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ON UNIQUENESS FOR EXISTENCE OF MINIMAL HYPERSURFACES WITH A GIVEN BOUNDARY IN A RIEMANNIAN MANIFOLD

By

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§ 0. Introduction.

Radó and Douglas proved the existence of a generalized parametric minimal surface S with any given closed Jordan curve Γ in R^3. Radó also proved in [6] that if Γ can be simply projected onto the boundary curve of a convex plane domain Ω, then S is expressed by a graph of a function of class C^2 on Ω. The starting point of our study is Radó's theorem stated above. The purpose of this paper is to investigate the uniqueness for existence of minimal hypersurfaces with a given boundary in a Riemannian manifold. Our main tool to the uniqueness is the minimum principle for solutions of quasilinear elliptic partial differential equations of second order.

In § 1 we define the notations which will be used without explanation in the later sections. For completeness in § 2 we give without proof the result (Theorem 2.1) connected with the minimum principle for solutions of a quasilinear elliptic partial differential equation of second order which was proved by the present author in [5]. In §§ 3, 4 we give some applications of Theorem 2.1 to geometry. In § 5, making use of results in §§ 3, 4, the uniqueness for existence of minimal hypersurfaces with a given boundary which is expressed by a graph of a function will be proved in the case where ambieant spaces are Riemannian product manifolds. In the last section we give some results related to the existence of minimal hypersurfaces with a given boundary in a Riemannian manifold.

§ 1. Definitions and Notations.

Throughout this paper we always assume that differentiable manifolds and apparatus on them are of class C^∞ and that manifolds are connected, unless otherwise stated.

Let M be a differentiable manifold of dimension n ($n \geqq 2$) and Ω a domain (that is, a connected open subset) with boundary $\partial\Omega$ in M. The domain Ω is said to have smooth boundary $\partial\Omega$ if for each point m of $\partial\Omega$ there exist an open neighborhood U of m in M and a coordinate system $(x_1, \cdots, x_n)$ on U such that

$$U\cap\bar{\Omega}=\{m'\in U; x_n(m')\geqq x_n(m)\}$$

where $\bar{\Omega}=\Omega\cup\partial\Omega$. If Ω is a domain with smooth boundary $\partial\Omega$ in M, then $\partial\Omega$ is an $(n-1)$-dimensional closed submanifold of class C^∞ in M.

Let M be an n-dimensional Riemannian manifold. We denote the Riemannian metric of M by $\langle\ ,\ \rangle$ and the Riemannian connection of M by ∇. Let m be a point of M and T_mM the tangent vector space of M at m. For each pair X and Y of T_mM we shall define a linear transformation $R(X, Y)$ of T_mM into itself as follows: Let Z be a tangent vector to M at m. Extend X, Y and Z to vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ on M, respectively. We put

$$R(X, Y)Z=(\nabla_{\tilde{X}}\nabla_{\tilde{Y}}\tilde{Z}-\nabla_{\tilde{Y}}\nabla_{\tilde{X}}\tilde{Z}-\nabla_{[\tilde{X},\tilde{Y}]}\tilde{Z})_m$$

where $[\tilde{X}, \tilde{Y}]=\nabla_{\tilde{X}}\tilde{Y}-\nabla_{\tilde{Y}}\tilde{X}$. Then we can easily show that $R(X, Y)Z$ is well-defined. For each plane σ in T_mM, the sectional curvature K_σ for σ is defined by

$$K_\sigma=\langle R(X_1, X_2)X_2, X_1\rangle$$

where $\{X_1, X_2\}$ is an orthonormal basis for σ. K_σ is independent of the choice of an orthonormal basis $\{X_1, X_2\}$. Let X be a unit tangent vector to M at m and $\{X, e_1, \cdots, e_{n-1}\}$ an orthonormal basis in T_mM. We put

$$\operatorname{Ric}(X)=\sum_i \langle R(e_i, X)X, e_i\rangle .$$

It is called the Ricci curvature of X at m.

Let f: $M\to N$ be an immersion of a differentiable manifold M of dimension n into a Riemannian manifold N of dimension $(n+1)$. We can induce a Riemannian metric on M so that f: $M\to N$ is isometric at each point of M. We shall denote the Riemannian metric induced on M by the same symbol $\langle\ ,\ \rangle$ that is the Riemannian metric of N. Let m be a point of M and let U be an open neighborhood of m in M which is mapped diffeomorphically into N by f. We identify U with $f(U)$. Then for each point m' of U the tangent vector space $T_{m'}M$ of M at m' can be regarded as a subspace of $T_{m'}N$. The normal space $T_{m'}M^\perp$ is the orthogonal complement of $T_{m'}M$ in $T_{m'}N$. Each vector of $T_{m'}M^\perp$ is called a normal vector to M at m'. Let ξ_m be a unit normal vector to M at m. We extend it to a vector field ξ on N such that ξ is a vector field of unit normal vectors to M in an open neighborhood of m in M. We put

$$A_{\xi_m}(X)=-\nabla_X\xi\ ,\quad X\in T_mM\ .$$

Then it is easily showed that A_{ξ_m} is well-defined and that for all $X \in T_m M$ $A_{\xi_m}(X) \in T_m M$. Let $\{X_1, \cdots, X_n\}$ be an orthonormal basis in $T_m M$. We put

$$H(m)=\frac{1}{n}\sum_i \langle A_{\xi_m}(X_i), X_i\rangle \, .$$

It is called the mean curvature (with respect to ξ_m) of M for the immersion f at m. If the mean curvature of M for the immersion f vanishes at each point of M, we say that $f: M \to N$ is a minimal immersion or M is a minimal hypersurface in N for the immersion f.

Let M be a Riemannian manifold and Ω a domain with smooth boundary $\partial\Omega$ in M. Throughout this paper, by the mean curvature of $\partial\Omega$ we always mean one of $\partial\Omega$ with respect to the inward unit normal vector to $\partial\Omega$.

A Riemannian manifold M is said to be homogeneous if the group of isometries of M is transitive on M.

Let M be an n-dimensional differentiable manifold. For an open subset V of M we denote by $C^k(V)$ the set of real-valued functions of class C^k on V where k is a positive integer. Let m be a point of M and U a local coordinate neighborhood* of m. Let $(x_1, \cdots, x_n)$ be a local coordinate system on U. For a $u \in C^2(U)$ we use the following convenient notations:

$$Du=(u_1, \cdots, u_n)=\left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right), \qquad u_{ij}=\frac{\partial^2 u}{\partial x_i \partial x_j} \quad (1 \leqq i, j \leqq n) \, .$$

§ 2. Quasilinear elliptic partial differential equation of second order.

Let Ω be a domain in the n-dimensional Euclidean space R^n. Let us consider on Ω a quasilinear elliptic partial differential equation of second order:

$$\sum_{i,j=1}^{n} A_{ij}(x, u, Du)u_{ij}=B(x, u, Du) \tag{2.1}$$

where A_{ij} $(1 \leqq i, j \leqq n)$ and B are real-valued continuous functions on $\Omega \times R \times R^n$ and $A_{ij}=A_{ji}$ $(1 \leqq i, j \leqq n)$. We denote by (x, t, p) a point of $\Omega \times R \times R^n$. Ellipticity of the equation (2.1) requires the following condition:

$$\sum_{i,j=1}^{n} A_{ij}(x, t, p)X_i X_j > 0 \qquad \text{on} \quad \Omega \times R \times R^n \tag{2.2}$$

for arbitrary non-vanishing real vector $X=(X_1, \cdots, X_n) \in R^n$.

* Throughout this paper we always assume that a local coordinate neighborhood is homeomorphic to an Euclidean open ball.

