}$ and so $s \in K_{q+j} \cap N$. Thus $R^q s \subseteq R^q(K_{q+j} \cap N) \subseteq K_q \cap N$. Finally,

$$R^q r_1 \cdots r_q m \subseteq R^q k + R^q s \subseteq K_q ,$$

so $R^{2q} m \subseteq K_q$. And $R^{2q} K_{q+j} \subseteq K_q$. But $j \in Z_+$ was arbitrary, so $R^{2q}(\bigcup_n K_n) \subseteq K_q$.

The analogous results for "Artinian" and "Noetherian" left R-modules are proved in the same way, so we leave these arguments to the reader.

Theorem 4. *Let J be a left ideal of the ring R. If the left R-modules J and R/J are almost Artinian (Noetherian), then the ring R is almost left Artinian (Noetherian).*

Proof. Proposition 7.

Again the corresponding statements for "Artinian" and "Noetherian" can be proved.

We apply Proposition 7 immediately.

Proposition 8. *Let R be a ring with nil radical W. Then R is almost left Artinian (Noetherian) if and only if W is nilpotent and each of the R-modules R/W, W/W^2, W^2/W^3, W^3/W^4, $\cdots$ is almost Artinian (Noetherian).*

Proof. Suppose R is almost left Artinian. Then the left R-modules R, W, W^2, W^3, $\cdots$ are almost Artinian. The quotient R-modules R/W, W/W^2, W^2/W^3, $\cdots$ are likewise almost Artinian. Of course, W is nilpotent by Lemma 2.

Now suppose each of R/W, W/W^2, W^2/W^3, $\cdots$ is almost Artinian and W is nilpotent. Say $W^{k-1} \neq (0) = W^k$. Then $W^{k-1} = W^{k-1}/W^k$ is almost Artinian. So are W^{k-2}/W^{k-1} and W^{k-2}, W^{k-3}/W^{k-2} and $W^{k-3}, \cdots, R/W$ and W, and finally R, by Proposition 7. So R is an almost Artinian R-module. R is an almost left Artinian ring.

The proof for "almost Noetherian" is quite analogous. We leave the details to the reader.

In Lemmas 5 and 6 and in Theorem A we will suppose that J is a proper left ideal of R, M is a left R-module such that $JM=(0)$, and we will assume that R/J is the sum of a family of minimal left R-modules of the form $\{I_\alpha/J\}_\alpha$ where each $I_\alpha \supseteq J$ is a left ideal satisfying $I_\alpha^2 \not\subseteq J$. Theorems A and B are the major results of the paper.

Lemma 5. *For each α and each $x \in I_\alpha \backslash J$, $Rx+J=I_\alpha$.*

Proof. Since $I_\alpha^2 \not\subseteq J$, there is a $y \in I_\alpha$ such that $I_\alpha y \not\subseteq J$. Let $K=\{x \in I_\alpha : Rx \subseteq J\}$. Then K is a left ideal of R and $J \subseteq K \subseteq I_\alpha$. Since I_α/J is a minimal R-module, either $K=J$ or $K=I_\alpha$. But $y \in I_\alpha \backslash K$, so $K=J$.

Thus $Rx \not\subseteq J$ for each $x \in I_\alpha \backslash J$. Now $Rx+J$ is a left ideal, and for each $x \in I_\alpha$, we again have that $Rx+J=J$ or I_α. But if $x \in I_\alpha \backslash J$, $Rx+J \neq J$ and $Rx+J=I_\alpha$.

Lemma 6. *For each $m \in RM$, $m \in Rm$. Thus $K=RK$ for any submodule K of RM.*

Proof. Any $m \in RM$ can be expressed $m=r_1m_1+\cdots+r_nm_n$ for some $r_i \in R$, $m_i \in M$. Moreover each r_i+J is a sum of elements of the form s_j+J where each s_j lies in one of the I_α. Since $Jm_i=(0)$ for all i, it follows that each r_im_i can be expressed as $r_im_i=s_1m_i+\cdots+s_km_i$ where each s_j lies in some I_α. We can suppose, without loss of generality, that $m=s_1m_1+\cdots+s_qm_q$ where each m_i lies in M and each s_i lies in some I_{α_i}.

