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CLOSED 4-MANIFOLDS COVERED BY THREE 4-BALLS

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1. Introduction.

K. Kobayashi and Y. Tsukui introduced the concept of the ball coverings of manifolds in [1]. For a manifold W, the minimum number of balls of the ball coverings of W is called the *covering number* of W and is denoted by $b(W)$.

In [1], they obtained the following result.

Theorem. *For a closed n-manifold W, we have* $2\leqq b(W)\leqq n+1$.

Clearly, a closed n-manifold W is an n-sphere if and only if $b(W)=2$. It is difficult, however, to determine the n-manifold W with $b(W)=2$ when the boundary of W is non-empty and $n\geqq 4$. The answer to this problem implies the classification of closed n-manifolds W with $b(W)=3$. Recently, Y. Tsukui obtained a complete answer to the problem under the situation $H_2(W)=0$ and $n=4$, in [2].

In the present paper, we are interested in the case $H_2(W)=Z$ (the additive group of integers) and $n=4$.

2. Preliminaries.

For a manifold W, we denote the boundary of W by $\dot{W}$ and the interior of W by $\mathring{W}$. For topological spaces X and Y, $X+Y$ means the disjoint union of X and Y, and $X\vee Y$ means a one point union of X and Y in the usual sense. For groups G and H, we denote the direct sum of G and H by $G+H$. For a group G and an integer p, pG means the direct sum $G+\cdots+G$ (p times).

Definition 1. The class $C(p, q)$ consists of connected compact bounded PL 4-manifolds W satisfying the following conditions;

(C.1) $b(W)=2$,

(C.2) $H_1(W)=pZ$,

(C.3) $H_2(W)=qZ$.

$\bar{C}(p, q)$ denotes the subclass of $C(p, q)$ defined by the condition;

(C.4) $\dot{W}=S^3$ (the 3-sphere).

For any $W\in C(p, q)$, there exist two 4-balls A and B such that $W=A\cup B$ and

$A\cap B$ is a compact 3-manifold. Usually, by F, we denote $A\cap B$ and call it the *attaching face* (of A and B). $W_F=(A, B; F)$ is called a *realization* of W.

Lemma 2.1. *For any realization W_F of W in $C(p, q)$, $H_{k-1}(F)=H_k(W)$ and $H_1(\dot{F})=2qZ$.*

Proof. Suppose $W_F=(A, B; F)$ is an arbitrary realization of W. The first half is an immediate consequence of the Mayer-Vietoris sequence since $W=A\cup B$, $F=A\cap B$ and A and B are both 4-balls. The latter half is shown as follows. First, we call the reader's attention to F being a subset of 3-sphere (for instance, $\dot{A}$) S^3. Put $E=\bar{S}^3-F$. Then, we conclude $H_1(\dot{F})=H_1(E)+H_1(F)$ using the Mayer-Vietoris sequence. On the other hand, applying the Alexander duality, we get $H_1(E)=H_1(S^3-F)=H^1(F)$. Since $H_1(F)=H_2(W)=qZ$, $H^1(F)=\mathrm{Hom}\,(H_1(F), Z)=\mathrm{Hom}\,(qZ, Z)=qZ$. Hence, we get $H_1(\dot{F})=2qZ$.

Remark 2.2. Suppose W_F is a realization of $W\in C(p, q)$. Then, the attaching face F is a subset of S^3 and is a disjoint union

$$F=F_0+F_1+\cdots+F_p$$

of connected compact 3-manifolds F_i $(i=0, 1, \cdots, p)$ with non-empty boundary $\dot{F}_i$, since $H_0(F)=H_1(W)=pZ$.

Let us consider a realization W_F of $W\in C(p, 1)$. The connected components of $\dot{F}$ are all 2-spheres but exactly one torus $S^1\times S^1$, since $H_1(\dot{F})=Z+Z$ and $\dot{F}$ is a disjoint union of closed surfaces. Without loss of generality, we assume $S^1\times S^1\subset\dot{F}_0$. Then, $\dot{F}$ is completely described as follows.

$$\dot{F}=\dot{F}_0+\dot{F}_1+\cdots+\dot{F}_p\,,$$
$$\dot{F}_0=S^1\times S^1+S^2_{01}+\cdots+S^2_{0r_0}\,,$$
$$\dot{F}_i=S^2_{i1}+\cdots+S^2_{ir_i}\,,\quad i=1, \cdots, p\,,$$

where S^2_{ij} is a 2-sphere and $r_0\geqq 0$, $r_i\geqq 1$ $(i\neq 0)$. Since each F_i $(i\neq 0)$ is a 3-dimensional connected submanifold of a 3-sphere and

$$\dot{F}_i=S^2_{i1}+\cdots+S^2_{ir_i}\,,$$

F_i is constructed by removing (r_i-1) small 3-balls from the interior of a large 3-ball. For details, it is shown by the induction on the number r_i using (3, 2)-Schoenflies theorem.