We set for a $u \in C^2(\Omega)$

$$L(u) = \sum_{i,j=1}^{n} A_{ij}(x, u, Du)u_{ij} - B(x, u, Du) .$$

We say that $u \in C^2(\Omega)$ is a supersolution of the equation (2.1) if $L(u) \leqq 0$.

Making use of E. Hopf's method ([1]) the author proved in [5] the following theorem.

Theorem 2.1. ([5]) *Suppose that for the equation* (2.1) *the inhomogeneous term B is of class C^1 for the variable $p = (p_1, \cdots, p_n)$ and that $B(x, t, 0) \leqq 0$ holds on $\Omega \times R \times \{0\}$. If $u \in C^2(\Omega)$ is a supersolution of the equation* (2.1), *then u can not take its minimum value in Ω unless u is constant.*

Under more general conditions we can prove the assertion of the above theorem (see § 2 in [5]).

§ 3. Applications of Theorem 2.1.

Let N be a Riemannian manifold of dimension $n+1$ ($n \geqq 2$) and let Ω be a domain with smooth boundary $\partial\Omega$ in N. We shall denote by $\mathscr{H}$ the mean curvature (with respect to the inward unit normal vector) of $\partial\Omega$. Let m_1 be a point of $\partial\Omega$. There exist an open neighborhood W of m_1 in $\partial\Omega$ and a positive τ such that the map Φ: $W \times (-\tau, \tau) \to N$ defined by

$$\Phi(m, t) = \exp_m t\xi_m , \qquad (m, t) \in W \times (-\tau, \tau) , \tag{3.1}$$

is imbedding and $\Phi(W \times (0, \tau)) \subset \Omega$ where $\exp_m$: $T_m N \to N$ is the exponential map at $m \in N$ and ξ_m is the inward unit normal vector to $\partial\Omega$ at m. We shall denote by $\langle\ ,\ \rangle$ the Riemannian metric of $W \times (-\tau, \tau)$ induced from by Φ. For a fixed $t \in (-\tau, \tau)$ we put

$$W_t = \{\Phi(m, t); m \in W\} ,$$

and for a fixed $m \in W$ we put

$$c_m(t) = \exp_m t\xi_m , \qquad t \in (-\tau, \tau) .$$

Then, by Gauss's lemma, the unit speed vector $\dot{c}_m(t)$ of the geodesic c_m is normal to W_t at $\Phi(m, t)$. For a fixed $t \in (-\tau, \tau)$ we shall denote by $\mathscr{H}_t(m)$ the mean curvature (with respect to $\dot{c}_m(t)$) of W_t at $\Phi(m, t)$, $m \in W$. Then it is clear $\mathscr{H}_0 = \mathscr{H}$.

Under the situations stated above we shall prove the following.

Lemma 3.1. *If the Ricci curvature of N is non-negative everywhere, then for a fixed $t \in (0, \tau)$*

$$\mathscr{H}_t(m) \geqq \mathscr{H}(m) \qquad \textit{for all} \quad m \in W\,.$$

Proof. Fix a $t \in (0, \tau)$. Let m be a point of W and $\{e_1, \cdots, e_n\}$ an orthonormal basis in $T_m W$. For each i $(1 \leqq i \leqq n)$ let us extend e_i to the parallel vector field $E_i(s)$, $s \in [0, t]$, along the geodesic c_m. Then $\{E_1(s), \cdots, E_n(s)\}$ is of course an orthonormal basis in $T_{c_m(s)} W_s$, $s \in [0, t]$. For each i $(1 \leqq i \leqq n)$ we can construct a variation $F^{(i)}$: $[0, t] \times (-a, a) \to N$ $(a > 0)$ of the geodesic c_m such that

$$F^{(i)}((s, 0)) = c_m(s)\,, \qquad F^{(i)}((0, \varepsilon)) \in W\,, \qquad F^{(i)}((t, \varepsilon)) \in W_t$$

and

$$\frac{\partial F^{(i)}}{\partial \varepsilon}((s, 0)) = E_i(s)$$

where $s \in [0, t]$, $\varepsilon \in (-a, a)$. We put

$$L^{(i)}(\varepsilon) = \int_0^t \left\langle \frac{\partial F^{(i)}}{\partial s}, \frac{\partial F^{(i)}}{\partial s} \right\rangle^{1/2} ds\,, \quad 1 \leqq i \leqq n\,.$$

Then we have

$$\frac{d^2 L^{(i)}}{d\varepsilon^2}(0) = -\langle \nabla_{E_i} \dot{c}_m, E_i \rangle (c_m(t)) + \langle \nabla_{E_i} \dot{c}_m, E_i \rangle (m)$$
$$- \int_0^t \langle R(E_i, \dot{c}_m) \dot{c}_m, E_i \rangle (c_m(s)) ds\,, \qquad 1 \leqq i \leqq n\,.$$

Since $(d^2 L^{(i)}/d\varepsilon^2)(0) \geqq 0$ for each i $(1 \leqq i \leqq n)$, we have

$$\sum_{i=1}^{n} \{-\langle \nabla_{E_i} \dot{c}_m, E_i \rangle (c_m(t)) + \langle \nabla_{E_i} \dot{c}_m, E_i \rangle (m)\} \geqq \int_0^t \mathrm{Ric}\,(\dot{c}_m(s)) ds\,.$$

The left-hand side of the above inequality is equal to $n(\mathscr{H}_t(m) - \mathscr{H}(m))$. Since the Ricci curvature of N is non-negative everywhere, we obtain $\mathscr{H}_t(m) \geqq \mathscr{H}(m)$. Thus we complete the proof.

Under the same situations stated at the beginning of this section, we shall continue the argument. Let U be a local coordinate neighborhood of m_1 in $\partial\Omega$ which is contained in W and $(x_1, \cdots, x_n, t)$ a local coordinate system on $U \times (-\tau, \tau)$. We put $g_{ij} = \langle \partial/\partial x_i, \partial/\partial x_j \rangle$, $1 \leqq i \leqq n$. We note $\langle \partial/\partial x_i, \partial/\partial t \rangle = 0$ $(1 \leqq i \leqq n)$ and $\langle \partial/\partial t, \partial/\partial t \rangle = 1$. We set

$$C^2(U; \tau) = \{u \in C^2(U);\ |u| < \tau \text{ on } U\}\,.$$

For a $u \in C^2(U; \tau)$ let us consider a hypersurface $S(u)$ in $U \times (-\tau, \tau)$ defined by

$$S(u) = \{(m, u(m)) \in U \times (-\tau, \tau);\ m \in U\}$$

If $u \equiv t$, $t \in (-\tau, \tau)$, we shall set $S(t) = U_t$. Put $X_i = \partial/\partial x_i + u_i(\partial/\partial t)$, $1 \leqq i \leqq n$. Then $X_1, \cdots, X_n$ are linearly independent tangent vector fields on $S(u)$. We set

(3.2) $$\bar{g}_{ij}=\langle X_i, X_j\rangle=g_{ij}+u_iu_j\ , \quad 1\leqq i,\ j\leqq n\ .$$

We can give a unit normal vector field η on $S(u)$ as follows: Put $\eta=\sum \eta^\alpha(\partial/\partial x_\alpha)$* where $\partial/\partial x_{n+1}=\partial/\partial t$. Then η is given by

(3.3) $$\eta^i=-\frac{1}{\sqrt{G}}\sum g^{ij}u_j \quad (1\leqq i\leqq n)\ , \qquad \eta^{n+1}=\frac{1}{\sqrt{G}}$$

where g^{ij} is the (i, j)-component of the inverse matrix of the matrix (g_{ij}) and

(3.4) $$G=1+\sum g^{ij}u_iu_j>0\ .$$

We shall denote by ∇ the Riemannian connection of $U\times(-\tau, \tau)$. Let H be the mean curvature of $S(u)$ with respect to η. Then H is given by

$$H=\frac{1}{n}\sum \bar{g}^{ij}\langle \nabla_{X_i}X_j, \eta\rangle$$

where $\bar{g}^{ij}$ is the (i, j)-component of the inverse matrix of the matrix $(\bar{g}_{ij})$. Using (3.3) we have