The proof is by induction on the least number of terms any such representation of m can have (call this the index of m). Suppose the index of m is 1. Say $m=r_1m_1$ where $r_1 \in I_{\alpha_1}$. By Lemma 5, $r_1 \in Rr_1+J$: then $m=r_1m_1=rr_1m_1=rm \in Rm$. Thus Lemma 6 holds for any element of RM of index 1. Suppose it holds for any element of index $<N$ $(N>1)$. Let index $m=N$. Say $m=r_1m_1+\cdots+r_Nm_N$, $r_i \in I_{\alpha_i}$. Now $r_1m_1 \neq 0$, $r_1 \in I_{\alpha_1}\backslash J$ and again $r_1 \in rr_1+J$ for some $r \in R$. Then m_1 does not occur in $(rm-m)$ so either $rm-m=0$ or index $(rm-m)<N$. In the former case $m=rm \in Rm$; in the latter case (by the induction hypothesis) there is an $s \in R$ such that $s(rm-m)=rm-m$ and $m=(r+s-sr)m \in Rm$. This completes the induction.

Theorem A. *The following are equivalent.*

(1) *M is almost Artinian.*

(2) *RM is Artinian.*

(3) *RM is Noetherian.*

(4) *M is almost Noetherian.*

Proof. (1)$\Rightarrow$(2). Since M is almost Artinian, so is the submodule RM. But for any submodule K of RM, $K=RK$ by Lemma 6. Thus RM is Artinian.

(2)$\Rightarrow$(1). For any contracting sequence of submodules $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$ of M, $RK_1 \supseteq RK_2 \supseteq \cdots \supseteq RK_n \supseteq \cdots$ are submodules of RM, and so $RK_q = RK_{q+k} \subseteq K_{q+k}$ for some $q \in Z_+$ and all $k \in Z_+$. So $RK_q \subseteq \bigcap_n K_n$.

(4)$\Rightarrow$(3). Since M is almost Noetherian, so is the submodule RM. But for any submodule K of RM, $K=RK$ by Lemma 6. Thus RM is Noetherian.

(3)$\Rightarrow$(4). For any expanding sequence of submodules $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$ of M, $RK_1 \subseteq RK_2 \subseteq \cdots \subseteq RK_n \subseteq \cdots$ are submodules of RM, and so $RK_{q+k} = RK_q \subseteq K_q$ for some $q \in Z_+$, and all $k \in Z_+$. So $R(\bigcup_n K_n) \subseteq K_q$.

It remains only to prove (2)$\Leftrightarrow$(3).

(2)$\Rightarrow$(3). Suppose RM is Artinian but not Noetherian. Let $K_0 = (0) \subsetneqq K_1 \subsetneqq K_2 \subsetneqq \cdots \subsetneqq K_n \subsetneqq \cdots$ be an expanding sequence of submodules of RM, no two equal. Choose $m_n \in K_n \backslash K_{n-1}$ for each $n \in Z_+$. By Lemma 6, $m_n = r_n m_n$ for some $r_n \in R$. But $r_n = s_1 + \cdots + s_k$ for some $s_i \in I_{\alpha_i}$ and so there is an $s_i \in$ some I_{α_i} such that $s_i m_n \in K_n \backslash K_{n-1}$. Thus for each $n \in Z_+$ there is an $m_n \in M$ and some $t_n \in$ some I_α such that $t_n m_n \in K_n \backslash K_{n-1}$. For each $n \in Z_+$ let N_n denote the submodule generated by $\{t_n m_n, t_{n+1} m_{n+1}, t_{n+2} m_{n+2}, \cdots\}$. Then $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ is a contracting sequence of submodules of RM. But RM is Artinian, so $N_q = N_{q+1}$ for some $q \in Z_+$. Say $t_q m_q = u_1 t_{q+1} m_{q+1} + u_2 t_{q+2} m_{q+2} + \cdots + u_p t_{q+p} m_{q+p}$ for appropriate $u_i \in R$ where $u_p t_{q+p} m_{q+p} \neq 0$. Then $u_p t_{q+p} \notin J$. By Lemma 5, $t_{q+p} \in u u_p t_{q+p} + J$ for some $u \in R$. Finally

$$t_{q+p} m_{q+p} = u u_p t_{q+p} m_{q+p} = u t_q m_q - u u_1 t_{q+1} m_{q+1} - \cdots - u u_{p-1} t_{q+p-1} m_{q+p-1}$$

and $t_{q+p} m_{q+p} \in K_{q+p-1}$ which is impossible.

