YOKOHAMA MATHEMATICAL
JOURNAL VOL. 26, 1978

MINIMAL MONOSYSTEMS

By

C. THAS

(Received June 8, 1978)

In this paper we give some results in connection with the mean curvature vector H of monosystems (i.e. submanifolds of the euclidean space E^m which are generated by a one-parameter family of linear spaces). Especially, if $H=0$ then we find restrictions for the dimension m of the euclidean space. Finally, we construct examples for each kind of minimal monosystem.

1. Introduction.

We shall assume throughout that all manifolds, maps, vector fields, etc.$\cdots$ are differentiable of class C^∞. Consider a general submanifold M of the euclidean space E^m. Suppose that $\bar{D}$ is the standard Riemann connection of E^m, while D is the Riemann connection of M. Then, if X and Y are vector fields of M and if V is the second fundamental form of M, we have by decomposing $\bar{D}_X Y$ in a tangent and a normal component

(1.1) $$\bar{D}_X Y = D_X Y + V(X, Y) .$$

Suppose that ξ is a normal vector field on M. If we decompose the field $\bar{D}_X\xi$ in a tangential component and a normal component, then we have the Weingarten equation

(1.2) $$\bar{D}_X\xi = -(A_\xi(X)) + D_X^\perp\xi .$$

A_ξ is at each point p of M a self adjoint linear map $M_p \to M_p$ and $D^\perp$ is a metric connection in the normal bundle $M^\perp$. If N is a subbundle of the normal bundle $M^\perp$ (i.e. N is a normal subbundle of M) and if $\bar{D}_X\eta$ has no component in the complementary normal subbundle $N^\perp$ orthogonal to N for each unit normal field η in N and each vector field X of M, then the subbundle N is said to be parallel ([2]). Suppose that X and Y are M-vector fields, while ξ is a normal vector field, then, if the standard metric tensor of E^m is denoted by $\langle , \rangle$,

(1.3) $$\langle V(X, Y), \xi\rangle = \langle A_\xi(X), Y\rangle .$$

If $\xi_1, \cdots, \xi_{m-\dim M}$ constitute an orthonormal base field of the normal bundle $M^\perp$, then we put

$$(1.4) \qquad \langle V(X, Y), \xi_i\rangle = V^i(X, Y) \qquad \text{or} \qquad V(X, Y) = \sum_{i=1}^{m-\dim M} V^i(X, Y)\xi_i .$$

The mean curvature vector H of M is given by

$$H = \sum_{i=1}^{m-\dim M} \frac{\operatorname{tr} A_{\xi_i}}{\dim M} \xi_i .$$

M is said to be minimal if $H=0$ at each point.

2. $(n+1)$-dimensional monosystems in E^m.

Assume that the base curve $r(s)$ of the monosystem M is an orthogonal trajectory of the n-dimensional generating spaces $(n \geq 1)$, which are spanned by the orthonormal base vectors $e_1(s), \cdots, e_n(s)$, then M can locally be represented by

$$r(s) + \sum_{i=1}^{n} l_i e_i(s) , \qquad l_i \in R , \quad i=1, \cdots, n .$$

Suppose that $e_1, \cdots, e_n, e$ is an orthonormal base field of M (i.e. e is the unit tangent vector of the orthogonal trajectories of the generating spaces). Then M is said to be k-developable if

$$(2.1) \qquad \operatorname{rank}[e, e_1, \cdots, e_n, \bar{D}_e e_1, \cdots, \bar{D}_e e_n] = 2n-k \quad \text{at each point} \quad p \in M .$$

Remark that if $k \geq 0$, then M contains singular points (which we leave out of consideration); in fact, if (2.1) holds, then each generating space contains a k-dimensional subspace S of singular points and the tangent spaces M_p and M_q at two non-singular points p and q of the same generating space are parallel in E^m (i.e. M_q is the parallel displacement in E^m of M_p from p to q; in a classical way we should say that M_p coincides with M_q) iff the $(k+1)$-dimensional spaces spanned by S and p and by S and q are the same ([5]). If $k=-1$ then the monosystem M is called non-developable; if $k=n-1$, then M is said to be total developable. Suppose that $X = \sum_{i=1}^{n} a^i e_i + ae$ and $Y = \sum_{i=1}^{n} b^i e_i + be$ are two M-vector fields. It is clear that, if V is the second fundamental form of M in E^m, we have

$$(2.2) \qquad V(e_i, e_j) = 0 , \quad i, j=1, \cdots, n .$$

So we find

$$(2.3) \qquad V(X, Y) = \sum_{i=1}^{n} (a^i b + b^i a) V(e, e_i) + ab V(e, e) .$$

The normal subbundle of $M^\perp$ spanned by the normal fields $V(e, e_i)$, $i=1, \cdots, n$ is denoted by F.

