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ON CONNES SPECTRUM Γ OF A TENSOR PRODUCT OF ACTIONS ON VON NEUMANN ALGEBRAS

By

YOSHIOMI NAKAGAMI and YUKIMASA OKA

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ABSTRACT. Let α^{j} $(j=1,2)$ be actions of a locally compact abelian group G on von Neumann algebras $\mathscr{M}_{j}$ satisfying $\alpha_{j}(\mathscr{M}_{j})'\cap(\mathscr{M}_{j}\times_{\alpha^{j}}G)=\mathscr{C}_{\mathscr{M}_{j}\times_{\alpha^{j}}G}$. If $(\alpha^{1}\otimes\alpha^{2})_{t}=\alpha_{t}^{1}\otimes\alpha_{t}^{2}$, then $\Gamma(\alpha^{1}\otimes\alpha^{2})$ is the set of all $p\in\hat{G}$ such that $\alpha_{p}^{1}\otimes\iota$ is trivial on the fixed point algebra of the center of $(\mathscr{M}_{1}\times_{\alpha^{1}}G)\bar{\otimes}(\mathscr{M}_{2}\times_{\alpha^{2}}G)$ with respect to the action $\hat{\alpha}_{p}=\hat{\alpha}_{p}^{1}\otimes\hat{\alpha}_{-p}^{2}$. Let β^{j} $(j=1,2)$ be ergodic actions of G on von Neumann algebras $\mathscr{N}_{j}$. If $H^{2}(G,\mathbb{T})=\{0\}$ and both β^{1} and β^{2} have invariant faithful normal states, then $(\mathscr{N}_{1}\bar{\otimes}\mathscr{N}_{2})^{\beta}$ is abelian, where $\beta_{t}=\beta_{t}^{1}\otimes\beta_{-t}^{2}$.

Introduction.

To each countable ergodic non singular transformation group $\mathfrak{G}$ acting on a standard measure space (Ω,μ) there corresponds the associated flow $\{A(t): t\in\mathbb{R}\}$ on the quotient space $\Omega\times\mathbb{R}/\tilde{\mathfrak{G}}$ given by $A(t)(\omega,s)=(\omega,s+t)$, (mod. $\mathfrak{G}$), where $\tilde{\mathfrak{G}}$ is the set of $\tilde{g}$: $(\omega,s)\mapsto(g\omega, s-\log(d\mu g/d\mu)(\omega))$ for $g\in\mathfrak{G}$. The associated flow is conjugate to Θ the dual action of the modular automorphism σ^{ϕ} restricted to the center $\mathscr{C}_{\mathscr{M}\times_{\sigma^{\phi}}\mathbb{R}}$ of the crossed product of $\mathscr{M}$ by σ^{ϕ}, where $\mathscr{M}$ is a factor $L^{\infty}(\Omega)\times\mathfrak{G}$ constructed from group measure space construction and ϕ is the weight on $\mathscr{M}$ induced from μ. On the other hand, Connes and Takesaki show that the covariant system $\{\mathscr{C}_{\mathscr{M}\times_{\sigma^{\phi}}\mathbb{R}}, \mathbb{R}, \Theta\}$ is equivalent (or conjugate) to the smooth flow of weights on $\mathscr{M}$ for all type III factors $\mathscr{M}$. Therefore, so far as we are concerned with type III factors, the smooth flow of weights is considered as a generalization of the associated flow to arbitrary von Neumann algebras.

Given a pair of countable ergodic non singular transformation groups $\mathfrak{G}_{j}$ on (Ω_{j},μ_{j}) for $j=1,2$, the associated flow of the product group $\mathfrak{G}_{1}\times\mathfrak{G}_{2}$ on $(\Omega_{1}\times\Omega_{2}, \mu_{1}\otimes\mu_{2})$ is conjugate to the joint flow of the associated flow $\{A_{j}(t): t\in\mathbb{R}\}$ of $\mathfrak{G}_{j}$, that is, the flow $\{A_{1}(t)\times id: t\in\mathbb{R}\}$ on the quotient space $X_{1}\times X_{2}/\tilde{\mathbb{R}}$, where $X_{j}=\Omega_{j}\times\mathbb{R}/\tilde{\mathfrak{G}}_{j}$ and $\tilde{\mathbb{R}}$ is the set of $\tilde{t}$: $(\zeta_{1},\zeta_{2})\mapsto(A_{1}(t)\zeta_{1}, A_{2}(-t)\zeta_{2})$. The same result is obtained for a smooth flow of weights as follows: the virtual spectrum $S_{v}(\mathscr{M}_{1}\bar{\otimes}\mathscr{M}_{2})$ is the closure of the product of $S_{v}(\mathscr{M}_{1})$ and $S_{v}(\mathscr{M}_{2})$. Therefore the joint flow or the closure of the product of vertual spectrums determines the type of the tensor product of factors. (See, [3, 4]) The main purpose of this paper is to show that if we assume the relative commutant property for α^j: $\alpha^j(\mathcal{M}_j)'\cap(\mathcal{M}_j\times_{\alpha^j}G)\subset(\mathcal{M}_j\times_{\alpha^j}G)'$, then the same type of results as above remains true for a pair of covariant systems $\{\mathcal{M}_j, G, \alpha^j\}$ of von Neumann algebras $\mathcal{M}_j$, a locally compact group G and actions of G on $\mathcal{M}_j$ in place of the covariant systems $\{\mathcal{M}_j, \mathbb{R}, \sigma^{\phi_j}\}$.

In §1 we shall recall the properties of Connes spectrum $\Gamma(\alpha)$ and the set $G(\alpha)$ of vanishing points in a certain sense, which are known as the S-set and the T-set for a modular automorphism.

In §2 the Connes spectrum $\Gamma(\alpha^1\otimes\alpha^2)$ of a tensor product of actions will be given as the kernel of the joint action of dual action $\hat{\alpha}^1$ and $\hat{\alpha}^2$ on the center $\mathcal{C}_{\mathcal{N}_1\bar{\otimes}\mathcal{N}_2}$, where $\mathcal{N}_j=\mathcal{M}_j\times_{\alpha^j}G$.

In §3 we shall give a sufficient condition for a joint action to have pure point spectrum. The crucial assumption is the existence of an invariant faithful normal state.