(3)$\Rightarrow$(2). Suppose RM is Noetherian but not Artinian. Let $K_0 \supsetneqq K_1 \supsetneqq K_2 \supsetneqq \cdots \supsetneqq K_n \supsetneqq \cdots$ be a contracting sequence of submodules of RM, no two equal. Choose $m_n \in K_n \backslash K_{n+1}$ for each $n \in Z_+$. By Lemma 6, $m_n = r_n m_n$ for some $r_n \in R$. But $r_n = s_1 + \cdots + s_k$ for some $s_i \in I_{\alpha_i}$ and so there is some $s_i \in$ some I_{α_i} such that $s_i m_n \in K_n \backslash K_{n+1}$. Thus for each $n \in Z_+$, there is an $m_n \in M$ and a $t_n \in$ some I_α such that $t_n m_n \in K_n \backslash K_{n+1}$. For each $n \in Z_+$, let N_n denote the submodule generated by $\{t_1 m_1, \cdots, t_n m_n\}$. Then $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ is an expanding sequence of submodules of RM. But RM is Noetherian, so $N_q=N_{q-1}$ for some $q\in Z_+$. So

$$t_q m_q = u_{q-1}t_{q-1}m_{q-1}+u_{q-2}t_{q-2}m_{q-2}+\cdots+u_{q-p}t_{q-p}m_{q-p}$$

for appropriate $u_i\in R$ where $u_{q-p}t_{q-p}m_{q-p}\neq 0$. Then $u_{q-p}t_{q-p}\notin J$. By Lemma 5, there is some $u\in R$ such that $t_{q-p}\in uu_{q-p}t_{q-p}+J$. Hence

$$t_{q-p}m_{q-p}=uu_{q-p}t_{q-p}m_{q-p}=ut_q m_q-uu_{q-1}t_{q-1}m_{q-1}-\cdots-uu_{q-p+1}t_{q-p+1}m_{q-p+1}$$

and $t_{q-p}m_{q-p}\in K_{q-p+1}$ which is impossible.

Theorem B. *Let $J_1,\cdots,J_n$ be left ideals of the ring R, each satisfying the hypothesis for J in Theorem A. Let N be a left R-module satisfying $J_1\cdots J_nN=(0)$. Then N is almost Artinian if and only if N is almost Noetherian.*

Proof. If N is either almost Noetherian or almost Artinian, the same is true of each of the R-modules $J_2\cdots J_nN/J_1\cdots J_nN=J_2\cdots J_nN$, $J_3\cdots J_nN/J_2\cdots J_nN$, $J_4\cdots J_nN/J_3\cdots J_nN,\cdots,N/J_nN$, and by Theorem A, all these R-modules are both almost Artinian and almost Noetherian. By Proposition 7, each of the R-modules $J_2\cdots J_nN$, $J_3\cdots J_nN$, $J_4\cdots J_nN,\cdots,J_nN$, N are both almost Artinian and almost Noetherian.

Theorems A and B have a number of consequences. We can completely characterize almost left Artinian rings in terms of nilpotent ideals and Artinian modules.

Theorem 5. *Let W be the nil radical of the ring R. Then R is almost left Artinian if and only if W is nilpotent and each of the left R-modules R/W, RW/W^2, $RW^2/W^3,\cdots,RW^n/W^{n+1},\cdots$ is Artinian.*

Proof. Let R be almost left Artinian. Then W is nilpotent, and R/W is a left Artinian ring. It follows from the classical Wedderburn structure theorem that $J=W$ satisfies the hypothesis of Theorem A; R/W is the sum of minimal submodules I_α/W such that $I_\alpha^2\not\subseteq W$. Indeed we can employ finitely many of those submodules. Now each of the R-modules R/W, W/W^2, $W^2/W^3,\cdots$ is almost Artinian since R is an almost Artinian left R-module. By Theorem A, each of RW/W^2, $RW^2/W^3,\cdots$ is Artinian, and R/W is also Artinian because the ring R/W is left Artinian by Theorem 1.

Now assume the condition holds. Then the ring R/W is left Artinian, and the left R-modules RW^n/W^{n+1} are Artinian. Theorem A again applies (by the Wedderburn structure theorem) and the left R-modules W^n/W^{n+1} are almost Artinian. By Proposition 8, R is almost left Artinian.