(3.5) $$nH\sqrt{G}=\sum \bar{g}^{ij}\{u_{ij}-\sum(\Gamma^k_{ij}+\Gamma^k_{i\,n+1}u_j)u_k+\Gamma^{n+1}_{ij}\}$$

where $\Gamma^\gamma_{\alpha\beta}$ $(1\leqq\alpha, \beta, \gamma\leqq n+1)$ are the Christoffel's symbols of the Riemannian connection ∇ with respect to the local coordinate system $(x_1, \cdots, x_n, x_{n+1})$, $x_{n+1}=t$. We note

(3.6) $$\bar{g}^{ij}=g^{ij}-u^iu^j/G\ , \quad 1\leqq i,\ j\leqq n\ ,$$

where we put

(3.7) $$u^i=\sum g^{ij}u_j\ , \quad 1\leqq i\leqq n\ .$$

Using (3.6) we can rewrite (3.5) in the form:

(3.8) $$\sum A_{ij}(m, u, Du)u_{ij}=B(m, u, Du, H)$$

where

(3.9) $$\begin{aligned} &A_{ij}=Gg^{ij}-u^iu^j\ , \quad 1\leqq i,\ j\leqq n\ ,\\ &B=nHG^{3/2}+\sum(Gg^{ij}-u^iu^j)\{\sum(\Gamma^k_{ij}+\Gamma^k_{i\,n+1}u_j)u_k-\Gamma^{n+1}_{ij}\}\ .\end{aligned}$$

We note that A_{ij} $(1\leqq i, j\leqq n)$ and B are continuous for the variable (m, u, Du) and B is of class C^1 for the variable $Du=(u_1, \cdots, u_n)$. If we regard H as a given continuous function on U in (3.8), it is a quasilinear partial differential equation of second order on U.

* Throughout this paper we always suppose that Greek indices $\alpha, \beta, \gamma, \cdots$ run over the range $1, 2, \cdots, n+1$ and that Latin indices $i, j, k, \cdots$ run over the range $1, 2, \cdots, n$, unless otherwise stated, and we take the summation for repeating indices.

Lemma 3.2. *Assume that the Ricci curvature of N is non-negative everywhere and that $\mathscr{H}$ satisfies the condition*

$$\mathscr{H} \geqq H_0 \quad on \quad U \tag{3.10}$$

where H_0 is a non-negative constant. Let H be a real-valued continuous function on U such that $|H| \leqq H_0$ on U. Suppose that $u \in C^2(U)$ is a solution of the equation (3.8) *such that $0 \leqq u < \tau$ on U. Then u can not take its minimum value in U unless u is constant.*

Proof. For a given continuous function H' on U, we set

$$L_{H'}(v) = \sum A_{ij}(m, v, Dv)v_{ij} - B(m, v, Dv, H')$$

where $v \in C^2(U)$ and A_{ij} $(1 \leqq i, j \leqq n)$ and B are given by (3.9). Since $L_H(u)=0$ and $|H| \leqq H_0$, we have

$$L_{H_0}(u) = L_{H_0}(u) - L_H(u) = n(H - H_0)G^{3/2} \leqq 0 .$$

Hence u is a supersolution of the equation $L_{H_0}(v)=0$. Now we put $B_{H_0}(m, v, Dv) = B(m, v, Dv, H_0)$ where $v \in C^2(U)$. Then for a fixed $t \in [0, \tau)$, from (3.4), (3.9) we have

$$\begin{aligned} B_{H_0}(m, t, 0) &= nH_0 - \sum g^{ij}((m, t))\Gamma_{ij}^{n+1}((m, t)) \\ &= n(H_0 - \mathscr{H}_t(m)) . \end{aligned}$$

Since by Lemma 3.1 and (3.10) the inequality

$$\mathscr{H}_t(m) \geqq \mathscr{H}(m) \geqq H_0$$

holds for all $m \in U$, $B_{H_0}(m, t, 0) \leqq 0$ holds for all $(m, t, 0) \in U \times [0, \tau) \times \{0\}$. Since $0 \leqq u < \tau$ on U, we can now apply Theorem 2.1 to the equation $L_{H_0}(v)=0$. Therefore the present lemma follows from Theorem 2.1.

We shall prove the following.

Theorem 3.1. *Let N be an $(n+1)$-dimensional Riemannian manifold with non-negative Ricci curvature and Ω a domain with smooth boundary $\partial\Omega$ in N. Let $\mathscr{H}$ be the mean curvature of $\partial\Omega$. Assume that $\mathscr{H}$ satisfies the condition*

$$\mathscr{H} \geqq H_0 \quad on \quad \partial\Omega \tag{3.11}$$

where H_0 is a non-negative constant. Let $f\colon M \to N$ be an immersion of an n-dimensional differentiable manifold M into N such that

$$f(M) \subset \bar{\Omega} \quad and \quad f(M) \cap \partial\Omega \neq \emptyset . \tag{3.12}$$

Suppose that the mean curvature H_M (defined up to a sign) of M for the immersion *f satisfies the condition*

(3.13) $$|H_M(m)| \leqq H_0 \quad \textit{for all} \quad m \in M .$$

Then $f(M)$ is contained in $\partial\Omega$.

Proof. We put $M' = \{m \in M; f(m) \in \partial\Omega\}$. By (3.12) M' is non-empty. Let m_0 be a point of M'. We put $f(m_0) = m_1$. Then there exist an open neighborhood W of m_1 in $\partial\Omega$ and a positive τ such that the map Φ: $W \times (-\tau, \tau) \to N$ defined by (3.1) is imbedding and $\Phi(W \times (0, \tau)) \subset \Omega$. Since by (3.12) $f(M)$ is tangent to $\partial\Omega$ at m_1, by the implicit function theorem, there exist a local coordinate neighborhood U ($U \subset W$) of m_1 in $\partial\Omega$ and a $u \in C^2(U, \tau)$ such that $f(M)$ is locally expressed by a hypersurface $S(u) = \{(m, u(m)) \in U \times (-\tau, \tau); m \in U\}$ in $W \times (-\tau, \tau)$. By (3.12) u satisfies the following condition

(3.14) $$0 \leqq u < \tau \quad \text{on} \quad U \quad \text{and} \quad u(m_1) = 0 .$$

We shall denote by H the mean curvature of $S(u)$ with respect to η which is defined by (3.3). By (3.13) H satisfies the following:

(3.15) $$|H(m)| \leqq H_0 \quad \text{for all} \quad m \in U .$$

We note that u is a solution of the equation (3.8) satisfying (3.14). Then from (3.11), (3.15) and Lemma 3.2 we see that $u \equiv 0$ on U. Thus, $S(u) = U \times \{0\}$. Hence there exists an open neighborhood V of m_0 in M such that $f(V) \subset \partial\Omega$. Therefore we have proved that M' is open in M. Since M' is closed in M, by the connectedness of M, $f(M) \subset \partial\Omega$. Thus we complete the proof.

Now, in the equation (3.8) let us consider the case $H=0$, that is, the following equation:

(3.16) $$\sum A_{ij}(m, u, Du) u_{ij} = B_0(m, u, Du)$$

where we put

(3.17) $$B_0(m, u, Du) = B(m, u, Du, 0) .$$

Lemma 3.3. *In Lemma 3.2, excluding the assumption for the Ricci curvature of N, suppose that*

(3.18) $$\mathcal{H}' > 0 \quad \textit{on} \quad \bar{U} .$$

Then

(1) *There exists a positive τ_1 such that $\tau_1 < \tau$ and*

(3.19) $$B_0(m, t, 0) < 0 \quad \textit{for all} \quad (m, t) \in U \times (-\tau_1, \tau_1) .$$

(2) *If* $u \in C^2(U)$ *is a solution of the equation* (3.16) *satisfying the condition*: $|u|<\tau_1$ *on* U, *then* u *is not constant and* u *can not take its minimum value in* U.