3. The class $\bar{C}(p, 1)$.

By $\hat{W}$, we denote the closed 4-manifold obtained by attaching a 4-ball C to an element W of $\bar{C}(p, q)$ at the boundary naturally. That is to say,

$$\hat{W}=W \cup C \quad \text{and} \quad W \cap C=\dot{W}=\dot{C}\,.$$

Obviously, $\hat{W}$ is uniquely determined by W in the sense of PL homeomorphism. We say $\hat{W}$ the *completion* of W.

Suppose $W_F=(A, B; F)$ is an arbitrary realization of $W \in \bar{C}(p, q)$. Then, $\hat{W}$ can be expressed as $\hat{W}=A \cup B \cup C$. Now, put $W'=A \cup C$ and $W''=B \cup C$. Then, each of W, W' and W'' is obtained from W by removing a 4-ball. Thus, W, W' and W'' are PL homeomorphic to each other by the homogeneity of manifold. Put $F'=A \cap C$ and $F''=B \cap C$. Then, $W_{F'}=(A, C; F')$ and $W_{F''}=(B, C; F'')$ are regarded as other realizations of W and are naturally determined by W_F.

Let us consider the expression $W_F=(A, B; F)$ a realization of W. Let A and B be 4-balls and F a compact 3-manifold. Let f_A: $F \to \dot{A}$ and f_B: $F \to \dot{B}$ be PL embeddings. Put $F_A=f_A(F)$ and $F_B=f_B(F)$. Then, $f=f_B \circ f_A^{-1}$; $F_A \to F_B$ is a PL homeomorphism and $W=A \cup_f B$ is a connected compact PL 4-manifold with $b(W)=2$. Conversely, any connected compact 4-manifold W with $b(W)=2$ can be obtained by the construction above. For this reason, we adopt the notation $((A, F_A), (B, F_B); F)_f$ (or shortened one $(A, B; F)_f$, if there is no confusion) for a realization W_F of W. In this sense, the completion $\hat{W}=A \cup B \cup C$ of $W \in \bar{C}(p, q)$ determines the following three realizations.

$$W_{{}_cF}=(A, B; {}_cF)_{f_c}=((A, {}_cF_A), (B, {}_cF_B); {}_cF)_{f_c}=W_F\,,$$
$$W_{{}_aF}=(B, C; {}_aF)_{f_a}=((B, {}_aF_B), (C, {}_aF_C); {}_aF)_{f_a}=W_{F'}\,,$$
$$W_{{}_bF}=(C, A; {}_bF)_{f_b}=((C, {}_bF_C), (A, {}_bF_A); {}_bF)_{f_b}=W_{F''}\,.$$

Note that, in W, there are equalities

$${}_aF_B={}_aF_C=\dot{B} \cap \dot{C}\,, \quad {}_bF_C={}_bF_A=\dot{C} \cap \dot{A} \quad \text{and} \quad {}_cF_A={}_cF_B=\dot{A} \cap \dot{B}\,.$$

Lemma 3.1. *In* W, ${}_a\dot{F}_B={}_b\dot{F}_C={}_c\dot{F}_A=\dot{A} \cap \dot{B} \cap \dot{C}$.

Proof. Since other cases hold similarly, we show ${}_c\dot{F}_A=\dot{A} \cap \dot{B} \cap \dot{C}$, as a typical case. From the construction, it is obvious that

$${}_cF_A={}_cF_B=\dot{A} \cap \dot{B}\,, \quad {}_c\dot{F}_A \subset \dot{W}=\dot{C} \quad \text{and} \quad {}_c\mathring{F}_A \subset \mathring{W}$$

where $W=A \cup B$. Thus, ${}_c\dot{F}_A \subset \dot{A} \cap \dot{B} \cap \dot{C}$. Conversely, take a point $x \in \dot{A} \cap \dot{B} \cap \dot{C}$ and assume $x \notin {}_c\dot{F}_A$. Since x belongs to $\dot{A} \cap \dot{B}$, $x \in {}_c\mathring{F}_A \subset \mathring{W}$. This implies $x \notin \dot{C}$, because $\mathring{W} \cap C$ is empty. This is a contradiction. This completes the proof.