Now we find that any almost left Artinian ring must also be almost left

Noetherian.

Theorem 6. *Let R be an almost left Artinian ring. Then any left R-module N is almost Artinian if and only if N is almost Noetherian. Thus in particular, any almost left Artinian ring must also be almost left Noetherian.*

Proof. Let W be the nil radical of R. Then W is nilpotent and R/W is a left Artinian ring. By the classical Wedderburn structure theorem, the left ideal $J=W$ satisfies the hypothesis of Theorem B. Then by Theorem B, N is almost Artinian if and only if N is almost Noetherian.

Finally, R is an almost Artinian left R-module. By the preceding paragraph, R must be an almost Noetherian left R-module. Then R is also an almost left Noetherian ring.

A point worth mentioning is that any expanding sequence of left ideals $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n \subseteq \cdots$ in any (almost) left Artinian ring R satisfying $R(\bigcup_n J_n) = \bigcup_n J_n$, must terminate.

Hopkins (*Annals of Math.*, 1939, pp. 712–730) proved Theorem 6 for unital modules and rings with identity. In that case, "almost Artinian (Noetherian)" is equivalent to "Artinian (Noetherian)".

Theorem 7. *Let $J_1, \cdots, J_n$ be left ideals of R satisfying the hypothesis of Theorem B. Suppose also that $J_1 \cdots J_n R = (0)$. Then R is almost left Artinian if and only if R is almost left Noetherian.*

Proof. Just let R be the left R-module N in Theorem B. The rest is clear.

Theorem 8. *Let $J_1, \cdots, J_n$ and R satisfy the hypothesis of Theorem 7 and suppose also $R^2 = R$ and $RJ_i = J_i$ $(i=1, \cdots, n)$. Then R is left Artinian if and only if R is left Noetherian.*

Proof. Let R be left Noetherian. Then the left R-modules $R/J_n R$ and $J_{k-1} \cdots J_n R / J_k \cdots J_n R$ $(k=1, \cdots, n)$ are almost Noetherian. By Theorem A, the left R-modules $RJ_{k-1} \cdots J_n R / J_k \cdots J_n R$ and $R^2/J_n R$ are Noetherian and Artinian. But $R^2 = R$ and $RJ_{k-1} = J_{k-1}$ so the left R-modules $R/J_n R$, $J_{k-1} \cdots J_n R / J_k \cdots J_n R$ are Artinian. By Proposition 8, R is an Artinian left R-module, so the ring R is left Artinian. The converse is analogous.

Theorem 9. *Let $J_1, \cdots, J_n$ and R satisfy the hypothesis of Theorem B. Suppose also $J_1 \cdots J_n = (0)$. Then any left R-module N is almost Artinian if and only if N is almost Noetherian.*

Proof. Theorem B.

Another point worth mentioning is that if W is the nil radical of a left Artinian ring R satisfying $W=RW$, then R must also be left Noetherian. This follows from Proposition 8, Theorem A and the Wedderburn structure theorem.

We will apply Theorems A and B again later. Even though every almost left Artinian ring is almost left Noetherian, the converse statement is false. For example, Z is Noetherian but not almost Artinian. However, we will prove a partial converse. For this purpose, we need some definitions.

We say that an ideal I of a ring R is *almost prime* if for any ideals A, B of R, $AB\subseteq I$ implies $A\subseteq I$ or $B\subseteq I$. Clearly any prime ideal must be almost prime. The reader can easily verify that if R is commutative, I is prime if and only if I is almost prime. However, in the ring of all n by n matrices over a field F ($n>1$), (0) is almost prime but not prime.

Let f be a homomorphism of the ring R onto the ring S. The reader can easily verify that an ideal J of S is almost prime in S if and only if $f^{-1}S$ is almost prime in R. Likewise, any ideal I of R containing the kernel of f is almost prime in R if and only if fI is almost prime in S.

If I is an almost prime ideal of R and if A and B are left ideals of R such that $AB\subseteq I$, then $A\subseteq I$ or $B\subseteq I$. To see this, note that $A+AR$ and $B+BR$ are ideals of R and $(A+AR)(B+BR)\subseteq I$.