In §4, as an application of the above results, we shall show that every result obtained in [4] for type III factors constructed from non singular transformation groups still holds for all type III factors.

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§1. Invariants $\Gamma(\alpha)$ and $G(\alpha)$.

Throughout this paper G is a locally compact abelian group and ρ is the regular representation. If α is an action of G on a von Neumann algebra $\mathcal{M}$, then the spectrum $sp(\alpha)$ is the intersection of all kernels $\{p\in\hat{G}: \hat{f}(p)=0\}$ of the Fourier transform $\hat{f}$ with $\alpha_f=0$, $f\in L^1(G)$, where $\alpha_f=\int f(t)\alpha_t dt$. For each projection e in $\mathcal{M}^\alpha$ we denote the restriction of α to the reduced von Neumann algebra $\mathcal{M}_e$ by α^e. The Connes spectrum $\Gamma(\alpha)$ of α is defined by

$$\Gamma(\alpha)=\bigcap\{sp(\alpha^e):\ e\in\mathcal{M}^\alpha,\ e\neq 0\}\,.$$

The Connes spectrum for a modular automorphism is the S-set $S(\mathcal{M})$ of $\mathcal{M}$, more precisely, $\Gamma(\sigma^\phi)=\{\log\lambda:\ \lambda\in S(\mathcal{M}),\ \lambda\neq 0\}$ for a faithful, semi-finite, normal weight ϕ on $\mathcal{M}$. For each unitary v in $\mathcal{M}$ we denote $Ad_v(x)=vav^*$. Another invariant $G(\alpha)$ is defined by

$$G(\alpha)=\{t\in G:\ \alpha_t=Ad_v,\ v\in\mathcal{M}^\alpha\}\,.$$

Let α_G be an isomorphism of $L^\infty(G)$ into $L^\infty(G)\bar{\otimes}L^\infty(G)$ with $(\alpha_G f)(s,t)=f(s+t)$ for $f\in L^\infty(G)$. An action α of G on $\mathcal{M}$ is then identified with an isomorphism α of

$\mathscr{M}$ into $\mathscr{M}\bar{\otimes}L^\infty(G)$ with $(\alpha\otimes\iota)\circ\alpha=(\iota\otimes\alpha_G)\circ\alpha$ by $(\alpha(x)\xi)(t)=\alpha_t(x)\xi(t)$ for $\xi\in\mathfrak{H}\otimes L^2(G)$. The crossed product $\mathscr{M}\times_\alpha G$ is defined as the von Neumann algebra generated by $\alpha(\mathscr{M})$ and $\mathbf{C}\otimes\mathscr{R}(G)$, where $\mathscr{R}(G)=\rho(G)''$. Let U be a unitary in $L^\infty(G)\bar{\otimes}L^\infty(\hat{G})$ defined by

$$(U\xi)(t,p)=\langle t,p\rangle\xi(t,p)\ ,\quad t\in G\ ,\quad p\in\hat{G}\ .$$

The action $\hat{\alpha}$ (dual to α) of $\hat{G}$ on $\mathscr{M}\times_\alpha G$ is defined by

$$\hat{\alpha}(y)=Ad_{1\otimes U}(y\otimes 1)\ .$$

Then the Connes spectrum $\Gamma(\alpha)$ of α coincides with the set $\operatorname{Ker}\hat{\alpha}\restriction\mathscr{C}_{\mathscr{M}\times_\alpha G}$, i.e.

$$\{p\in\hat{G}\colon\ \hat{\alpha}_p=\iota \text{ on } \mathscr{C}_{\mathscr{M}\times_\alpha G}\}$$

[3, Theorem 3.3.2], where $\mathscr{C}_{\mathscr{N}}$ denotes the center of $\mathscr{N}$. Moreover, $G(\alpha)$ of α coincides with a subset of the point spectrum of $\hat{\alpha}$:

$$\{t\in G\colon\ \exists \text{ a unitary } u\in\mathscr{C}_{\mathscr{M}\times_\alpha G} \text{ such that } \hat{\alpha}_p(u)=\langle t,p\rangle u \text{ for all } p\in\hat{G}\}$$

Indeed, we have the following proposition.

Proposition 1.1. *If α is an action of G on $\mathscr{M}$, then the following three conditions are equivalent*:

(i) $t\in G(\alpha)$;

(ii) $t\in G(\hat{\hat{\alpha}})$; *and*

(iii) $\hat{\alpha}_p(u)=\overline{\langle t,p\rangle}u$ *for some unitary u in* $\mathscr{C}_{\mathscr{M}\times_\alpha G}$.

Proof. (i)⇒(ii): By Takesaki's duality

$$\{(\mathscr{M}\times_\alpha G)\times_{\hat{\alpha}}\hat{G},\hat{\hat{\alpha}}\}\cong\{\mathscr{M}\bar{\otimes}\mathscr{L}(L^2(G)),\tilde{\alpha}\}\ ,$$

where $\tilde{\alpha}_t=\alpha_t\otimes\lambda_t$ and $\lambda_s(x)=\rho(s)^*x\rho(s)$. Suppose that $t\in G(\alpha)$. There exists a unitary $v\in\mathscr{M}^\alpha$ such that $\alpha_t=Ad_v$. Therefore $v\otimes\rho(t)^*\in\mathscr{M}\bar{\otimes}\mathscr{L}(L^2(G))$, $\tilde{\alpha}_s(v\otimes\rho(t)^*)=v\otimes\rho(t)^*$ for all s and $\tilde{\alpha}_t=Ad_{v\times\rho(t)^*}$ on $\mathscr{M}\bar{\otimes}\mathscr{L}(L^2(G))$. Thus $t\in G(\tilde{\alpha})$ and hence $t\in G(\hat{\hat{\alpha}})$.

(ii)⇒(i): Suppose that $t\in G(\hat{\hat{\alpha}})$. Since then $t\in G(\tilde{\alpha})$, there exists a unitary $w\in(\mathscr{M}\bar{\otimes}\mathscr{L}(L^2(G)))^{\tilde{\alpha}}$ such that $\tilde{\alpha}_t=Ad_w$. Then $\alpha_t\otimes\iota=Ad_{(1\otimes\rho(t))w}$ is an inner automorphism of $\mathscr{M}\bar{\otimes}\mathscr{L}(L^2(G))$ and hence $(1\otimes\rho(t))w\in\mathscr{M}\otimes\mathbf{C}$. Therefore $(1\otimes\rho(t))w$ is of the form $v\otimes 1$ for some unitary $v\in\mathscr{M}$, and hence $w=v\otimes\rho(t)^*$. Since $\tilde{\alpha}_s(w)=w$, it follows that $v\in\mathscr{M}^\alpha$ and $\alpha_t=Ad_v$. Thus $t\in G(\alpha)$.