Proof. For a fixed $t \in (-\tau, \tau)$ we put $S(t)=\{(m, t) \in U\times(-\tau, \tau); m \in U\}$ and denote by H_t the mean curvature of $S(t)$ with respect to the unit normal vector $\partial/\partial t$. Then, from (3.4), (3.8) and (3.9), we have

$$B_0(m, t, 0)=-\sum g^{ij}((m, t))\Gamma_{ij}^{n+1}((m, t))=-nH_t(m)\ , \quad (m, t) \in U\times(-\tau, \tau)\ .$$

Since by (3.18) $H_0=\mathscr{H}$ is positive on $\bar{U}$, there exists a positive τ_1 ($\tau_1<\tau$) such that for each $t \in (-\tau_1, \tau_1)$ $H_t>0$ holds on U. Thus (1) is showed. The latter follows from (3.19) and Theorem 2.1.

Theorem 3.2. *Let* N *be an* $(n+1)$*-dimensional Riemannian manifold and* Ω *a domain with smooth boundary* $\partial\Omega$ *in* N. *Assume that the mean curvature* $\mathscr{H}$ *of* $\partial\Omega$ *satisfies the condition*:

$$\mathscr{H}>0 \qquad \text{on} \quad \partial\Omega\ . \tag{3.20}$$

Let f: $M\rightarrow N$ *be an immersion of an* n*-dimensional differentiable manifold* M *into* N *such that*

$$f(M)\subset\bar{\Omega}\ . \tag{3.21}$$

Suppose that M *is a minimal hypersurface in* N *for the immersion* f. *Then* $f(M)$ *can not contact to* $\partial\Omega$.

Proof. Suppose for contradiction that there exists a point m_0 of M such that $f(m_0) \in \partial\Omega$. We put $f(m_0)=m_1$. We can take an open neighborhood W of m_1 in $\partial\Omega$ and a positive τ such that the map Φ: $W\times(-\tau, \tau)\rightarrow N$ defined by (3.1) is imbedding and $\Phi(W\times(0, \tau))\subset\Omega$. Since $f(M)$ is tangent to $\partial\Omega$ at m_1, by implicit function theorem, there exist a local coordinate neighborhood U ($U\subset W$) of m_1 in $\partial\Omega$ and a $u \in C^2(U, \tau)$ such that $f(M)$ is locally expressed by a hypersurface $S(u)=\{(m, u(m)) \in U\times(-\tau, \tau); m \in U\}$ in $W\times(-\tau, \tau)$. We note that by (3.21) u satisfies the condition:

$$0\leqq u<\tau \qquad \text{on} \quad U \quad \text{and} \quad u(m_1)=0\ . \tag{3.22}$$

Since $S(u)$ is minimal in $W\times(-\tau, \tau)$, u is a solution of the equation (3.16) on U which takes its minimum value at $m_1 \in U$. Then, by (3.20), (3.22) and Lemma 3.3, we have a contradiction.

§ 4. Hypersurfaces in a Riemannian manifold admitting a Killing vector field.

A vector field X on a Riemannian manifold N is called a Killing vector field if the local 1-parameter group of local transformations generated by X in an open neighborhood of each point of N consists of local isometries.

Theorem 4.1. *Let N be an $(n+1)$-dimensional Riemannian manifold and Ω a domain with smooth boundary $\partial\Omega$ in N. Suppose that the mean curvature $\mathscr{H}$ of $\partial\Omega$ satisfies the condition:*

$$\mathscr{H} \geqq H_0 \quad on \quad \partial\Omega \tag{4.1}$$

where H_0 is a non-negative constant. Let $f: M \to N$ be an immersion of an n-dimensional differentiable manifold M into N such that

$$f(M) \subset \bar{\Omega} \quad and \quad f(M) \cap \partial\Omega \neq \emptyset \ . \tag{4.2}$$

Suppose that the mean curvature H_M (defined up to a sign) of M for the immersion f satisfies

$$|H_M(m)| \leqq H_0 \quad for\ all \quad m \in M \ . \tag{4.3}$$

Moreover assume that for a point m_0 of M such that $f(m_0) \in \partial\Omega$ there exists a Killing vector field X in an open neighborhood of $f(m_0)$ in N which is transversal to $\partial\Omega$ on an open subset in $\partial\Omega$ containing $f(m_0)$. Then there exists an open neighborhood V of m_0 in M such that $f(V) \subset \partial\Omega$.

Proof. Put $f(m_0)=m_1$. Let $\{\phi_t\}$, $|t|<\tau'$, be the local 1-parameter group of local transformations generated by X in an open neighborhood of m_1 in N. Since X is transversal to $\partial\Omega$ about m_1, we can take an open neighborhood W of m_1 in $\partial\Omega$ and a positive τ such that the map Φ: $W \times (-\tau, \tau) \to N$ defined by

$$\Phi(m, t) = \phi_t(m) \ , \qquad (m, t) \in \tilde{W} := W \times (-\tau, \tau) \ ,$$

is imbedding. Then we can assume, taking τ sufficiently small if necessary,

$$\Phi(W \times (0, \tau)) \subset \Omega \ . \tag{4.4}$$

We denote by $\langle \ , \ \rangle$ the Riemannian metric of $\tilde{W}$ induced from N by Φ and denote by Γ the Riemannian connection of $\tilde{W}$. Since by (4.2) $f(M)$ is tangent to $\partial\Omega$ at m_1, by the implicit function theorem, there exist a local coordinate neighborhood U ($U \subset W$) of m_1 in $\partial\Omega$ and a $u \in C^2(U, \tau) = \{v \in C^2(U); |v| < \tau\}$ such that $f(M)$ is locally expressed by a hypersurface $S(u) = \{(m, u(m)) \in \tilde{W}; m \in U\}$ in $\tilde{W}$. From (4.2) and (4.4) we see that u satisfies the following:

(4.5) $$0 \leqq u < \tau \quad \text{and} \quad u(m_1)=0 .$$

Let $(x_1, \cdots, x_n, t)$ be a local coordinate system on $U\times(-\tau, \tau)$. We put

$$g_{\alpha\beta}=\left\langle \frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right\rangle, \quad 1\leqq\alpha,\ \beta\leqq n+1 ,$$

where $\partial/\partial x_{n+1}=\partial/\partial t$. Since for each t ($|t|<\tau$) ϕ_t is isometric, $g_{\alpha\beta}$ ($1\leqq\alpha, \beta\leqq n+1$) are independent of t, $|t|<\tau$. For simplicity, in what follows, we shall use the following convenient notations:

(4.6) $$g_i=g_{in+1} \quad (1\leqq i\leqq n) \quad \text{and} \quad g=g_{n+1n+1} .$$

Since by (4.5) $Du(m_1)=0$, by taking U sufficiently small if necessary, we may suppose from the beginning that the inequality

(4.7) $$1+\sum g_i u^i>0$$

holds on U where we put

(4.8) $$u^i=\sum g^{ij}u_j , \quad 1\leqq i\leqq n ,$$

where g^{ij} is the (i, j)-component of the inverse matrix (g_{ij}), $1\leqq i,\ j\leqq n$. We put $X_i=\partial/\partial x_i+u_i(\partial/\partial x_{n+1})$ ($1\leqq i\leqq n$). Then $X_1, \cdots, X_n$ are linearly independent tangent vector fields on $S(u)$. We set