We say that a proper left ideal J of a ring R is *almost maximal* in R if the left R-module R/J is the sum of a family of minimal submodules of the form I_α/J where $I_\alpha\supseteq J$ is a left ideal of R. We say that J is *finitely almost maximal* if R/J is the sum of finitely many minimal submodules. If R/J is the sum of just *one* minimal submodule, then of course J is just a maximal left ideal of R.

We note that if J is an almost prime ideal that is also an almost maximal left ideal, then $I_\alpha^2\not\subseteq J$ for each α, so J satisfies the hypothesis of Theorem A. We will make use of this fact later.

Let f be a homomorphism of the ring R onto the ring S. The reader can easily verify that a left ideal J of S is almost maximal in S if and only if $f^{-1}S$ is almost maximal in R. Likewise, a left ideal I of R containing the kernel of f is almost maximal in R if and only if fI is almost maximal in S.

It is worth mentioning that if R has an identity e, any almost maximal left ideal J is finitely almost maximal. To see this, let $e+J=(x_1+J)+\cdots+(x_n+J)$ where $x_i\in I_{\alpha_i}$. Then for any $r\in R$,

$$r+J=(r+J)(e+J)=(rx_1+J)+\cdots+(rx_n+J)\ ,$$

so $R/J=I_{\alpha_1}/J+\cdots+I_{\alpha_n}/J$.

Our next order of business is to prove that in any nonnilpotent almost left Noetherian ring, there is a proper almost prime ideal.

Lemma 7. *Let R be an almost left Noetherian ring with nil radical* (0). *Then there exists a proper almost prime ideal that is also the left annihilator of some nonzero ideal of R.*

Proof. We may assume, without loss of generality, that there exist nonzero ideals A and B such that $AB=(0)$; for otherwise (0) is the left annihilator of R and (0) is almost prime. So there exist $I\neq(0)$, $J\neq(0)$ such that $IJ=(0)$ and a $q\in Z_+$ such that $R^qA\subseteq I$ for any nonzero ideals A, B such that $A\supseteq I$ and $AB=(0)$. We may assume, without loss of generality, that I is the left annihilator of J by extending I if necessary. It remains only to prove that I is almost prime and $I\neq R$.

Suppose C, D are ideals such that $CD\subseteq I$ and $D\nsubseteq I$. Then DJ is an ideal and $CDJ\subseteq IJ=(0)$ and $DJ\neq(0)$. So $(C+I)D\subseteq I$ and $(C+I)DJ=(0)$. Since $I\subseteq C+I$ it follows that $R^q(C+I)\subseteq I$ and $R^qC\subseteq I$. So $R^qCJ=(0)$. Now

$$(CJ)^{q+1}\subseteq R^qCJ=(0)$$

and since R has no nonzero nilpotent ideals, $CJ=(0)$. But I is the left annihilator of J, so $C\subseteq I$. This proves that I is almost prime. Of course $I\neq R$; for otherwise $J^2\subseteq RJ=(0)$ and $J=(0)$, contrary to the choice of J.

The only almost prime ideal in a nilpotent ring R is of course R. But for nonnilpotent rings we have the following.

Lemma 8. *Let R be a nonnilpotent almost left Noetherian ring. Then there is a proper almost prime ideal I in R such that for some nonnilpotent ideal J of R, IJ is nilpotent.*

Proof. Let W be the nil radical of R. Then R/W is an almost left Noetherian ring with nil radical (0), and by Lemma 7, there is a proper ideal $I\supseteq W$ such that I/W is a proper almost prime ideal of R/W, and also an ideal $J\supsetneqq W$ such that $(I/W)(J/W)=(0)$ in R/W. It follows that I is a proper almost prime ideal of R, J is not nilpotent in R, but $IJ\subseteq W$ is nilpotent since W is.

Now we will prove that (0) is the product of finitely many such ideals in an almost left Noetherian ring.

Lemma 9. *Let R be a nonnilpotent almost left Noetherian ring. Then there exist finitely many proper almost prime ideals $J_1,\cdots,J_n$ such that $J_1\cdots J_n=(0)$.*

Proof. Suppose (0) is not such a product. Let $\mathscr{I}$ be the family of all ideals of R that do not contain the product of finitely many proper almost prime ideals. Then $(0)\in\mathscr{I}$ so $\mathscr{I}$ is nonvoid.