Lemma 1. *M is k-developable iff the normal subbundle F is $(n-k-1)$-dimensional.*

Proof. Suppose that we have (2.1). We know that, if D is the Riemann connection of M, $\bar{D}_e e_i = D_e e_i + V(e, e_i)$, $i=1, \cdots, n$. But $D_e e_i$ is a linear combination of the fields $e_1, \cdots, e_n, e$ and so we may replace the fields $\bar{D}_e e_i$ by $V(e, e_i)$ in (2.1). Now, the tangent space spanned by $e, e_1, \cdots, e_n$ is at each point normal to F and thus we find $n+1+\dim F = 2n-k$ or $\dim F = n-k-1$, q.d.e.

As a corollary we have that M can only be k-developable if $m-n-1 \geq n-k-1$ or $m \geq 2n-k$.

Lemma 2. *Consider the orthonormal base field $e_1, \cdots, e_n, e$ in a neighbourhood of a point p of M. The Riemannian curvature K_σ in the two-dimensional direction σ of M_p spanned by the vectors $(e_i)_p$ and e_p is given by*

$$K_\sigma = -\langle \bar{D}_{e_i} e, \bar{D}_{e_i} e \rangle_p , \quad i=1, \cdots, n . \tag{2.4}$$

Proof. Suppose that R is the curvature tensor of M, then

$$K_\sigma = \langle e_i, R(e_i, e) e \rangle_p .$$

But from the Gauss equation, we see that

$$\langle e_i, R(e_i, e) e \rangle_p = \langle V(e_i, e_i), V(e, e) \rangle_p - \langle V(e_i, e), V(e_i, e) \rangle_p ,$$

and we know that $V(e_i, e_i)=0$, $i=1, \cdots, n$. Moreover we have

$$\langle \bar{D}_{e_i} e, e_j \rangle = -\langle e, \bar{D}_{e_i} e_j \rangle = 0 , \quad i, j=1, \cdots, n$$

and

$$\langle \bar{D}_{e_i} e, e \rangle = -\langle e, \bar{D}_{e_i} e \rangle = 0 , \quad i=1, \cdots, n .$$

This means that $\bar{D}_{e_i} e$ is a normal vector field or

$$\bar{D}_{e_i} e = V(e_i, e) , \tag{2.5}$$

and this completes the proof.

Suppose that $\xi_1, \cdots, \xi_{m-n-1}$ is an orthonormal base field of the normal bundle $M^\perp$, then we have the following Weingarten equations

$$\begin{cases} \bar{D}_{e_1} \xi_j = \sum_{i=1}^{n} a_{1j}^i e_i + b_{1j} e + \sum_{r=1}^{m-n-1} C_{1j}^r \xi_r \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ \bar{D}_{e_n} \xi_j = \sum_{i=1}^{n} a_{nj}^i e_i + b_{nj} e + \sum_{r=1}^{m-n-1} C_{nj}^r \xi_r \\ \bar{D}_e \xi_j = \sum_{i=1}^{n} b_j^i e_i + b_j e + \sum_{r=1}^{m-n-1} C_j^r \xi_r \end{cases} \quad j=1, \cdots, m-n-1 . \tag{2.6}$$

Because of (1.3) and (2.2) we find

$$\langle V(e_s, e_k), \xi_j\rangle = \langle A_{\xi_j}(e_s), e_k\rangle = -a^k_{sj} = 0\ , \quad s, k=1, \cdots, n \quad \text{and} \quad j=1, \cdots, m-n-1\ .$$

Moreover, the linear maps A_{ξ_j} are self adjoint and thus $b^i_j = b_{ij}$ $i=1, \cdots, n$ and $j=1, \cdots, m-n-1$. So the matrix of A_{ξ_j} has the form

$$-\begin{pmatrix} 0 \cdots 0 & b_{1j} \\ \vdots \quad\ \ \vdots & \vdots \\ 0 \cdots 0 & b_{nj} \\ b_{1j} \cdots b_{nj} & b_j \end{pmatrix} \quad j=1, \cdots, m-n-1$$

and this means det $A_{\xi_j}=0$ if $n \geq 2$, from which we have:

Corollary 1. *If $n \geq 2$, then the Lipschitz-Killing curvature of M is zero at each point in each normal direction.*

Using (1.3) again, we find

$$\langle V(e_s, e), \xi_j\rangle = \langle A_{\xi_j}(e_s), e\rangle = -b_{sj}\ ,$$

and if the Riemannian curvature of M in the (variable) two-dimensional direction spanned by e_s and e is denoted by $K(e_s, e)$, then (2.4) together with (2.5) gives

$$K(e_s, e) = -\sum_{j=1}^{m-n-1} (b_{sj})^2\ . \tag{2.7}$$

Remark that $K(e_s, e_i)=0$, $s, i=1, \cdots, n$ for each generating space is totally geodesic in E^m. Because of this, the Ricci curvature of M in the direction of e_s is equal to $K(e_s, e)$ and thus, from (2.7) we find immediately

Corollary 2. *The scalar curvature r of M is given by*

$$r = -2\sum_{i=1}^{n} \sum_{j=1}^{m-n-1} (b_{ij})^2\ . \tag{2.8}$$

Because of (1.3) we have

$$\langle V(e, e), \xi_j\rangle = \langle A_{\xi_j}(e), e\rangle = -b_j\ , \quad j=1, \cdots, m-n-1$$

and from the definition of the mean curvature vector we find at once that

$$H = \frac{V(e, e)}{n+1}\ . \tag{2.9}$$

Corollary 3. *The monosystem M is minimal iff each orthogonal trajectory of the generating spaces is an asymptotic line of M.*

Theorem 1. *If the $(n+1)$-dimensional k-developable monosystem M is minimal, then M is necessarily a submanifold of an E^{2n-k}.*

Proof. From Lemma 1, we already know that the codimension of M is at least $n-k-1$. We have two cases:

a. First, suppose that the normal subbundle F is zero-dimensional, then, because of Lemma 1, $k=n-1$, i.e. M is total developable. From (2.4) and (2.5) we see that $K(e_i, e)=0$, $i=1, \cdots, n$ and this means that the scalar curvature r of M is identically zero. A result of Takahashi says that in this case the minimal monosystem must be totally geodesic, i.e. M is a part of a $(n+1)$-dimensional linear space, which was our claim in this case. (This follows also immediately from (2.3), because now $V(e, e)=V(e, e_i)=0$, $i=1, \cdots, n$ and thus $V(X, Y)=0$ for each two vector fields X and Y of M).

b. Next suppose that $F \neq 0$ and that $\xi_1, \cdots, \xi_{m-n-1}$ is an orthonormal normal base field of $M^{\perp}$ such that $\xi_1, \cdots, \xi_{n-k-1}$ constitute an orthonormal base field of F. Consider the equations (2.6) in this case. Since $\langle V(e_s, e), \xi_j \rangle=-b_{sj}$, $s=1, \cdots, n$ $j=1, \cdots, m-n-1$, we have immediately

$$b_{sr}=0\,, \qquad s=1, \cdots, n \quad \text{and} \quad r=n-k, \cdots, m-n-1\,.$$

But $H=0$ and hence $\operatorname{tr} A_{\xi_j}=0$, $j=1, \cdots, m-n-1$, and so we get

$$A_{\xi_{n-k}}=\cdots=A_{\xi_{m-n-1}}=0\,. \tag{2.10}$$

Since $V(e, e)=0$, we see, because of (2.3), that $V(X, Y) \in F$ for each two vector fields X and Y of M. But F is spanned by $\xi_1, \cdots, \xi_{n-k-1}$ and thus, if we decompose $V(X, Y)$ such as in (1.4), we find

$$V^{n-k}(X, Y)=\cdots=V^{m-n-1}(X, Y)=0 \tag{2.11}$$

for each two vector fields X and Y of M.