Let us consider the 3-sphere $\dot{A}$. Note that ${}_bF_A$ and ${}_cF_A$ are submanifolds of $\dot{A}$ satisfying

$${}_bF_A \cup {}_cF_A = \dot{A} \qquad \text{and} \qquad {}_bF_A \cap {}_cF_A = {}_b\dot{F}_A \cap {}_c\dot{F}_A = {}_b\dot{F}_A = {}_c\dot{F}_A\ .$$

We say that the pair $({}_bF_A, {}_cF_A)$ is the *splitting* of $\dot{A}$ determined by the completion $\hat{W} = A \cup B \cup C$.

Hereafter, we deal with the manifold $W \notin \bar{C}(p, 1)$. For the simplicity, we confuse ${}_bF_A$ and ${}_bF$, and write $F = {}_bF_A = {}_bF$ and $F' = {}_cF_A = {}_cF$. Therefore, (F, F') means the splitting $({}_bF_A, {}_cF_A)$.

Lemma 3.2. *For the splitting* (F, F') *of* $\dot{A}$, $F_0 \cap F'_0 = \dot{F}_0 \cap \dot{F}'_0 = S^1 \times S^1$.

Proof. It is trivial that $F_0 \cap F'_0 = \dot{F}_0 \cap \dot{F}'_0$. We claim that $\dot{F}_0$ and $\dot{F}'_0$ have $S^1 \times S^1$ in common. Recall that each of $\dot{F}$ and $\dot{F}'$ has unique torus component. Let T and T' denote the torus components of $\dot{F}$ and $\dot{F}'$, respectively. Then T and T' belong to $\dot{F}_0$ and $\dot{F}'_0$, respectively. Now, we have $T = T'$ because $\dot{F} = \dot{F}'$. This means $T = S^1 \times S^1 \subset \dot{F}_0 \cap \dot{F}'_0$. The torus T divides $\dot{A}$ into two connected components $\mathring{X}$ and $\mathring{Y}$ such that $X \cup Y = \dot{A}$ and $X \cap Y = T$. Without loss of generality, we assume $F_0 \subset X$ and $F'_0 \subset Y$ since F_0 and F'_0 are connected and $\mathring{F}_0 \cap \mathring{F}'_0 = \emptyset$. Therefore, $\dot{F}_0 \cap \dot{F}'_0 = F_0 \cap F'_0 \subset X \cap Y = T$, completing the proof.

For the splitting (F, F'), F_0 has r_0 2-spheres S^2_{0j} $(j = 1, \cdots, r_0)$ as its boundary components. We will cap off these r_0 boundary components by 3-balls D^3_j. Since F_0 is connected and is contained in the 3-sphere $\dot{A}$, we can take the 3-balls D_j in $\dot{A} - \mathring{F}_0$ by the aid of (3, 2)-Schönflies theorem. We denote the resulting 3-manifold in $\dot{A}$ by $\hat{F}_0$. That is to say,

$$\hat{F}_0 = F_0 \cup D^3_1 \cup \cdots \cup D^3_{r_0} \subset \dot{A}\ ,$$
$$F_0 \cap D^3_j = S^2_{0j}\ ,$$
$$D^3_j \cap D^3_k = \emptyset \quad (j \neq k)\ .$$

Similarly, we construct $\hat{F}'_0$ from F'_0 in the same 3-sphere $\dot{A}$. We call the pair $(\hat{F}_0, \hat{F}'_0)$ the *capping* of the splitting (F_0, F'_0).

Since $\hat{F}_0$ is a submanifold of the 3-sphere $\dot{A}$ and the boundary component of $\hat{F}_0$ is just a torus, $\hat{F}_0$ should be the exterior of some knot (may be trivial) in $\dot{A}$. Similarly, $\hat{F}'_0$ is also the exterior of some (other) knot it $\dot{A}$. Each of $\hat{F}_0$ and $\hat{F}'_0$ has a common torus $T = S^1 \times S^1 \subset \dot{A}$ as the boundary by Lemma 3.2. Since T divides the 3-sphere $\dot{A}$ into two components and $\hat{F}_0 \neq \hat{F}'_0$, we obtain the following lemma.

Lemma 3.3. $\hat{F}_0 \cup \hat{F}'_0 = \dot{A}$ *and* $\hat{F}_0 \cap \hat{F}'_0 = T$.

Corollary 3.4. *One of* $\hat{F}_0$ *and* $\hat{F}'_0$ *is homeomorphic to the solid torus* $S^1 \times D^2$.