(ii)⇒(iii): Suppose that $t\in G(\hat{\hat{\alpha}})$. There exists a unitary $u\in\mathscr{C}_{\mathscr{M}\times_\alpha G}$ such that $\hat{\hat{\alpha}}_t=Ad_{\hat{\alpha}(u)}$ on $(\mathscr{M}\times_\alpha G)\times_{\hat{\alpha}}\hat{G}$. Since

$$\langle t,p\rangle 1\otimes 1\otimes\rho(p)=\hat{\hat{\alpha}}_t(1\otimes 1\otimes\rho(p))=\hat{\alpha}(u)(1\otimes 1\otimes\rho(p))\hat{\alpha}(u)^*\ ,$$

it follows that

$$\hat{\alpha}(\hat{\alpha}_p(u))=Ad_{1\otimes 1\otimes\rho(p)}(\hat{\alpha}(u))=\overline{\langle t, p\rangle}\hat{\alpha}(u)$$

and hence that $\hat{\alpha}_p(u)=\overline{\langle t, p\rangle}u$.

(iii)⇒(ii): Suppose that $\hat{\alpha}_p(u)=\overline{\langle t, p\rangle}u$ for some unitary $u\in\mathscr{C}_{\mathscr{M}\times_\alpha G}$. Then

$$Ad_{1\otimes 1\otimes\rho(p)}(\hat{\alpha}(u))=\overline{\langle t, p\rangle}\hat{\alpha}(u)$$

and hence that

$$Ad_{\hat{\alpha}(u)}(1\otimes 1\otimes\rho(p))=\langle t, p\rangle(1\otimes 1\otimes\rho(p))=\hat{\hat{\alpha}}_t(1\otimes 1\otimes\rho(p)) .$$

Since $u\in\mathscr{C}_{\mathscr{M}\times_\alpha G}$, $Ad_{\hat{\alpha}(u)}=\iota$ on $\hat{\alpha}(\mathscr{M}\times_\alpha G)$ and hence $\hat{\hat{\alpha}}_t=Ad_{\hat{\alpha}(u)}$ on $(\mathscr{M}\times_\alpha G)\times_{\hat{\alpha}}\hat{G}$. Since $\hat{\hat{\alpha}}_s(\hat{\alpha}(u))=\hat{\alpha}(u)$, we have $t\in G(\hat{\hat{\alpha}})$. q.e.d.

The set $G(\alpha)$ for a modular automorphism is the T-set $T(\mathscr{M})$ of $\mathscr{M}$. If $\mathscr{M}$ is a factor and $\hat{G}/\Gamma(\alpha)$ is compact, then $G(\alpha)=\Gamma(\alpha)^{\perp}$, [2, 5].

Corollary 1.2. *If β is a dual action of $\hat{G}$ on $\mathscr{N}$, then the following two conditions are equivalent*:

(i) $t\in G(\hat{\beta})$; *and*

(ii) $\beta_p(v)=\overline{\langle t, p\rangle}v$ *for some unitary* v *in* $\mathscr{C}_{\mathscr{N}}$.

If α is integrable and has the relative commutant property, i.e. $\alpha(\mathscr{M})'\cap(\mathscr{M}\times_\alpha G)=\mathscr{C}_{\mathscr{M}\times_\alpha G}$, then $(\mathscr{M}^\alpha)'\cap\mathscr{M}=\mathscr{C}_{\mathscr{M}^\alpha}$ by [7] and hence α is center fixing.

Proposition 1.3. *The following two conditions are equivalent*:

(i) $\mathscr{M}$ *is a factor; and*

(ii) α *is center fixing and* $\hat{\alpha}$ *is ergodic on* $\mathscr{C}_{\mathscr{M}\times_\alpha G}$.

Proof. By Takesaki's duality we have

$$(\mathscr{M}\times_\alpha G)\times_{\hat{\alpha}}\hat{G}\cong\mathscr{M}\bar{\otimes}\mathscr{L}(L^2(G)) .$$

Therefore $\mathscr{M}$ is a factor if and only if $(\mathscr{M}\times_\alpha G)\times_{\hat{\alpha}}\hat{G}$ is a factor. Therefore the condition (i) is equivalent to the fact that $\Gamma(\hat{\alpha})=G$ and $\hat{\alpha}$ is ergodic on $\mathscr{C}_{\mathscr{M}\times_\alpha G}$ by [3, Theorem 3.3.4]. By [6, Theorem 6.1] $\Gamma(\hat{\alpha})=G$ if and only if α is center fixing on $\mathscr{M}$. q.e.d.

§ 2. Connes spectrum of a tensor product.

Let α be an action of G on $\mathscr{M}$ and H a closed subgroup of G. We denote by α^H the restriction of α to H and by $\mathscr{M}\times_\alpha H$ the von Neumann subalgebra generated by $\alpha(\mathscr{M})$ and $\mathbf{C}\otimes\rho(H)''$. Then $\mathscr{M}\times_{\alpha^H}H$ is isomorphic to $\mathscr{M}\times_\alpha H$ by the correspondence:

$$(2.1)\qquad \begin{cases} \alpha^H(x) \mapsto \alpha(x) \\ 1\otimes\rho^H(r) \mapsto 1\otimes\rho(r)\ , \end{cases}$$

where ρ^H is the regular representation of H on $L^2(H)$, [8, Theorem 4.2]. Moreover,

$$(2.2)\qquad \mathscr{M}\times_\alpha H=\{y\in\mathscr{M}\times_\alpha G:\ \hat{\alpha}_p(y)=y \text{ for all } p\in H^\perp\}\ ,$$

[9, Theorem 7.1].