If $\bar{R}$ is the curvature tensor of E^m and if Z is an other vector field of M, then the Codazzi equation says

$$\begin{aligned}(\bar{R}(X, Y)Z)^N= &\sum_{j=1}^{m-n-1} \{(D_X V^j)(Y, Z)-(D_Y V^j)(X, Z)\}\xi_j \\ &+\sum_{j=1}^{m-n-1} V^j(Y, Z)D_X^{\perp}\xi_j-\sum_{j=1}^{m-n-1} V^j(X, Z)D_Y^{\perp}\xi_j=0\,.\end{aligned} \tag{2.12}$$

Suppose that

$$D_{e_i}^{\perp}\xi_l=\sum_{h=1}^{n-k-1} C_{il}^h\xi_h+\sum_{r=n-k}^{m-n-1} C_{il}^r\xi_r\,, \quad i=1, \cdots, n, \ l=1, \cdots, n-k-1\,. \tag{2.13}$$

From (2.11) and (2.12) we find

$$\begin{aligned}(\bar{R}(e_i, e)e_s)^N= &\sum_{l=1}^{n-k-1} \{\cdots\}\xi_l+\sum_{l=1}^{n-k-1} V^l(e, e_s)D_{e_i}^{\perp}\xi_l \\ &-\sum_{l=1}^{n-k-1} V^l(e_i, e_s)D_e^{\perp}\xi_l=0\,, \quad i, s=1, \cdots, n\end{aligned} \tag{2.14}$$

But $V(e_i, e_s)=0$, $i, s=1, \cdots, n$ and thus (2.14) together with (2.13) gives

$$\sum_{l=1}^{n-k-1} C_{il}^{r} V^{l}(e, e_s)=0\ , \qquad i, s=1, \cdots, n \quad \text{and} \quad r=n-k, \cdots, m-n-1\ .$$

Now, fix in this expression i and r and let s be variable, then we find a system of n homogeneous linear equations with $n-k-1$ unknowns C_{il}^{r} $(l=1, \cdots, n-k-1)$. The matrix of this system is

$$[V^{l}(e, e_s)]\ , \quad l=1, \cdots, n-k-1;\ s=1, \cdots, n\ ,$$

and its rank is at each point of M equal to $n-k-1$ because M is k-developable. So, all the unknowns of this system must be zero and we find

(2.15) $$C_{il}^{r}=0 \quad i=1, \cdots, n;\ l=1, \cdots, n-k-1;\ r=n-k, \cdots, m-n-1\ .$$

Next we have

(2.16) $$(\bar{R}(e, e_i)e)^{N}=\sum_{l=1}^{n-k-1}\{\cdots\}\hat{\xi}_l+\sum_{l=1}^{n-k-1} V^{l}(e_i, e)D_{e}^{\perp}\hat{\xi}_l - \sum_{l=1}^{n-k-1} V^{l}(e, e)D_{e_i}^{\perp}\hat{\xi}_l=0\ , \quad i=1, \cdots, n\ .$$

But $V(e, e)=0$, and if we put

$$D_{e}^{\perp}\hat{\xi}_l=\sum_{h=1}^{n-k-1} C_{l}^{h}\hat{\xi}_h+\sum_{r=n-k}^{m-n-1} C_{l}^{r}\xi_r\ , \quad l=1, \cdots, n-k-1\ ,$$

we find from (2.16)

$$\sum_{l=1}^{n-k-1} C_{l}^{r} V^{l}(e_i, e)=0\ , \quad i=1, \cdots, n;\ r=n-k, \cdots, m-n-1\ .$$

And here again this means analogously that

(2.17) $$C_{l}^{r}=0\ , \quad l=1, \cdots, n-k-1;\ r=n-k, \cdots, m-n-1\ .$$

Now, (2.15) together with (2.17) says that for each unit normal field η in F and for each vector field X of M, $D_{X}^{\perp}\eta$ has no component in the complementary subbundle $F^{\perp}$; i.e. the normal subbundle F is parallel. If we identify all the tangent spaces of E^{m} with E^{m} itself, then, since F is parallel and because of (2.10), we see that the $(2n-k)$-dimensional subspaces of E^{m} spanned at each point of M by the tangent space and the normal space F, are independent of the choice of the point p of M. This completes the proof.

Corollary 4. *In the euclidean space E^{m} there are at most $(m-1)/2$ (resp. $(m-2)/2$) different types* (i.e. *generated by linear spaces of different dimension*) *of minimal not-totally geodesic monosystems which do not lie in any E^{l} with $l<m$ if m is odd* (*resp. even*). *Non-developable minimal monosystems of dimension $n+1$ can only exist in E^{2n+1}.*

Proof. If the $(n+1)$-dimensional minimal monosystem is k-developable, then Theorem 1 says that we must have $2n-k=m$. So the minimal value of k is -1 (resp. 0) if m is odd (resp. even), and thus the minimal value of n is $(m-1)/2$ (resp. $m/2$). It is clear that the maximal value of n is $m-2$ (and then $k=m-4$). If m is odd (resp. even), we have at most $m-2-(m-1)/2+1$ (resp. $m-2-m/2+1$) possibilities and this completes the proof.