(4.9) $$\bar{g}_{ij}=\langle X_i, X_j\rangle=g_{ij}+g_iu_j+g_ju_i+gu_iu_j , \quad 1\leqq i,\ j\leqq n .$$

We can now give a unit normal vector field $\eta=\sum \eta^\alpha\partial/\partial x_\alpha$ on $S(u)$ in the form:

(4.10) $$\eta^i=-\frac{1}{\sqrt{G}}\sum a^{ij}a_j \quad (1\leqq i\leqq n) , \qquad \eta^{n+1}=\frac{1}{\sqrt{G}}$$

where

(4.11)
$$a^{ij}=g^{ij}-u^ig^j(1+\sum g_ku^k)^{-1} , \qquad a_j=g_j+gu_j ,$$
$$g^j=\sum g^{jk}g_k , \quad 1\leqq i,\ j\leqq n ,$$
$$G=\sum g_{ij}a^{ik}a^{jl}a_ka_l-2\sum g_ia^{ik}a_k+g>0 .$$

Let H be the mean curvature of $S(u)$ with respect to η. It is given by

$$H=\frac{1}{n}\sum \bar{g}^{ij}\langle \nabla_{X_i}X_j, \eta\rangle$$

where $\bar{g}^{ij}$ is the (i, j)-component of the inverse matrix of the matrix $(\bar{g}_{ij})$. By (4.3) H satisfies the inequality

(4.12) $$|H|\leqq H_0 \quad \text{on} \quad U .$$

Using (4.10) we have

$$nH\sqrt{G}=\sum \bar{g}^{ij}\{(g-\sum a^{kl}a_l g_k)u_{ij}+\sum (g_{\alpha n+1}-\sum a^{kl}a_l g_{\alpha k})(\Gamma^{\alpha}_{ij}+\Gamma^{\alpha}_{in+1}u_j)\} \tag{4.13}$$

where $\Gamma^{\alpha}_{\beta\gamma}$ $(1\leqq\alpha, \beta, \gamma\leqq n+1)$ are the Christoffel's symbols of the Riemannian connection ∇ with respect to the local coordinate system $(x_1, \cdots, x_{n+1})$, $x_{n+1}=t$. We put

$$\bar{G}^{ij}=\bar{g}^{ij}\det(\bar{g}_{ij}), \quad 1\leqq i, j\leqq n. \tag{4.14}$$

Then (4.13) can be rewritten in the form:

$$\sum A_{ij}(m, Du)u_{ij}=B(m, Du, H) \tag{4.15}$$

where

$$\begin{aligned} A_{ij}&=(g-\sum a^{kl}a_l g_k)\bar{G}^{ij}, \quad 1\leqq i, j\leqq n, \\ B&=nH\sqrt{G}\det(\bar{g}_{ij})-\sum \bar{G}^{ij}(\Gamma^{\alpha}_{ij}+\Gamma^{\alpha}_{in+1}u_j)(g_{\alpha n+1}-\sum g_{\alpha k}a^{kl}a_l). \end{aligned} \tag{4.16}$$

It is evident that A_{ij} $(1\leqq i, j\leqq n)$ and B are continuous for the variable (m, Du) and that B is of class C^1 for the variable $Du=(u_1, \cdots, u_n)$. We note $\langle\eta, \partial/\partial t\rangle=(g-\sum a^{kl}a_l g_k)/\sqrt{G}$ which never vanish on $S(u)$. Since $\langle\eta(m_1), \partial/\partial t(m_1)\rangle>0$ and $S(u)$ is connected, $g-\sum a^{kl}a_l g_k>0$ on $S(u)$. Therefore if in (4.15) we regard H as a given continuous function on U, (4.15) is a quasilinear elliptic partial differential equation of second order on U.

Now, since g_{ij} $(1\leqq i, j\leqq n)$ are independent of $t\in(-\tau, \tau)$, it follows from (4.9) that for each fixed $t\in(-\tau, \tau)$ the mean curvature H_t of $S(t)=\{(m, t)\in U\times(-\tau, \tau); m\in U\}$ is equal to $\mathscr{H}$. Hence from (4.11), (4.14) and (4.15) we obtain for a fixed $t\in(-\tau, \tau)$

$$\mathscr{H}=H_t=\frac{1}{n}(g-\sum g^{kl}g_k g_l)^{-1/2}\sum g^{ij}\Gamma^{\alpha}_{ij}(g_{\alpha n+1}-g_{\alpha k}g^{kl}g_l). \tag{4.17}$$

We put $B_{H_0}(m, Du)=B(m, Du, H_0)$. From (4.11), (4.14) and (4.16) we have

$$B_{H_0}(m, 0)=nH_0(g-\sum g^{kl}g_k g_l)^{1/2}\det(g_{ij})-\sum g^{ij}\Gamma^{\alpha}_{ij}(g_{\alpha n+1}-\sum g_{\alpha k}g^{kl}g_l)\det(g_{ij}).$$

Then, by (4.1) and (4.17)

$$B_{H_0}(m, 0)=n(H-\mathscr{H})(g-\sum g^{ij}g_i g_j)^{1/2}\det(g_{ij})\leqq 0. \tag{4.18}$$

Now for a given continuous function H' on U we set

$$L_{H'}(v)=\sum A_{ij}(m, Dv)v_{ij}-B(m, Dv, H')$$

where $v\in C^2(U)$ and A_{ij} $(1\leqq i, j\leqq n)$ and B are given by (4.16). Since $L_H(u)=0$, we have

$$L_{H_0}(u)=L_{H_0}(u)-L_H(u)=n(H-H_0)\sqrt{G}\det(\bar{g}_{ij})\,.$$

Then, by (4.12), u is a supersolution of the equation $L_{H_0}(v)=0$. By (4.18) we can now apply Theorem 2.1 to the equation $L_{H_0}(v)=0$. Then from (4.5) we have $u\equiv 0$ on U. Thus $S(u)=U\times\{0\}$. Hence there exists an open neighborhood V of m_0 in M such that $f(V)\subset\partial\Omega$. We complete the proof.

In Theorem 4.1, if N is a homogeneous Riemannian manifold, for each point m of $\partial\Omega$ there exists a Killing vector field on M which is transversal to $\partial\Omega$ on an open neighborhood of m in $\partial\Omega$. Making use of Theorem 4.1 and a similar method as in the proof of Theorem 3.1, we have the following.

Theorem 4.2. *Let N be an $(n+1)$-dimensional homogeneous Riemannian manifold and Ω a domain with smooth boundary $\partial\Omega$ in N. Suppose that the mean curvature $\mathscr{H}$ of $\partial\Omega$ satisfies the condition: $\mathscr{H}\geqq H_0$ on $\partial\Omega$ where H_0 is a non-negative constant. Let f: $M\to N$ be an immersion of an n-dimensional differentiable manifold M into N such that $f(M)\subset\bar{\Omega}$ and $f(M)\cap\partial\Omega\neq\emptyset$. Suppose that the mean curvature H_M (defined up to a sign) of M for the immersion f satisfies the condition: $|H_M(m)|\leqq H_0$ for all $m\in M$. Then $f(M)$ is contained in $\partial\Omega$.*

In what follows, for a Riemannian manifold M we shall denote by $\tilde{M}$ the Riemannian product manifold of M and the real line R.