Let $\{I_\alpha\}$ be a family of ideals in $\mathscr{I}$ that is totally ordered by inclusion. Then there is a $q\in Z_+$ and an I_{α_0} in $\{I_\alpha\}$ such that for any $I_\alpha\supseteq I_{\alpha_0}$, $R^qI_\alpha\subseteq I_{\alpha_0}$. But for each α, $I_\alpha\subseteq I_{\alpha_0}$ or $I_{\alpha_0}\subseteq I_\alpha$. Hence $R^qI_\alpha\subseteq I_{\alpha_0}$ for all α, and $R^q(\bigcup_\alpha I_\alpha)\subseteq I_{\alpha_0}$.

Now $\bigcup_\alpha I_\alpha\in\mathscr{I}$; for otherwise there exist proper almost prime ideals $J_1,\cdots,J_n$ such that $J_1\cdots J_n\subseteq\bigcup_\alpha I_\alpha$ and

$$(J_1\cdots J_n)^{q+1}\subseteq R^q(\bigcup_\alpha I_\alpha)\subseteq I_{\alpha_0}\,,$$

contrary to $I_{\alpha_0}\in\mathscr{I}$. So any chain of ideals in $\mathscr{I}$ has an upper bound in $\mathscr{I}$. By Zorn's axiom, $\mathscr{I}$ contains a maximal ideal; call it K.

But $K\neq R$ since R contains a proper almost prime ideal. (This is the only place in the argument we use Lemma 8, but it is crucial.) So K is proper and K cannot be almost prime. Thus there exist ideals K_1, K_2 such that $K_1K_2\subseteq K$ but $K_1\not\subseteq K$, $K_2\not\subseteq K$. Then $(K_1+K)(K_2+K)\subseteq K$ and $K\subseteq K_1+K$, $K\subseteq K_2+K$. Since K is maximal in $\mathscr{I}$, $K_1\notin\mathscr{I}$ and $K_2\notin\mathscr{I}$. Say $J_1,\cdots,J_n$, $L_1,\cdots,L_m$ are proper almost prime ideals such that

$$J_1\cdots J_n\subseteq K_1\,,\qquad L_1\cdots L_m\subseteq K_2\,.$$

Finally, $J_1\cdots J_nL_1\cdots L_m\subseteq K_1K_2\subseteq K$ and this is contrary to $K\in\mathscr{I}$.

Theorem 10. *Let R be a ring and suppose $J_1,\cdots,J_n$ are proper almost prime ideals that are also almost maximal left ideals such that $J_1\cdots J_n=(0)$. Then any left R-module N is almost Artinian if and only if N is almost Noetherian.*

Proof. Any proper almost prime ideal that is also an almost maximal left ideal satisfies the hypothesis of Theorem A. The rest follows from Theorem B.

Theorem 11. *Let R be an almost left Noetherian ring. Then R is almost left Artinian if and only if every proper almost prime ideal is an almost maximal left ideal of R.*

Proof. We can suppose, without loss of generality, that R is not nilpotent. For if it were, R would be almost left Artinian and no proper almost prime ideal would exit in R.

Now suppose every proper almost prime ideal is a maximal left ideal. By Lemma 9, there exist such ideals $J_1\cdots,J_n$ satisfying $J_1\cdots J_n=(0)$. By Theorem 10, R is an almost Artinian left R-module, and so R is an almost left Artinian ring.

Suppose R is almost left Artinian. Let I be a proper almost prime ideal. It follows that the ring R/I has (0) nil radical. By Theorem 1, R/I is left Artinian, has an identity, and by the classical Wedderburn structure theorem, (0) is a finitely almost maximal left ideal of R/I. It follows that I is a finitely almost maximal left ideal of R.

Corollary 3. *Let R be a left Noetherian ring such that for every $r \in R$, $r \in Rr$. Then R is left Artinian if and only if every proper almost prime ideal of R is an almost maximal left ideal of R.*

Proof. Since $r \in Rr$ for each $r \in R$, we have that $RJ=J$ for each left ideal J of R. The rest reduces to Theorem 11.

Before we produce special cases for commutative rings, we need another lemma.