Theorem 2. *If the mean curvature vector $H\neq 0$ of the $(n+1)$-dimensional k-developable monosystem M is at each point of M a vector of the normal subbundle F, then M is necessarily a submanifold of an E^{2n-k}.*

Proof. If $M\subset E^m$, we know that since M is k-developable, $m\geq 2n-k$. Moreover, if we choose the orthonormal field $\xi_1, \cdots, \xi_{m-n-1}$ such as in the proof of Theorem 1, then, because $(n+1)H=V(e, e)\in F$, we find again

$$V^{n-k}(X, Y)=\cdots=V^{m-n-1}(X, Y)=0 ,$$

for each two vector fields X and Y of M.

Next, if we have the expression (2.13), then we find (2.15) in the same way as before.

Since the vector fields $D^{\perp}_{e_i}\xi_l$, $i=1, \cdots, n$; $l=1, \cdots, n-k-1$ have no component in the complementary subbundle $F^{\perp}$, we find, because of (2.16), again (2.17) and this completes the proof.

Remarks 1. If the normal subbundle $F+H$ spanned by F and H of the $(n+1)$-dimensional k-developable monosystem M is parallel in the normal bundle, then M is contained in an euclidean space of dimension $2n-k+1$. This is a corollary of the facts that $F+H$ is at most $(n-k)$-dimensional and that $F+H$ is also the subbundle spanned by $V(X, Y)$ for each two vector fields X and Y of M.

2. In [2] the following theorem is proved: let M be an $(n+1)$-dimensional submanifold of E^m and N be an $(m-n-2)$-dimensional normal subbundle of $M^{\perp}$. If N is non-parallel and if M is umbilical with respect to N, then M is a locus of n-spheres, where an n-sphere means a hypersphere or a hyperplane of an euclidean $(n+1)$-space.

If M is in such case a locus of hyperplanes, then there are restrictions: suppose that $\xi_1, \cdots, \xi_{m-n-1}$ is an orthonormal base field of $M^{\perp}$ such that N is spanned by $\xi_1, \cdots, \xi_{m-n-2}$. If M is umbilical with respect to N (i.e. A_η is proportional to the unity transformation for each vector field η in N), then we get at once (see the form of the matrix of A_{ξ_j} in the equation (2.6)) $A_{\xi_1}=\cdots=A_{\xi_{m-n-2}}=0$

and so $H//N^{\perp}$. Consider the normal subbundle F. For each vector field ξ in F, we have $A_{\xi}\neq 0$ and because of this, it is necessary that $F\perp N$ or $F=N^{\perp}$. But if $F\neq 0$, we now find $H\in F$ and the Theorems 1 and 2 say that M is a hypersurface in some E^{n+2} which means that $N^{\perp}$ (and thus N) is parallel. So the only possibilities are: M is totally geodesic ($H=F=0$) or M is totally developable ($F=0$) and $H\neq 0$ in E^{m} with $m>n+2$.

3. Construction of minimal monosystems.

Suppose that $x^{1}, \cdots, x^{m}$ is the standard coordinate system of the euclidean space E^{m}. Consider the euclidean space E^{p} as the subspace of E^{m} determined by $x^{p+1}=\cdots=x^{m}=0$. Assume that M is an r-dimensional submanifold of E^{p}, which is locally given by the parametric representation

$$x^{i}=f^{i}(u_{1}, \cdots, u_{r}), \quad i=1, \cdots, p .$$

Construct the following $(m-p+r)$-dimensional submanifold M' of E^{m}

$$x^{i}=f^{i}(u_{1}, \cdots, u_{r}), \quad i=1, \cdots, p;\ x^{j}=l_{j};\ j=p+1, \cdots, m;\ l_{j}\in R .$$