Let α^1 and α^2 be actions of G on $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively. We denote

$$(\alpha^1\otimes\alpha^2)_t=\alpha^1_t\otimes\alpha^2_t \qquad\text{and}\qquad (\alpha^1\times\alpha^2)_{(s,t)}=\alpha^1_s\otimes\alpha^2_t\ .$$

The former $\alpha^1\otimes\alpha^2$ is an action of G on $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$ and the latter $\alpha^1\times\alpha^2$ is an action of $G\times G$ on $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$.

Lemma 2.1. *If $\mathscr{N}_j=\mathscr{M}_j\times_{\alpha^j}G$ and $\hat{\alpha}_p=\hat{\alpha}^1_p\otimes\hat{\alpha}^2_{-p}$, then*

$$(2.3)\qquad \{(\mathscr{M}_1\bar{\otimes}\mathscr{M}_2)\times_{\alpha^1\otimes\alpha^2}G,\ (\alpha^1\otimes\alpha^2)\hat{}\ \}\cong\{(\mathscr{N}_1\bar{\otimes}\mathscr{N}_2)^{\hat{\alpha}},\ \hat{\alpha}^1\otimes\iota\}\ .$$

Proof. Put $\mathscr{M}=\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$ and $H=\{(s,t)\in G\times G:\ s=t\}$. Let V be the isometry which maps naturally $L^2(\mathfrak{H}_1;G)\otimes L^2(\mathfrak{H}_2;G)$ onto $L^2(\mathfrak{H}_1\otimes\mathfrak{H}_2;G\times G)$. Then there is an isomorphism γ of $\mathscr{M}\times_{\alpha^1\otimes\alpha^2}G$ onto $V(\mathscr{M}\times_{\alpha^1\times\alpha^2}H)V^{-1}$ such that

$$\gamma:\ \begin{cases} (\alpha^1\otimes\alpha^2)(x_1\otimes x_2)\mapsto\alpha^1(x_1)\otimes\alpha^2(x_2) \\ 1_{\mathscr{M}_1}\otimes 1_{\mathscr{M}_2}\otimes\rho(r)\mapsto 1_{\mathscr{M}_1}\otimes\rho(r)\otimes 1_{\mathscr{M}_2}\otimes\rho(r)\ . \end{cases}$$

Moreover, γ gives an equivalence of covariant systems:

$$(2.4)\qquad \{\mathscr{M}\times_{\alpha^1\otimes\alpha^2}G,\ (\alpha^1\otimes\alpha^2)\hat{}\ \}\cong\{V(\mathscr{M}\times_{\alpha^1\times\alpha^2}H)V^{-1},\ \hat{\alpha}^1\otimes\iota\}\ .$$

Really, it suffices to check this on the generators. Since $\hat{\alpha}^j_p(\alpha^j(x))=\alpha^j(x)$ and $\hat{\alpha}^j_p(1_{\mathscr{M}_j}\otimes\rho(r))=\overline{\langle r,p\rangle}1_{\mathscr{M}_j}\otimes\rho(r)$, it follows that

$$\begin{aligned}
\gamma\circ(\alpha^1\otimes\alpha^2)\hat{}_p((\alpha^1\otimes\alpha^2)(x_1\otimes x_2)) &= \gamma\circ(\alpha^1\otimes\alpha^2)(x_1\otimes x_2)\\
&=(\hat{\alpha}^1_p\otimes\iota)\circ\gamma((\alpha^1\otimes\alpha^2)(x_1\otimes x_2))\\
\gamma\circ(\alpha^1\otimes\alpha^2)\hat{}_p(1_{\mathscr{M}_1}\otimes 1_{\mathscr{M}_2}\otimes\rho(r)) &= \overline{\langle r,p\rangle}(1_{\mathscr{M}_1}\otimes\rho(r)\otimes 1_{\mathscr{M}_2}\otimes\rho(r))\\
&=(\hat{\alpha}^1_p\otimes\iota)\circ\gamma(1_{\mathscr{M}_1}\otimes 1_{\mathscr{M}_2}\otimes\rho(r))\ .
\end{aligned}$$

q.e.d.

Here we recall recent results due to Paschke. If α is an action of G dual to some action of $\hat{G}$, then $\alpha(\mathscr{M})'\cap(\mathscr{M}\times_\alpha G)=\alpha(\mathscr{C}_{\mathscr{M}})$ is equivalent to $(\mathscr{M}^\alpha)'\cap\mathscr{M}=\mathscr{C}_{\mathscr{M}^\alpha}$, [7, Theorems 10 and 11].

Lemma 2.2. *If α^1 and α^2 satisfy the relative commutant property: $\alpha^j(\mathscr{M}_j)'\cap(\mathscr{M}_j\times_{\alpha^j}G)=\mathscr{C}_{\mathscr{M}_j\times_{\alpha^j}G}$, then*

$$(2.5)\qquad \mathscr{C}_{(\mathscr{N}_1\bar{\otimes}\mathscr{N}_2)^{\hat{\alpha}}}=(\mathscr{C}_{\mathscr{N}_1\bar{\otimes}\mathscr{N}_2})^{\hat{\alpha}}\ ,$$

where $\mathscr{N}_j=\mathscr{M}_j\times_{\alpha^j}G$ and $\hat{\alpha}_p=\hat{\alpha}^1_p\otimes\hat{\alpha}^2_{-p}$.

Proof. Put $\mathcal{N}=\mathcal{N}_1\bar{\otimes}\mathcal{N}_2$. Then $\hat{\alpha}$ is an action of $\hat{G}$ on $\mathcal{N}$. Since

$$H=\{(t, t);\ t\in G\}=\{(p, -p):\ p\in\hat{G}\}^{\perp}$$

in $G\times G$, it follows from (2.1) and (2.2) that the crossed product $\mathcal{N}\times_{\hat{\alpha}}\hat{G}$ is isomorphic to the fixed point subalgebra

$$(\mathcal{N}\times_{(\hat{\alpha}^1\times\hat{\alpha}^2)}(\hat{G}\times\hat{G}))^{\hat{\hat{\alpha}}^1\otimes\hat{\hat{\alpha}}^2}$$

and the generators $\hat{\alpha}(y)$ and $1\otimes\rho(p)$ correspond to $(\hat{\alpha}^1\times\hat{\alpha}^2)(y)$ and $1\otimes\rho(p)\otimes\rho(-p)$, respectively, [9, Theorem 7.1]. Therefore the inclusion relation

$$\hat{\alpha}(\mathcal{N})'\cap(\mathcal{N}\times_{\hat{\alpha}}\hat{G})\subset\hat{\alpha}(\mathcal{N}) \tag{2.6}$$

is equivalent to the inclusion relation

$$(\hat{\alpha}^1\times\hat{\alpha}^2)(\mathcal{N})'\cap(\mathcal{N}\times_{(\hat{\alpha}^1\times\hat{\alpha}^2)}(\hat{G}\times\hat{G}))^{\hat{\hat{\alpha}}^1\otimes\hat{\hat{\alpha}}^2}\subset(\hat{\alpha}^1\times\hat{\alpha}^2)(\mathcal{N})\,. \tag{2.7}$$