Theorem 3.5. *For a completion* $W=A\cup B\cup C$ *of* $W\in\bar{C}(p,1)$, *one of three 3-spheres* $\dot{A}$, $\dot{B}$ *and* $\dot{C}$ *has a splitting* (F,F') *such that both* $\hat{F}_0$ *and* $\hat{F}'_0$ *are homeomorphic to the solid torus* $S^1\times D^2$.

Proof. We use the full notation of the splitting within this proof. First, we claim that ${}_a\hat{F}_{B0}$, ${}_b\hat{F}_{C0}$ and ${}_c\hat{F}_{A0}$ are homeomorphic to ${}_a\hat{F}_{C0}$, ${}_b\hat{F}_{A0}$ and ${}_c\hat{F}_{B0}$, respectively, but ${}_a\hat{F}_{B0}\neq{}_a\hat{F}_{C0}$, ${}_b\hat{F}_{C0}\neq{}_b\hat{F}_{A0}$ and ${}_c\hat{F}_{A0}\neq{}_c\hat{F}_{B0}$ as subsets in W. Let us consider the splitting $({}_bF_A, {}_cF_A)$ of $\dot{A}$. By Lemma 3.4, one of ${}_b\hat{F}_{A0}$ and ${}_c\hat{F}_{A0}$, say ${}_b\hat{F}_{A0}$, is homeomorphic to the solid torus. If ${}_c\hat{F}_{A0}$ is also homeomorphic to the solid torus, we choose $({}_bF_A, {}_cF_A)$ as a required splitting. We consider the case that ${}_c\hat{F}_{A0}$ is not homeomorphic to the solid torus, and we consider the splitting $({}_aF_C, {}_bF_C)$ of $\dot{C}$. Since ${}_b\hat{F}_{A0}$ and ${}_b\hat{F}_{C0}$ are homeomorphic and ${}_b\hat{F}_{A0}$ is the solid torus, ${}_b\hat{F}_{C0}$ is the solid torus. Now, it is sufficient to show that ${}_a\hat{F}_{C0}$ is homeomorphic to the solid torus. Suppose not. Let us consider the splitting $({}_cF_B, {}_aF_B)$ of $\dot{B}$. Since ${}_c\hat{F}_{B0}$ and ${}_a\hat{F}_{B0}$ are homeomorphic to ${}_c\hat{F}_{A0}$ and ${}_a\hat{F}_{C0}$, both ${}_c\hat{F}_{B0}$ and ${}_a\hat{F}_{B0}$ are not homeomorphic to the solid torus. This is a contradiction to Lemma 3.4. This completes the proof.

Remark 3.6. For the splitting obtained in Theorem 3.5, both $\hat{F}_0$ and $\hat{F}'_0$ are trivial solid tori in the 3-sphere.

4. Realizations.

In this section, we give some properties with respect to the form of the attaching face. The typical result is Theorem 4.7 which mentions that a manifold $W\in\bar{C}(p,1)$ has a realization $W_F=(A,B;F)$ such that F has a 3-ball as a connected component.

Without loss of generality, we may assume that the splitting (F,F') of $\dot{A}$ satisfies the condition of Theorem 3.5, namely, both F_0 and F'_0 are homeomorphic to the solid torus $S^1\times D^2$. Remember that F is the attaching face of a realization $W_F=(A,B;F)$ of $W\in\bar{C}(p,1)$ and that $\dot{F}$ is characterized by

$$\dot{F}=\dot{F}_0+\dot{F}_1+\cdots+\dot{F}_p\,,$$
$$\dot{F}_0=S^1\times S^1+(S^2_{01}+S^2_{02}+\cdots+S^2_{0r_0})\,,$$
$$\dot{F}_i=S^2_{i1}+S^2_{i2}+\cdots+S^2_{ir_i}\,,\quad i=1,2,\cdots,p\,,$$

where S^2_{ij} is a 2-sphere and $r_0\geqq 0$, $r_i\geqq 1$ $(i\neq 0)$.

Proposition 4.1. *For any realization* $W_F=(A,B;F)$ *of* $W\in\bar{C}(p,1)$, *the inequality* $r_0\leqq p$ *holds.*