Let M be an n-dimensional Riemannian manifold and Ω a domain in M. For a $u\in C^2(\Omega)$ let us consider a hypersurface $S(u)$ in $\tilde{M}$ defined by

$$S(u)=\{(m,\,u(m))\in\tilde{M};\ m\in\Omega\}\,. \tag{4.19}$$

We set

$$S_+(u)=\{(m,\,t)\in\Omega\times R;\ m\in\Omega,\ t\geqq u(m)\}\,,$$
$$S_-(u)=\{(m,\,t)\in\Omega\times R;\ m\in\Omega,\ t\leqq u(m)\}\,. \tag{4.20}$$

We can give a unit normal vector field η on $S(u)$ as follows: Let m be a point of Ω and U a local coordinate neighborhood of m in M which is contained in Ω. Let $(x_1,\cdots,x_n,t)$ be a local coordinate system on $U\times R$. We put $g_{ij}=\langle\partial/\partial x_i,\,\partial/\partial x_j\rangle$ $(1\leqq i,j\leqq n)$ where $\langle\ ,\ \rangle$ denotes the Riemannian metric of $\tilde{M}$. We denote by g^{ij} the (i,j)-component of the inverse marix of marix (g_{ij}), $1\leqq i,\ j\leqq n$. Now we put $\eta=\sum\eta^i(\partial/\partial x_i)+\eta^{n+1}(\partial/\partial t)$. Then η is given by

$$\eta^i=-(\sum g^{ij}u_j)/\sqrt{G}\quad(1\leqq i\leqq n)\,,\qquad \eta^{n+1}=1/\sqrt{G} \tag{4.21}$$

where

$$G=1+\sum g^{ij}u_iu_j\,. \tag{4.22}$$

It is easy to see that η is globally defined on $S(u)$.

Under the above notations we shall prove the following.

Proposition 4.1. *Let M be an n-dimensional Riemannian manifold and Ω a domain in M. For a $u\in C^2(\Omega)$ let $S(u)$ be a hypersurface in $\tilde{M}$ defined by* (4.19) *and $\mathscr{H}$ the mean curvature of $S(u)$ with respect to η which is defined by* (4.21). *Suppose that $\mathscr{H}\geqq 0$ holds everywhere. Then u can not take its maximum value in Ω unless u is constant.*

Proof. Suppose that u takes its maximum value at a point m_0 in Ω. Put $c=u(m_0)$ and $m_1=(m_0, c)\in S(u)$. We now consider a hypersurface $S(c)$ in $\tilde{M}$. Then $S(c)\subset S_+(u)$ and $S(c)$ is tangent to $S(u)$ at m_1. For the domain $S_+(u)$ η is the inward unit normal vector field on the boundary hypersurface $S(u)$. We note that $\partial/\partial t$ is a Killing vector field on $\tilde{M}$ which is transversal to $S(u)$ at each point of $S(u)$. Therefore we can now apply Theorem 4.1 to the present case. Then it is easy to see $u\equiv c$ on Ω. The proof is complete.

As a corollary of Proposition 4.1 we have

Corollary 4.1. *Let M be an n-dimensional compact Riemannian manifold. For a $u\in C^2(M)$ let $S(u)$ be a hypersurface in $\tilde{M}$ defined by* (4.19) *and $\mathscr{H}$ the mean curvature of $S(u)$ with respect to η which is defined by* (4.21). *Suppose that $\mathscr{H}\geqq 0$ holds everywhere. Then u is constant.*

Let M be an n-dimensional Riemannian manifold and Ω a domain in M. We shall denote by $C^0(\bar{\Omega})$ the set of real-valued continuous functions on $\bar{\Omega}$ where $\bar{\Omega}$ stands for the closure of Ω.

Proposition 4.2. *Let M be an n-dimensional Riemannian manifold and Ω a compact domain with boundary $\partial\Omega$ in M. For $u, v\in C^2(\Omega)\cap C^0(\bar{\Omega})$ let $S(u)$ and $S(v)$ be hypersurfaces in $\tilde{M}$ defined by* (4.19) *respectively. Suppose that $S(u)$ and $S(v)$ are minimal hypersurfaces in $\tilde{M}$. If $u=v$ on $\partial\Omega$, then $u=v$ in Ω.*

Proof. Suppose for contradiction that u and v are distinct. Put $h(m):=|u(m)-v(m)|$, $m\in\Omega$. Since $h=0$ on $\partial\Omega$, h must take its maximum value in Ω. Let m_0 be a point of Ω where h takes its maximum value. Put $c=h(m_0)$ (>0) Without loss of generality we may assume that $v(m_0)=u(m_0)+c$. Then $S(v)$ is contained in $S_-(u+c)$ and $S(v)$ is tangent to $S(u+c)$ at $m_1=(m_0, u(m_0)+c)$. We note that $\partial/\partial t$ is a Killing vector field on $\tilde{M}$ which is transversal to $S(u+c)$ at each point of $S(u+c)$ and that $S(u+c)$ is a minimal hypersurface in $\tilde{M}$ (because $S(u)$ is minimal in $\tilde{M}$). Therefore we can now apply Theorem 4.1 to the present case. Then we see $v=u+c$ on Ω. Since u, v are continuous on $\bar{\Omega}$, we obtain $v=u+c$ on $\bar{\Omega}$ which is a contradiction.

§ 5. Uniqueness for minimal hypersurfaces with a given boundary.

Throughout this section, for a Riemannian manifold M we shall denote by $\tilde{M}$ the Riemannian product manifold of M and the real line R and denote by π_M (resp. π_R) the natural projection of $\tilde{M}$ onto M (resp. R). Let M be an n-dimensional Riemannian manifold. We consider a compact domain Ω with smooth boundary $\partial\Omega$ in M having the following properties:

(5.1)
(1) Ω is homeomorphic to the unit open ball $D=\{x\in R^n;\ \|x\|<1\}$ in R^n where $\|\ \ \|$ denotes the standard Euclidean norm of R^n.
(2) $\partial\Omega$ is homeomorphic to the $(n-1)$-dimensional unit sphere $\partial D=\{x\in R^n;\ \|x\|=1\}$ in R^n.
(3) The mean curvature $\mathcal{H}$ of $\partial\Omega$ is non-negative everywhere.

For a $u\in C^2(\Omega)\cap C^0(\bar{\Omega})$ we set

$$S(u)=\{(m,\ u(m))\in\tilde{M};\ m\in\Omega\}\ ,$$
$$\partial S(u)=\{(m,\ u(m))\in\tilde{M};\ m\in\partial\Omega\}\ . \tag{5.2}$$

Moreover, for a $u\in C^2(\Omega)\cap C^0(\bar{\Omega})$ let us consider a continuous map Φ: $\bar{D}\to\tilde{M}$ satisfying the following conditions:

(5.3)
(1) $\Phi_{|D}$: $D\to\tilde{M}$ is a minimal immersion of class C^2.
(2) $\Phi_{|\partial D}$: $\partial D\to\partial S(u)$ is a homeomorphism.

In what follows we shall use the notations described above without mention.

By virtue of the degree theory of continuous maps we can easily show the following.

Lemma 5.1. *Let M be an n-dimensional Riemannian manifold and Ω a compact domain with smooth boundary $\partial\Omega$ in M having the properties* (1) *and* (2) *in* (5.1). *For a $u\in C^2(\Omega)\cap C^0(\bar{\Omega})$ let $S(u)$ be a hypersurface in $\tilde{M}$ defined by* (5.2) *and let Φ: $\bar{D}\to\tilde{M}$ be a continuous map such that $\Phi_{|\partial D}$: $\partial D\to\partial S(u)$ is a homeomorphism. Then there exists a point x of D such that $\Phi(x)\notin\partial\Omega\times R$.*

Theorem 5.1. *Let M be an n-dimensional Riemannian manifold whose Ricci curvature is non-negative everywhere. Let Ω be a compact domain with smooth boundary $\partial\Omega$ in M having the properties* (1), (2) *and* (3) *in* (5.1). *For a $u\in C^2(\Omega)\cap C^0(\bar{\Omega})$ let $S(u)$ be a hypersurface in $\tilde{M}$ defined by* (5.2) *and let Φ: $\bar{D}\to\tilde{M}$ be a con-*

tinuous map satisfying the conditions (1) *and* (2) *in* (5.3). *Suppose that* $S(u)$ *is a minimal hypersurface in* $\tilde{M}$ *and that* $\Phi(\bar{D})\subset\bar{\Omega}\times R$. *Then* $\Phi_{|D}$: $D\to\tilde{M}$ *is imbedding of class* C^2 *and* $\Phi(D)=S(u)$.