On the other hand, as we assume the relative commutant property for α^j, we have

$$(\mathcal{N}_j^{\hat{\alpha}^j})'\cap\mathcal{N}_j\subset\mathcal{C}_{\mathcal{N}_j}\,.$$

Since $\hat{\alpha}^j$ is a dual action of $\hat{G}$ on $\mathcal{N}_j$, we have

$$\hat{\alpha}^j(\mathcal{N}_j)'\cap(\mathcal{N}_j\times_{\hat{\alpha}^j}\hat{G})\subset\hat{\alpha}^j(\mathcal{N}_j)$$

by [5, Theorem 10], and hence

$$(\hat{\alpha}^1\times\hat{\alpha}^2)(\mathcal{N})'\cap(\mathcal{N}\times_{(\hat{\alpha}^1\times\hat{\alpha}^2)}(\hat{G}\times\hat{G}))\subset(\hat{\alpha}^1\times\hat{\alpha}^2)(\mathcal{N})\,.$$

This implies (2.7) and hence (2.6).

Finally we shall show (2.5). Since $\hat{\alpha}$ is dual, the inclusion (2.6) implies that

$$(\mathcal{N}^{\hat{\alpha}})'\cap\mathcal{N}=\mathcal{C}_{\mathcal{N}} \tag{2.8}$$

by [5, Theorem 11]. Therefore, $\mathcal{C}_{\mathcal{N}^{\hat{\alpha}}}\subset\mathcal{C}_{\mathcal{N}}$ and hence $\mathcal{C}_{\mathcal{N}^{\hat{\alpha}}}\subset(\mathcal{C}_{\mathcal{N}})^{\hat{\alpha}}$. The converse inclusion is clear. q.e.d.

Combining the above two lemmas, we can prove our main theorem.

Theorem 2.3. *If α^j $(j=1,2)$ is an action of G on $\mathcal{M}_j$ and have the relative commutant property, then*

$$\Gamma(\alpha^1\otimes\alpha^2)=\{p\in\hat{G}:\ \hat{\alpha}_p^1\otimes\iota \text{ is trivial on } (\mathcal{C}_{\mathcal{N}_1\bar{\otimes}\mathcal{N}_2})^{\hat{\alpha}}\}\,,$$

where $\mathcal{N}_j=\mathcal{M}_j\times_{\alpha^j}G$ and $\hat{\alpha}_p=\hat{\alpha}_p^1\otimes\hat{\alpha}_{-p}^2$.

Proof. By virtue of [3, Theorem 3.3.2] we have

$$\Gamma(\alpha^1\otimes\alpha^2)=\{p\in\hat{G}:\ (\alpha^1\otimes\alpha^2)\hat{}_p=\iota \text{ on } \mathcal{C}_{\mathcal{M}\times_{\alpha^1\otimes\alpha^2}G}\}\,,$$

where $\mathscr{M}=\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$. By Lemma 2.1, $(\alpha^1\otimes\alpha^2)\widehat{}_p=\iota$ on the center of $\mathscr{M}\times_{\alpha^1\otimes\alpha^2}G$ if and only if $\hat{\alpha}^1_p\otimes\iota$ is trivial on the center of $(\mathscr{N}_1\bar{\otimes}\mathscr{N}_2)^{\hat{\alpha}}$. Since α^1 and α^2 have the relative commutant property, $\mathscr{C}_{(\mathscr{N}_1\bar{\otimes}\mathscr{N}_2)^{\hat{\alpha}}}=(\mathscr{C}_{\mathscr{N}_1\bar{\otimes}\mathscr{N}_2})^{\hat{\alpha}}$ by Lemma 2.2. Thus our theorem is proved. q.e.d.

Corollary 2.4. *If α^1 and α^2 have the relative commutant property, then $\Gamma(\alpha^1)+\Gamma(\alpha^2)\subset\Gamma(\alpha^1\otimes\alpha^2)$.*

§ 3. Joint action with pure point spectrum.

In this section we assume that von Neumann algebras are σ-finite. To begin with we shall discuss the pure point spectrum of an action.

For each action α of G on a von Neumann algebra $\mathscr{N}$ we denote the set of all eigenvalues of α by $\Sigma(\alpha)$ and the eigenspace corresponding to $p\in\Sigma(\alpha)$ by $\mathscr{N}^\alpha(p)$.

Definition 3.1. An action α of G on $\mathscr{N}$ is said to have ***pure point spectrum for a faithful normal state*** ϕ on $\mathscr{N}$ (resp. to have ***pure point spectrum***) if the linear span of $\mathscr{N}^\alpha(p)$, $p\in\Sigma(\alpha)$ is dense in $L^2(\mathscr{N},\phi)$ (resp. σ-strongly dense in $\mathscr{N}$).

The topology on $\mathscr{N}$ induced from L^2-norm defined by $\|x\|_\phi=\phi(x^*x)^{1/2}$ is weaker than the σ-strong topology on $\mathscr{N}$. Therefore, if α has pure point spectrum, then α has pure point spectrum for all faithful normal states on $\mathscr{N}$. The converse holds as the following:

Proposition 3.2. *If either*

(i) *α has pure point spectrum for all faithful normal states on $\mathscr{N}$; or*

(ii) *α has pure point spectrum for some α-invariant faithful normal state on $\mathscr{N}$ and $\Sigma(\alpha)$ is closed,*

then α has pure point spectrum.

Proof. Let $\mathscr{N}_0$ be the linear span of $\mathscr{N}^\alpha(p)$, $p\in\Sigma(\alpha)$.