Lemma 3. *If M is minimal in E^{p}, then M' is minimal in E^{m}.*

Proof. Suppose that $e_{1}, \cdots, e_{r}$ is an orthonormal base field of M and that $\bar{V}$ is the second fundamental tensor of M' in E^{m}. The normal vector field

$$H(M; M', E^{m})=\frac{1}{r}\sum_{i=1}^{r}\bar{V}(e_{i}, e_{i})$$

is called the relative mean curvature vector of M with respect to M' and E^{m}([1]). If H (resp. $\bar{H}$) is the mean curvature vector of M in E^{p} (resp. in M'), then we have

$$H=\bar{H}+H(M; M', E^{m}) .$$

If M is minimal in E^{p} (and thus in E^{m}), then $H=0$ and so $\bar{H}=0$ and $H(M; M', E^{m})=0$ at each point of M. Complete the orthonormal base field $e_{1}, \cdots, e_{r}$ of M to an orthonormal base field $e_{1}, \cdots, e_{r}, e_{r+1}, \cdots, e_{m-p+r}$ of M' (at the points of M). The mean curvature vector H' of M' in E^{m} is given by (at the points of M)

$$H'=\frac{1}{m-p+r}\left(\sum_{i=1}^{r}\bar{V}(e_{i}, e_{i})+\sum_{j=r+1}^{m-p+r}\bar{V}(e_{j}, e_{j})\right) .$$

Suppose that $\partial/\partial x^{1}, \cdots, \partial/\partial x^{m}$ is the standard coordinate base field of E^{m}. Because of the construction of M', it is clear that we may put (at the points of M) $e_{j}=\partial/\partial x^{p-r+j}$, $j=r+1, \cdots, m-p+r$ and it is trivial that each of these fields determine at each point of M an asymptotic direction of M'. So if M is minimal in E^p, then we have $H'=0$ at each point of M.

Next, consider any point $q(u_1^0, \cdots, u_r^0, l_{p+1}^0, \cdots, l_m^0)$ of M'. The submanifold $\bar{M}$ of the euclidean space $x^j=l_j^0$, $j=p+1, \cdots, m$, represented by

$$x^i=f^i(u_1, \cdots, u_r)\ , \quad i=1, \cdots, p;\ x^j=l_j^0;\ j=p+1, \cdots, m\ ,$$

is minimal, since M is minimal. But at the point q, the vectors $(\partial/\partial x^{p-r+j})_q$, $j=r+1, \cdots, m-p+r$ are again an orthonormal base field of the normal space $\bar{M}_q^{\perp}$ in M_q' and they determine asymptotic directions of M' at q. From all this, we see that $H_q'=0$ at each point q of M' and this completes the proof.

We now construct minimal monosystems. If the $(n+1)$-dimensional minimal monosystem M is k-developable, then we know that $M\subset E^{2n-k}$. So, consider E^{2n-k} with its standard coordinate system $x^1, \cdots, x^{2n-k}$.

a. $2n-k=3$. Then $n=1$ and $k=-1$, i.e. M is non-developable and in this case we know that the helicoid is the only minimal non-developable ruled surface in E^3.

b. $2n-k=4$. Then Corollary 4 says that we have only one (non-trivial) possibility: *a* minimal monosystem generated by planes. Because of Lemma 3, we have the following example (using the helicoid as minimal submanifold of E^3):

$$x^1=l_1 \cos s\ , \quad x^2=l_1 \sin s\ , \quad x^3=as\ , \quad x^4=l_2\ , \quad a=\text{constant}\neq 0,\ l_1, l_2 \in R\ .$$

This manifold is C^∞ and 0-developable (the only (common) singular point in each generating plane is infinite). Moreover each orthogonal trajectory of the generating planes is a circular helix (in some E^3) or a straight line.

c. $2n-k=5$. Then there are two possibilities: minimal monosystems generated by planes $(n=2, k=-1)$ or by three-dimensional spaces $(n=3, k=1)$. The first are non-developable (see later). Using Lemma 3, we find the following example for the second kind:

$$x^1=l_1 \cos s\ , \quad x^2=l_1 \sin s\ , \quad x^3=as\ , \quad x^4=l_2\ , \quad x^5=l_3\ , \quad l_1, l_2, l_3 \in R\ .$$

This manifold is C^∞ and 1-developable (the common line of singular points in each three-dimensional generating space is infinite). Here again (such as in all the following examples) the orthogonal trajectories are circular helices or straight lines. Next, consider the general case of non-developable minimal monosystems in E^{2n+1}. Put

$$x^{i-1}=l_{i/2} \cos s\ , \quad x^i=l_{i/2} \sin s\ , \quad x^{2n+1}=as\ , \quad l_{i/2} \in R\ , \quad i=2, 4, 6, \cdots, 2n\ .$$