Proof. By the condition (3) in (5.1) we see that the mean curvature (with respect to the inward unit normal vector) of the boundary $\partial\Omega\times R$ of $\Omega\times R$ is non-negative everywhere. Suppose that there exists a point x of D such that $\Phi(x)\in\partial\Omega\times R$. Since $\Phi(D)$ is a minimal hypersurface in $\tilde{M}$ such that $\Phi(D)\subset\bar{\Omega}\times R$, by virtue of Theorem 3.1 we see that $\Phi(D)$ must be contained in $\partial\Omega\times R$. This contradicts Lemma 5.1. Thus we have proved $\Phi(D)\subset\Omega\times R$. Next we shall prove $\Phi(D)=S(u)$. Suppose for contradiction that $\Phi(D)$ is not contained in $S(u)$. We consider a continuous function h on $\bar{D}$ defined by $h(x)=|\pi_R(\Phi(x))-u(\pi_M(\Phi(x)))|$, $x\in\bar{D}$. By the condition (2) in (5.3) $h=0$ on ∂D. Therefore h must take its maximum value in D. Let x' be a point of D where h takes its maximum value. Put $c=h(x')$ $(c>0)$ and $m'=\pi_M(\Phi(x'))$. We now assume $\pi_R(\Phi(x'))=u(m')+c$. We set $D'=\{x\in D;\ \Phi(x)\in S(u+c)\}$. Then D' is clearly closed in D. Let x_0 be a point of D'. We put $m_0=\pi_M(\Phi(x_0))$ and $m_1=(m_0, u(m_0)+c)$. Then $\Phi(D)$ is contained in $S_-(u+c)$ which is defined by (4.20) and $\Phi(D)$ is tangent to $S(u+c)$ at m_1. Since $S(u+c)$ is minimal in M and $\partial/\partial t$ (the natural vector field on R) is a Killing vector field on $\tilde{M}$ which is transversal to $S(u+c)$ at each point of $S(u+c)$, we can now apply Theorem 4.1 to the present case. By virtue of Theorem 4.1 there exists an open neighborhood V of x_0 in D such that $\Phi(V)\subset S(u+c)$. Thus we have proved that D' is open in D. By the connectedness of D we obtain $\Phi(D)\subset S(u+c)$. Since Φ is continuous, $\Phi(\partial D)\subset\overline{S(u+c)}$. This contradicts $\Phi(\partial D)=\partial S(u)$ because $\overline{S(u+c)}\cap\partial S(u)=\emptyset$. In the case of $\pi_R(\Phi(x'))=u(m')-c$ we can deduce a contradiction by a similar argument as above. Hence $\Phi(D)=S(u)$. Since by the condition (1) in (5.3) Φ is locally homeomorphic, we see that Φ is $1:1$. Thus we complete the proof.

Using Theorems 4.1, 4.2 and a similiar method as in the proof of Theorem 5.1 we have the following.

Theorem 5.2. *Let* M *be an* n*-dimensional homogeneous Riemannian manifold. Let* Ω *be a compact domain with smooth boundary* $\partial\Omega$ *in* M *having the properties* (1), (2) *and* (3) *in* (5.1). *For a* $u\in C^2(\Omega)\cap C^0(\bar{\Omega})$ *let* $S(u)$ *be a hypersurface in* $\tilde{M}$ *defined by* (5.2) *and let* Φ: $\bar{D}\to\tilde{M}$ *be a continuous map satisfying the conditions* (1) *and* (2) *in* (5.3). *Suppose that* $S(u)$ *is a minimal hypersurface in* $\tilde{M}$ *and that* $\Phi(\bar{D})\subset\bar{\Omega}\times R$. *Then* $\Phi_{|D}$: $D\to\tilde{M}$ *is imbedding of class* C^2 *and* $\Phi(D)=S(u)$.

Let M be an n-dimensional simply connected, complete Riemannian manifold whose sectional curvature is non-positive everywhere. We shall denote by ρ the distance function of M. We set

$$B_r(m)=\{m'\in M;\ \rho(m, m')\leqq r\}\ ,\qquad \partial B_r(m)=\{m'\in M;\ \rho(m, m')=r\}$$

where r is positive. We now consider a compact domain Ω with smooth boundary $\partial\Omega$ in M having the following properties:

(5.4)
(1) Ω has the properties (1), (2) in (5.1).
(2) For each point m of $\partial\Omega$ there exists a closed metric ball $B_r(m')$ such that $\bar{\Omega}\subset B_r(m')$ and $m\in\partial B_r(m')$.

We note that by the condition (2) in (5.4) the mean curvature $\mathscr{H}$ of $\partial\Omega$ is positive everywhere.

Theorem 5.3. *Let M be an n-dimensional simply connected, complete Riemannian manifold whose sectional curvature is non-positive everywhere. Let Ω be a compact domain with smooth boundary $\partial\Omega$ in M having the properties* (1) *and* (2) *in* (5.4). *For a $u\in C^2(\Omega)\cap C^0(\bar{\Omega})$ let $S(u)$ be a hypersurface in $\tilde{M}$ defined by* (5.2) *and let Φ: $\bar{D}\to\tilde{M}$ be a continuous map satisfying the conditions* (1), (2) *in* (5.3). *Suppose that $S(u)$ is a minimal hypersurface in $\tilde{M}$. Then $\Phi_{|D}$: $D\to\tilde{M}$ is imbedding of class C^2 and $\Phi(D)=S(u)$.*

Proof. Suppose that there is a point x of D such that $\Phi(x)\in\tilde{M}-\bar{\Omega}\times R$. Then there exists a point $x_0\in D$ such that $\rho_0:=\rho(\pi_M(\Phi(x_0)), \bar{\Omega})\geqq\rho(\pi_M(\Phi(x)), \bar{\Omega})$, $x\in\bar{D}$. Of course $\rho_0>0$. Let m_1 be a point of $\partial\Omega$ such that $\rho_0=(\pi_M(\Phi(x_0)), m_1)$. By the assumption we can take a closed metric ball $B_r(m')$ which has the properties: $\bar{\Omega}\subset B_r(m')$ and $m_1\in\partial B_r(m')$. It is easy to see that $\Phi(D)\subset B_{r_1}(m')\times R$ and $\Phi(x_0)\in\partial B_{r_1}(m')\times R$ where $r_1=\rho(m', \pi_M(\Phi(x_0)))=r+\rho_0$. But this contradicts Theorem 3.2 because the mean curvature (with respect to the inward unit normal vector) of $\partial B_{r_1}(m')\times R$ is positive everywhere. Thus we have proved $\Phi(\bar{D})\subset\Omega\times R$. Since by the condition (2) in (5.4) the mean curvature of $\partial\Omega\times R$ is positive everywhere, from Theorem 3.2 we see $\Phi(D)\subset\Omega\times R$. Then, using the same argument as in the proof of Theorem 5.1, we can complete the proof.

§ 6. Remarks on the existence of minimal hypersurfaces with a given boundary.

In this section we shall use the notations in § 5. Let Ω be a domain with boundary $\partial\Omega$ of class C^2 in the n-dimensional Euclidean unit sphere S^n having the following properties:

(6.1) (1) Ω has the properties (1) and (2) in (5.1) and $\bar{\Omega}\subset S^n_+:=\{x\in S^n;\ x_{n+1}\geqq 0\}$.
(2) For each point m of $\partial\Omega$ there exists a closed metric ball $B_{\pi/2}(m')$ of radius $\pi/2$ centred at $m'\in S^n$ such that $\bar{\Omega}\subset B_{\pi/2}(m')$ and $m\in\partial B_{\pi/2}(m')$.