(i) Suppose that $\mathscr{N}_0$ is dense in $L^2(\mathscr{N},\phi)$ for all faithful normal states ϕ on $\mathscr{N}$. If ψ is a normal positive linear form on $\mathscr{N}$, then $L^2(\mathscr{N},\phi+\psi)=L^2(\mathscr{N},\|\phi+\psi\|^{-1}(\phi+\psi))$ and hence $\mathscr{N}_0$ is dense in $L^2(\mathscr{N},\psi+\phi)$. Since $\mathscr{N}\subset L^2(\mathscr{N},\psi+\phi)$, if $x\in\mathscr{N}$, then for any $\varepsilon>0$ and for any normal positive linear form ψ on $\mathscr{N}$ there exists an element $y\in\mathscr{N}_0$ such that $\|y-x\|_{\psi+\phi}<\varepsilon$. Since $\|z\|_\psi\le\|z\|_{\psi+\phi}$, $\mathscr{N}_0$ is σ-strongly dense in $\mathscr{N}$.

(ii) If ϕ is an α-invariant faithful normal state on $\mathscr{N}$, then there exists a unitary representation U of G on $L^2(\mathscr{N},\phi)$ such that

$$(U_t x|y)_\phi=(\alpha_t(x)|y)_\phi\,,\qquad x,y\in\mathscr{N}\,.$$

Then $\Sigma(U)=\Sigma(\alpha)$, where $\Sigma(U)$ denotes the set of all eigenvalues of U whose eigenvectors belong to $\mathcal{N}$. The spectrum $sp(U)$ is defined as the intersection of all kernels of $\hat{f}$ with $U_f=0$, $f\in L^1(G)$, where $U_f=\int f(t)U_t dt$. Since $\alpha_f=0$ is equivalent to $U_f=0$, $sp(\alpha)=sp(U)$. If α has pure point spectrum for ϕ, then $\overline{\Sigma(U)}=sp(U)$. Since $\Sigma(\alpha)=\overline{\Sigma(\alpha)}$ by assumption, we have $\Sigma(\alpha)=sp(\alpha)$. Thus $\mathcal{N}_0$ is σ-strongly dense in $\mathcal{N}$. q.e.d.

Now we shall go into the main part of this section. We begin with the definition of a joint action, [4].

Definition 3.3. Let β^1 and β^2 be actions of G on von Neumann algebras $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, and $\beta_t=\beta_t^1\otimes\beta_{-t}^2$ on $\mathcal{N}_1\bar{\otimes}\mathcal{N}_2$. Since $\beta_t^1\otimes\iota=\iota\otimes\beta_t^2$ on $(\mathcal{N}_1\bar{\otimes}\mathcal{N}_2)^\beta$, the action $\beta^1\otimes\iota$ of G on $(\mathcal{N}_1\bar{\otimes}\mathcal{N}_2)^\beta$ is called the *joint action* of β^1 and β^2.

Theorem 3.4. *Let β^j $(j=1,2)$ be an ergodic action of G on $\mathcal{N}_j$ and ϕ_j a faithful normal state on $\mathcal{N}_j$. If either*

(a) *ϕ_j is β^j-invariant for $j=1,2$; or*

(b) *ϕ_1 is β^1-invariant and β^1 has pure point spectrum for ϕ_1,*

then

(i) *the joint action γ of β^1 and β^2 is ergodic;*

(ii) *the restriction of $\phi_1\otimes\phi_2$ to $(\mathcal{N}_1\bar{\otimes}\mathcal{N}_2)^\beta$ is γ-invariant;*

(iii) *γ has pure point spectrum for the restriction of $\phi_1\otimes\phi_2$ to $(\mathcal{N}_1\bar{\otimes}\mathcal{N}_2)^\beta$;*

(iv) *$sp(\gamma)=\Sigma(\beta^1)\cap\Sigma(\beta^2)$; and*

(v) *if $H^2(G,\mathbb{T})=\{0\}$, then $(\mathcal{N}_1\bar{\otimes}\mathcal{N}_2)^\beta$ is abelian.*

Proof. First we notice that each eigen space of an ergodic action is the set of all scalar multiples of some unitary.

(i) If $(\beta_t^1\otimes\iota)(x)=x$ for $x\in(\mathcal{N}_1\bar{\otimes}\mathcal{N}_2)^\beta$, then $(\beta_{t-s}^1\otimes\beta_s^2)(x)=x$ for all $t,s\in G$. Since β^1 and β^2 are ergodic, x is a scalar multiple of the identity.

(ii) Since $\beta_t^1\otimes\iota=\iota\otimes\beta_t^2$ on $(\mathcal{N}_1\bar{\otimes}\mathcal{N}_2)^\beta$, it is clear.

(iii) and (iv) By Lemmas 3.5 and 3.6 below.

(v) By (i) and (iii). q.e.d.

Lemma 3.5. *Let β^j and ϕ_j be as in Theorem* 3.4. *If ϕ_j is β^j-invariant for $j=1,2$, then the joint action γ has pure point spectrum for $\phi_1\otimes\phi_2$ and $sp(\gamma)=\Sigma(\beta^1)\cap\Sigma(\beta^2)$.*

Proof. Since ϕ_j is β^j-invariant, we may assume that $\mathcal{N}_j$ acts canonically on $L^2(\mathcal{N}_j,\phi_j)$ and β^j agrees on $\mathcal{N}_j$ with a unitary representation $U^{(j)}$ of G on

$L^2(\mathcal{N}_j, \phi_j)$:

$$(U_t^{(j)}x|y)_{\phi_j}=(\beta_t^j(x)|y)_{\phi_j}, \qquad x, y \in \mathcal{N}_j .$$

Let $\{E^{(j)}(p): p \in \hat{G}\}$ be the spectral resolution of $U^{(j)}$. Let E be the projection of $L^2(\mathcal{N}_1\bar{\otimes}\mathcal{N}_2, \phi_1\otimes\phi_2)$ onto the closure of $(\mathcal{N}_1\bar{\otimes}\mathcal{N}_2)^\beta$ in $L^2(\mathcal{N}_1\bar{\otimes}\mathcal{N}_2, \phi_1\otimes\phi_2)$, and P the projection onto the closed linear span of $\{u_p^{(1)}\otimes u_p^{(2)}: p \in \Sigma(\beta^1)\cap\Sigma(\beta^2)\}$, where $u_p^{(j)}$ are the normalized eigenvectors of β^j belonging to $p \in \Sigma(\beta^j)$. Then $P\leq E$.