It is at once clear that this is a C^∞ monosystem M which is non-developable:

call it a generalized helicoid. Put $l_1=\cdots=l_n=0$, then we find the straight line $x^j=0$, $j=1, \cdots, 2n$, $x^{2n+1}=as$ and this line is clearly an orthogonal trajectory of the generating spaces and a geodesic line of M, so it is also the line of striction of M ([5]). The orthogonal trajectory through the point $p(l_1^0, \cdots, l_n^0, s_0)$ is given by

$$x^{i-1}=l_{i/2}^0 \cos s\,, \qquad x^i=l_{i/2}^0 \sin s\,, \qquad x^{2n+1}=as\,, \quad i=2, 4, \cdots, 2n\,.$$

It is not difficult to see that this curve is a circular helix in some E^3 and this fact gives immediately that all the orthogonal trajectories are asymptotic curves of M, which means that M is minimal.

d. $2n-k=6$. There are two possibilities: $n=4$ and $k=2$ or $n=3$ and $k=0$. For the first kind, we use a helicoid in E^3:

$$\begin{aligned} &x^1=l_1 \cos s\,, \qquad x^2=l_1 \sin s\,, \qquad x^3=as\,, \qquad x^4=l_2\,, \\ &x^5=l_3\,, \qquad x^6=l_4\,, \qquad l_i \in R\,, \quad i=1, \cdots, 4\,. \end{aligned}$$

For the second type, we take a generalized helicoid in E^5 and use again Lemma 3:

$$\begin{aligned} &x^1=l_1 \cos s\,, \qquad x^2=l_1 \sin s\,, \qquad x^3=l_2 \cos s\,, \qquad x^4=l_2 \sin s\,, \\ &x^5=as\,, \qquad x^6=l_3\,, \qquad l_i \in R\,, \quad i=1, 2, 3\,. \end{aligned}$$

e. We are now able to give a general example, valid for all kinds of minimal monosystems in E^{2n-k}

$$\begin{aligned} &x^{i-1}=l_{i/2} \cos s\,, \qquad x^i=l_{i/2} \sin s\,, \quad i=2, 4, 6, \cdots, 2(n-k-1) \\ &x^{2(n-k)-1}=as\,, \quad a=\text{constant}\neq 0,\ s \in R \\ &x^{2(n-k)+j}=l_{n-k+j}\,, \quad j=0, 1, \cdots, k,\ l_r \in R,\ r=1, \cdots, n\,. \end{aligned}$$

We find here all the possible cases: e.g. suppose that $2n-k=23$, then we have only to put the following values in the general example: ($n=11$, $k=-1$), ($n=12$; $k=1$), ($n=13$; $k=3$), ($n=14$; $k=5$), ($n=15$; $k=7$), ($n=16$; $k=9$), ($n=17$; $k=11$), ($n=18$; $k=13$), ($n=19$; $k=15$), ($n=20$; $k=17$), ($n=21$; $k=19$) (remark that even the case ($n=22$; $k=21$) is contained in this general example: we obtain a totally geodesic manifold).

Bibliography

[1] B. Y. Chen: *Geometry of submanifolds.* Marcel Dekker, New York 1973.

[2] B. Y. Chen and K. Yano: *Submanifolds umbilical with respect to a non-parallel normal subbundle.* Kodai Math. Sem. Rep. **25** (1973).

[3] M. Spivak: *A comprehensive introduction to differential geometry.* Publish or perish inc., Boston 1970.

[4] T. Takahashi: *Minimal immersions of Riemannian manifolds.* J. Math. Soc. Japan, **18** (1966), 380–385.

[5] C. Thas: *Een (lokale) studie van de* $(m+1)$*-dimensionale variëteiten van de* n*-dimensionale euklidische ruimte* R^n $(n \geq 2m+1$ *en* $m \geq 1)$, *beschreven door een eendimensionale familie van* m*-dimensionale lineaire ruimten*. Med. Kon. Acad. Wet. Lett. Sch. K. van België, Jaargang XXXVI, 1974, nr. 4.

Seminar of Higher Geometry
University of Ghent
Krijgslaan 271
B-9000 GENT
Belgium