Lemma 10. *Let R be a commutative almost Artinian ring. Then any proper prime ideal I of R is maximal.*

Proof. The ring R/I is without zero divisors, but R/I is almost Artinian. By Theorem 1, R/I is Artinian and has an identity e. Take any nonzero $x \in R/I$. The descending chain of ideals

$$(x) \supseteq (x^2) \supseteq (x^3) \supseteq \cdots \supseteq (x^n) \supseteq \cdots$$

must terminate. Say $(x^n)=(x^{n+1})$. So $rx^{n+1}=x^n$ for some $r \in R/I$. Since R/I has no zero divisors, it follows that $(rx-e)x^n=0$ and $rx=e$. Thus every nonzero element of R/I is invertible and R/I is a field. Hence I is a maximal ideal of R.

Theorem 12. *Let R be a commutative almost Noetherian ring. Then the following are equivalent.*

(1) *Every proper prime ideal is a maximal ideal.*

(2) *Every proper prime ideal is an almost maximal ideal.*

(3) *R is almost Artinian.*

Proof. (1)$\Rightarrow$(2). Clear.

(2)$\Rightarrow$(3). Theorem 11.

(3)$\Rightarrow$(1). Suppose R is almost Artinian and I is a proper prime ideal. Then I is maximal by Lemma 10.

By Theorem 3, a commutative ring is almost Artinian if and only if it is the direct sum of an Artinian ring with identity and a nilpotent ring. Hence

Corollary 4. *Let R be a commutative (almost) Noetherian ring. Then the* *following are equivalent.*

(1) *Every proper prime ideal is a maximal ideal.*

(2) *Every proper prime ideal is an almost maximal ideal.*

(3) *R is a direct sum of an Artinian ring with identity and a nilpotent ring. Thus in particular, when R does not have a nilpotent direct summand $\neq (0)$, R is Artinian if and only if every proper prime ideal of R is maximal in R.*

Proof. Theorem 12.

Cohen (*Duke Math. Journal*, 1950, pp. 27–42) proved the equivalence of (1) and (3) for rings with identity. In this case, the nilpotent summand vanishes.

Theorem 5 provided an alternative definition of "almost left Artinian" ring. We conclude with yet another.

Proposition 9. *Let R be a ring. Then R is almost left Artinian if and only if there is an integer $q \in Z_+$ such that for any contracting sequence of left ideals $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_n \supseteq \cdots$, the contracting sequence of left ideals $R^q J_1 \supseteq R^q J_2 \supseteq \cdots \supseteq R^q J_n \supseteq \cdots$ terminates.*

Proof. First assume the condition holds. Then for any contracting sequence of left ideals $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_n \supseteq \cdots$ there is an $m \in Z_+$ such that $R^q J_m = R^q J_{m+k} \subseteq J_{m+k}$ for all $k \in Z_+$. Thus R is almost left Artinian.

Now assume R is almost left Artinian. Let $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_n \supseteq \cdots$ be a contracting sequence of left ideals. Also $R \supseteq R^2 \supseteq \cdots \supseteq R^n \supseteq \cdots$ is a contracting sequence of left ideals that must almost terminate. Say $R^q \cdot R^q \subseteq R^{q+k}$ for all $k \in Z_+$. Then $R^{2q} \subseteq R^{2q+1}$ and it follows that $R^{2q} = \bigcap_n R^n$. But the sequence (J_n) also almost terminates, and it follows that there is a $p \in Z_+$ such that $R^{2q} J_p \subseteq J_{p+k}$ for all $k \in Z_+$.

In other words, for all $k \in Z_+$,

$$R^{2q} J_p = R^{2q} \cdot R^{2q} J_p \subseteq R^{2q} J_{p+k} \subseteq R^{2q} J_p$$

and

$$R^{2q} J_p = R^{2q} J_{p+k} .$$

Since the choice of q is independent of the sequence (J_n), the condition follows.

References

[1] Burton, D.: *A First Course in Rings and Ideals*, Addison-Wesley, 1970.

[2] Cohen, I. S.: *Commutative Rings With Restricted Minimum Condition*, Duke Mathematical Journal, **17**, 27–42 (1950).

[3] Gray, Mary: *A Radical Approach to Algebra*, Addison-Wesley, 1970.

[4] Hopkins, C.: *Rings With Minimum Condition for Left Ideals*, Annals of Mathematics, **40**, 712–730 (1939).

[5] Jacobson, N.: *The Structure of Rings*, American Mathematical Society, Providence, R. I., 1956.

[6] Levy, Lawrence: *Artinian, Nonnoetherian Rings*, to appear.

Department of Mathematics
Portland State University
Portland, Oregon 97207
U.S.A.