Now we consider the stereographic projection $\Psi: S^n-\{m_0\}\to T_{m_1}S^n$ from the south pole m_0 onto the tangent space of S^n at the north pole m_1. We identify $T_{m_1}S^n$ with R^n and we put $\Psi(\Omega)=\tilde{\Omega}$. Then the line element of $S^n-\{m_0\}$ is given by

$$ds^2=a^2dx^2\,,\qquad a=4/(4+\|x\|^2)\quad (x\in R^n)\,. \tag{6.2}$$

Let ξ (resp. $\tilde{\xi}$) be the inward unit normal vector field of $\partial\Omega$ in S^n (resp. $\partial\tilde{\Omega}$ in R^n) and let $\mathscr{H}$ (resp. $\tilde{\mathscr{H}}$) be the mean curvature of $\partial\Omega$ in S^n (resp. $\partial\tilde{\Omega}$ in R^n) with respect to ξ (resp. $\tilde{\xi}$). Since $\xi=a\tilde{\xi}$, we have

$$\mathscr{H}(\Psi^{-1}(y))=\tilde{\mathscr{H}}(y)/a+\frac{1}{2}(\tilde{\xi}\cdot y)\,,\qquad y\in\partial\tilde{\Omega} \tag{6.3}$$

where the dot denotes the inner product in R^n. We note that by the condition (2) in (6.1) $\mathscr{H}\geqq 0$ on $\partial\Omega$. Therefore we have

$$\tilde{\mathscr{H}}(y)\geqq -\frac{a}{2}(\tilde{\xi}\cdot y)\,,\quad y\in\partial\Omega\,. \tag{6.4}$$

For a $u\in C^2(\Omega)$ let us consider a hypersurface $S(u)$ in $S^n\times R$ defined by (5.2). We suppose that $S(u)$ is minimal in $S^n\times R$. Then we see that $v=u\circ\Psi^{-1}$ is a solution of the following equation on $\tilde{\Omega}$:

$$\begin{aligned}&\sum\{(a^2+\|Dv\|^2)\delta_{ij}-v_iv_j\}v_{ij}\\ &\qquad=-\frac{a}{2}\sum\{(a^2+\|Dv\|^2)\delta_{ij}-v_iv_j\}(x_iv_j+x_jv_i-\delta_{ij}Dv\cdot x)\,.\end{aligned} \tag{6.5}$$

We can easily check the solvability conditions in Theorem 14.3 of J. Serrin's paper [8]. Therefore, by J. Serrin's theorem (Theorem 14.3 in [8]), for a given $f\in C^2(\partial\tilde{\Omega})$ there exists exactly one $v\in C^2(\bar{\tilde{\Omega}})$ such that v is a solution of (6.5) in $\tilde{\Omega}$ and $v=f$ on $\partial\tilde{\Omega}$ (We note that in the case where quasilinear partial differential equations of second order whose coefficients are independent of unknown function the uniqueness of solution with same boundary value is well-known). Then we have the following.

Theorem 6.1. *Let Ω be a domain with boundary $\partial\Omega$ of class C^2 in the n-dimensional Euclidean unit sphere S^n having the properties* (1) *and* (2) *in* (6.1). *For a given $h\in C^2(\partial\Omega)$ the following hold*:

(a) *There exists exactly one* $u \in C^2(\bar{\Omega})$ *such that the hypersurface* $S(u)$ *in* $S^n \times R$ *defined by* (5.2) *is minimal and* $\partial S(u)=\{(m, h(m)) \in S^n \times R;\ m \in \partial\Omega\}$.

(b) *For this u let* $\Phi: \bar{D} \to S^n \times R$ *is a continuous map satisfying the conditions* (1) *and* (2) *in* (5.3). *If* $\Phi(\bar{D}) \subset S^n_+ \times R$, *then* $\Phi_{|D}: D \to S^n \times R$ *is imbedding of class* C^2 *and* $\Phi(D)=S(u)$.

Proof. (a) follows from J. Serrin's theorem stated above. We shall give a proof of (b). It is sufficient to show $\Phi(\bar{D}) \subset \bar{\Omega} \times R$ (see Theorem 5.2). Suppose for contradiction that for some point x' of D $\Phi(x') \in S^n_+ \times R - \bar{\Omega} \times R$. Let m be a point of $\partial\Omega$ such that $\rho(\pi_{S^n}(\Phi(x')), m)=\rho(\pi_{S^n}(\Phi(x')), \bar{\Omega})$. Then by the condition (2) in (6.1) we can take a closed metric ball $B_{\pi/2}(m')$ satisfying the condition: $\bar{\Omega} \subset B_{\pi/2}(m')$ and $m \in \partial B_{\pi/2}(m')$. Since $\pi_{S^n}(\Phi(\bar{D})) \subset S^n_+$ and $\bar{\Omega} \subset S^n_+ \cap B_{\pi/2}(m')$, we see that there exists a motion φ in S^n such that $\pi_{S^n}(\Phi(\bar{D})) \subset \varphi(S^n_+)$ and for some $x \in D$ $\pi_{S^n}(\Phi(x)) \in \varphi(\partial S^n_+)$ where $\partial S^n_+=\{x \in S^n;\ x_{n+1}=0\}$. Hence $\Phi(\bar{D}) \subset \varphi(S^n_+) \times R$ and $\Phi(D)$ is tangent to the boundary $\varphi(\partial S^n_+) \times R$ of $\varphi(S^n_+) \times R$. Then by Theorem 4.2 and the continuity of Φ, $\Phi(\bar{D}) \subset \varphi(\partial S^n_+) \times R$. Since by the hypothesis $\Phi(\partial D)=\partial S(u) \subset \partial\Omega \times R$, we see $\partial\Omega=\partial S^n_+=\varphi(\partial S^n_+)$. Thus we have $\Omega=S^n_+$, so $\Phi(x') \subset \bar{\Omega} \times R$. This is a contradiction.

By a similar argument as above, we can prove the following.

Theorem 6.2. *Let M be an n-dimensional simply connected, complete Riemannian manifold with constant curvature K where* $K=0$ *or* $K<0$. *Let* Ω *be a compact domain with boundary* $\partial\Omega$ *of class* C^2 *in M having the following properties*:

(1) Ω *has the properties* (1) *and* (2) *in* (5.1).

(2) *For each point m of* $\partial\Omega$ *there exists a supporting hyperplane of* $\partial\Omega$ *in M passing through m. Then for a given* $h \in C^2(\partial\Omega)$ *the following hold*:

(a) *There exists exactly one* $u \in C^2(\bar{\Omega})$ *such that the hypersurface* $S(u)$ *in* $M \times R$ *defined by* (5.2) *is minimal and* $\partial S(u)=\{(m, h(m)) \in M \times R;\ m \in \partial\Omega\}$.

(b) *For this u let* $\Phi: \bar{D} \to M \times R$ *is a continuous map satisfying the conditions* (1) *and* (2) *in* (5.3). *Then* $\Phi_{|D}: D \to M \times R$ *is imbedding of class* C^2 *and* $\Phi(D)=S(u)$.

Remark 1. In Theorem 6.2, in the case of $M=R^n$ the existence of non-parametric minimal hypersurfaces with a given boundary was proved by H. Jenkins and J. Serrin (see [8]) and in the case where M is the hyperbolic space the existence follows from J. Serrin's theorem (see Theorem 14.3 in [8]) using a similar argument as in the case of $M=S^n$.

Remark 2. J. Serrin, R. Gulliver and J. Spruck extended Radó's theorem stated in the introduction to surfaces of constant mean curvature in R^3 ([7], [3]).

Our results obtained in § 5 can be extended to hypersurfaces of constant mean curvature under some conditions.

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