We want to show that $E(1-P)=0$. First we notice that each pair of E, P, $U_s^{(1)}\otimes 1$ and $1\otimes U_t^{(2)}$ mutually commutes. We put $F=E(1-P)$. Since $(U_s^{(1)}\otimes 1)E=(U_{s-t}^{(1)}\otimes U_t^{(2)})E$, we have

$$\int \langle s, p\rangle(E^{(1)}(dp)\otimes 1)F=\iint \langle s, p\rangle\langle t, q-p\rangle(E^{(1)}(dp)\otimes E^{(2)}(dq))F .$$

If $f, g \in L^1(G)$ and $\hat{g}(e)=1$, then Fubini theorem implies

$$\int \hat{f}(p)(E^{(1)}(dp)\otimes 1)F=\iint \hat{f}(p)\hat{g}(q-p)(E^{(1)}(dp)\otimes E^{(2)}(dq))F . \tag{3.1}$$

We may assume that $\hat{f}$ and $\hat{g}$ have compact carriers and $0\leq\hat{f}\leq 1$, $0\leq\hat{g}\leq 1$. The spectral measure $E^{(1)}(dp)$ or $E^{(2)}(dq)$ is continuous at every point $p \in \hat{G}$ over $FL^2(\mathcal{N}_1\bar{\otimes}\mathcal{N}_2, \phi_1\otimes\phi_2)$. Therefore, if the carrier of $\hat{g}$ converges to the unit of $\hat{G}$, then the right hand side of (3.1) converges to 0. Because, the right hand side of (3.1) is the integration by a product measure. Hence the left hand side of (3.1) vanishes for any $\hat{f}$ with $0\leq\hat{f}\leq 1$. If $\hat{f}\uparrow 1$, then $F=0$. q.e.d.

The proof of the following lemma is essentially a copy of [4, Lemma 1.(1)].

Lemma 3.6. *Let β^j and ϕ_j be as in Theorem 3.4. If ϕ_1 is β^1-invariant and β^1 has pure point spectrum for ϕ_1, then the joint action γ has pure point spectrum for $\phi_1\otimes\phi_2$ and $sp(\gamma)=\Sigma(\beta^1)\cap\Sigma(\beta^2)$.*

Proof. Since β^j is ergodic, we can choose a unitary $u_p^{(j)} \in \mathcal{N}_j$ with $\beta_t^j(u_p^{(j)})=\overline{\langle t, p\rangle}u_p^{(j)}$ for $p \in \Sigma(\beta^j)$. Since $\beta_t(u_p^{(1)}\otimes u_p^{(2)})=u_p^{(1)}\otimes u_p^{(2)}$, it suffices to show that the set of all $u_p^{(1)}\otimes u_p^{(2)}$ with $p \in \Sigma(\beta^1)\cap\Sigma(\beta^2)$ is dense in $L^2((\mathcal{N}_1\bar{\otimes}\mathcal{N}_2)^\beta, \phi_1\otimes\phi_2)$.

For this we suppose that $x \in (\mathcal{N}_1\bar{\otimes}\mathcal{N}_2)^\beta$ and

$$\langle(u_p^{(1)}\otimes u_p^{(2)})^* x, \phi_1\otimes\phi_2\rangle=0 , \qquad p \in \Sigma(\beta^1)\cap\Sigma(\beta^2) . \tag{3.2}$$

Let R and L be the right and the left slice mappings on $\mathcal{N}_1\bar{\otimes}\mathcal{N}_2$:

$$\langle R_\phi(x), \psi\rangle=\langle x, \phi\otimes\psi\rangle=\langle L_\psi(x), \phi\rangle , \qquad \phi \in \mathcal{N}_{1,*} , \quad \psi \in \mathcal{N}_{2,*} .$$

We put $x_p=R_{\phi_1 u_p^{(1)*}}(x)$ for $p \in \Sigma(\beta^1)$. Then $x_p \in \mathcal{N}_2$. Since $\beta_t(x)=x$ and ϕ_1 is β^1-invariant, it follows that

$$\langle\beta_t^2(x_p), \psi\rangle = \langle(u_p^{(1)}\otimes 1)^* \beta_t(x), (\phi_1\otimes\psi)\circ\beta_{-t}\rangle$$
$$= \langle\langle t, p\rangle x_p, \psi\rangle\,, \qquad \psi\in\mathcal{N}_{2,*}\,.$$

Therefore $x_p=\lambda u_p^{(2)}$ for some $\lambda\in\mathbb{C}$. If $p\notin\Sigma(\beta^2)$, then $x_p=0$. If $p\in\Sigma(\beta^1)\cap\Sigma(\beta^2)$, then (3.2) implies that

$$\lambda=\langle x_p, \phi_2 u_p^{(2)*}\rangle=\langle x, \phi_1 u_p^{(1)*}\otimes\phi_2 u_p^{(2)*}\rangle=0\,.$$

Consequently, $x_p=0$ for all $p\in\Sigma(\beta^1)$. Therefore

$$\langle L_\psi(x)u_p^{(1)*}, \phi_1\rangle=\langle x_p, \psi\rangle=0\,, \qquad p\in\Sigma(\beta^1)\,.$$

Since β^1 has pure point spectrum for ϕ_1 by assumption, $L_\psi(x)=0$ for all $\psi\in\mathcal{N}_{2,*}$ and hence $x=0$. q.e.d.

We shall apply Theorem 3.4 to the results obtained in the previous section.

Corollary 3.7. *Let α^j be an action of G on a factor $\mathcal{M}_j$ having the relative commutant property. If either*

(a) *$\hat{\alpha}^j$ has an invariant faithful normal state on the center of $\mathcal{N}_j=\mathcal{M}_j\times_{\alpha^j}G$ for $j=1,2$: or*

(b) *$\hat{\alpha}^1$ or $\hat{\alpha}^2$ has pure point spectrum for some invariant faithful normal state on the center,*

then the joint action of $\hat{\alpha}^1\restriction\mathcal{C}_{\mathcal{N}_1}$ and $\hat{\alpha}^2\restriction\mathcal{C}_{\mathcal{N}_2}$ has pure point spectrum for some invariant faithful normal state.

Finally we shall give some examples.

Example 1. Let $\mathcal{M}$ be a factor and $\{\mathcal{M}, G, \alpha\}$ a covariant system. If $\Gamma(\alpha)=\hat{G}$, then $\mathcal{M}\times_\alpha G$ is a factor, [3, Corollary 3.3.4]. Therefore, $\{\mathcal{M}\times_\alpha G, \hat{G}, \hat{\alpha}\}$ is a covariant system with $\Sigma(\hat{\alpha}\restriction\mathcal{C}_{\mathcal{M}\times_\alpha G})=\{0\}$.

Example 2. Let Λ be a subgroup of G endowed with a discrete topology and $\hat{\Lambda}$ its dual group. Let Θ be an action of $\hat{G}$ on $L^\infty(\hat{\Lambda})$ defined by

$$(\Theta_p f)(\omega)=f(\omega+p)\,, \qquad \omega\in\hat{\Lambda}\,,\quad p\in\hat{G}\,,$$

where $\hat{G}$ is embedded in $\hat{\Lambda}$ by the canonical homomorphism. Let τ be the faithful normal state on $L^\infty(\hat{\Lambda})$ induced from the normalized Haar measure. Then τ is Θ-invariant. Here, we set $f_t(\omega)=\langle t, \omega\rangle$ for $t\in\Lambda$. Then $f_t\in L^\infty(\hat{\Lambda})$, $\Theta_p(f_t)=\langle t, p\rangle f_t$ and $\{f_t\colon t\in\Lambda\}$ spans $L^\infty(\hat{\Lambda})$. Therefore Θ has an invariant faithful normal state and pure point spectrum. For instance, if $G=\mathbb{R}$, $\Lambda=\mathbb{Z}$ and $\hat{\Lambda}=\mathbb{T}$, then Θ is ergodic. Moreover, $\{L^\infty(\hat{\Lambda}), \hat{G}, \Theta\}$ is a covariant system with $\Sigma(\Theta)=\Lambda$ and $\operatorname{Ker}\Theta=\Lambda^\perp$.

Example 3. We combine the above two examples. Let $\mathcal{N}=(\mathcal{M}\times_\alpha G)\bar{\otimes}L^\infty(\hat{\Lambda})$

and $\beta=\hat{\alpha}\otimes\Theta$. Then $\{\mathscr{N}, \hat{G}, \beta\}$ is a covariant system and β is dual to some action, [6]. Therefore $\{\mathscr{N}\times_\beta\hat{G}, G, \hat{\beta}\}$ is also a covariant system with $\Gamma(\hat{\beta})=\mathrm{Ker}\,\beta\restriction\mathscr{C}_{\mathscr{N}}=\mathrm{Ker}\,\Theta=\Lambda^\perp$, (By [3]) and $\Sigma(\beta\restriction\mathscr{C}_{\mathscr{N}})=\Sigma(\hat{\alpha}\restriction\mathscr{C}_{\mathscr{M}\times_\alpha G})+\Sigma(\Theta)=\Lambda$. If Θ is ergodic, then $G(\hat{\beta})=\Lambda$ by Corollary 1.2.

In particular, if Θ is ergodic, then β is ergodic on $\mathscr{C}_{\mathscr{N}}$. If $\Gamma(\beta)=G$ in addition, then $\mathscr{N}\times_\beta\hat{G}$ is a factor.

§4. Applications to type III factors.

In this section, we define a new type for σ-finite von Neumann algebras and apply the results in the previous sections to the type classification of tensor products of factors.

Now, we define a new type of factors of type III by the following.

Definition 4.1. Let α be an action of G on a σ-finite factor $\mathscr{M}$. For a subgroup Λ of $\hat{G}$, $\{\mathscr{M}, \alpha\}$ is of type Λ, if $\hat{\alpha}\restriction\mathscr{C}_{\mathscr{M}\times_\alpha G}$ admits an invariant faithful normal state ϕ and has pure point spectrum Λ for ϕ. In particular, if α is a modular automorphism and $\{\mathscr{M}, \alpha\}$ is of type Λ, we say that $\mathscr{M}$ is *of type* III^Λ.

We note that $\mathrm{III}^{(2\pi/\log\lambda)\mathbf{Z}}=\mathrm{III}_\lambda$, $0<\lambda<1$, and $\mathrm{III}^{\{0\}}=\mathrm{III}_1$. Then we obtain the following:

Theorem 4.2. (1) *Let $\mathscr{M}_1$ be a σ-finite factor of type III^Λ and $\mathscr{M}_2$ be any σ-finite factor. Then $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$ is of type $\mathrm{III}^{\Lambda\cap T(\mathscr{M}_2)}$.*

(2) *Let $\mathscr{M}_j$ $(j=1,2)$ be a σ-finite factor with a modular automorphism group σ^j. If $\{\mathscr{C}_{\mathscr{M}_j\times_{\sigma^j}\mathbf{R}}, \hat{\sigma}^j\}$ $(j=1,2)$ admits an invariant faithful normal state, then $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$ is of type $\mathrm{III}^{T(\mathscr{M}_1)\cap T(\mathscr{M}_2)}$.*

The proof is evident from Corollary 3.7, since σ^j $(j=1,2)$ has the relative commutant property [3, Theorem 2.5.1].

Corollary 4.3. [4] *Let $\mathscr{M}$ be a σ-finite factor and $T(\mathscr{M})$ be the T-set of $\mathscr{M}$.*

(1) *If the tensor product of $\mathscr{M}$ and a σ-finite factor of type III_λ, $0<\lambda<1$, is isomorphic to $\mathscr{M}$, $\mathscr{M}$ is of type $\mathrm{III}_{\lambda^{1/k}}$ for some integer k, or of type III_1.*

(2) *The tensor product of $\mathscr{M}$ and a σ-finite factor of type III_λ, $0<\lambda<1$, is also of type III_λ if and only if $2\pi/\log\lambda\in T(\mathscr{M})$.*

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Department of Mathematics
Yokohama City University
Yokohama 236, Japan
and
Department of Mathematics
Kumamoto University
Kumamoto 860, Japan