Proof. Let (F, F') be the splitting of $\dot{A}$. If $p=0$, we have $F=F_0$ and $F'=F'_0$. This implies $F_0 \cup F'_0 = F \cup F' = \dot{A}$, and hence $\hat{F}_0 = F_0$ and $\hat{F}'_0 = F'_0$. Thus $\dot{F}_0$ should be just $S^1 \times S^1$. That is, $r_0 = 0 = p$. Now, we assume $p \geqq 1$. Let us consider the characterization of $\dot{F}_0$ above and

$$F_0 = F_0 + (D_1^3 + D_2^3 + \cdots + D_{r_0}^3) .$$

Where D_j^3 is a 3-ball and $F_0 \cap D_j^3 = \dot{F}_0 \cap \dot{D}_j^3 = S_{0j}^2$. Since

$$F \cup F' = (F_0 + (F_1 + \cdots + F_p))(F'_0 + (F'_1 + \cdots + F'_p)) = \dot{A} ,$$

and S_{0j}^2 belongs to $\dot{F}_0$, exactly one of $F'_0, \cdots, F'_p$ has S_{0j}^2 as a boundary component. Because $F'_0 \cap S_{0j}^2 = \varnothing$ follows from $F_0 \cap F'_0 = \dot{F}_0 \cap \dot{F}'_0 = S^1 \times S^1$ by Lemma 3.2, F'_0 can not contain S_{0j}^2 as a boundary component. Hence one of $F'_1, \cdots, F'_p$, say $F'_{i(j)}$, has S_{0j}^2 as its boundary component. Since $F'_{i(j)}$ is connected, $F'_{i(j)}$ should be contained in D_j^3. $F'_{i(j)}$ does not contain other 2-spheres S_{0k}^2 because $D_j^3 \cap D_k^3$ is empty if $j \neq k$. Therefore, r_0 can not exceed p.

A splitting (F, F') of $\dot{A}$ induces two realizations W_F and $W_{F'}$. In the following, we choose a good realization from W_F and $W_{F'}$.

Proposition 4.2. *For $p \geqq 1$, there exists a realization W_F of $W \in \bar{C}(p, 1)$ such that $r_0 \geqq 1$ and $\hat{F}_0$ is a solid torus $S^1 \times D^2$.*

Proof. For a completion $\hat{W} = A \cup B \cup C$, we can take a splitting (F, F') of $\dot{A}$ such that both $\hat{F}_0$ and $\hat{F}'_0$ are homeomorphic to the solid torus by Theorem 3.5. Let us consider the characterizations

$$\dot{F}_0 = S^1 \times S^1 + (S_{01}^2 + \cdots + S_{0r_0}^2) ,$$
$$\dot{F}'_0 = S^1 \times S^1 + (S_{01}'^2 + \cdots + S_{0r'_0}'^2) .$$

If we assume $r_0 = 0 = r'_0$, we have $\dot{F}_0 = \dot{F}'_0 = S^1 \times S^1$. Hence $\hat{F}_0 = F_0$ and $\hat{F}'_0 = F'_0$. Then $\hat{F}_0 \cup \hat{F}'_0 = F_0 \cup F'_0 = \dot{A}$. This implies $F = F_0$ and $F' = F'_0$. Thus, we have $p=0$. This contradicts to our hypothesis $p \geqq 1$. Therefore, either $r_0 \geqq 1$ or $r'_0 \geqq 1$ holds. If $r_0 \geqq 1$, we take W_F as a required realization of W. If $r_0 = 0$, $W_{F'}$ is a required one.

Proposition 4.3. *For the realization W_F of Theorem 4.2, we obtain*

$$F_0 \simeq S^1 \vee (S_1^2 \vee \cdots \vee S_{r_0}^2) \quad \textit{and}$$
$$F_i \simeq \begin{cases} \textit{one point}, & \textit{if } r_i = 1 , \\ S_1^2 \vee \cdots \vee S_{r_i - 1}^2 , & \textit{if } r_i \geqq 2 , \end{cases}$$

where $\simeq$ means "homotopically equivalent to".

Proof. F_0 is obtained from a solid torus by removing r_0 small disjoint 3-balls in its interior. Thus, the first half part of the proposition is obtained by collapsing F_0 naturally. By the same way, the latter half is an immediate consequence of the characterization of F_i in section 2.

Corollary 4.4. *For the realization W_F of Theorem 4.2, we obtain*

$$H_2(F_0)=r_0Z\,, \qquad \text{and}$$
$$H_2(F_i)=(r_i-1)Z\,, \quad i=1,\cdots,p\,.$$

Proposition 4.5. *For a manifold W of $\bar{C}(p, q)$, we obtain $H_3(W; Z_2)=pZ_2$.*

Proof. Using the Poincaré duality of Z_2-coefficient, we have

$$H_3(W; Z_2)=H^1(W, \dot{W}; Z_2)=H^1(W, S^3; Z_2)=\mathrm{Hom}\,(H_1(W, S^3), Z_2)\,.$$

On the other hand, the exact sequence of the pair (W, S^3) shows $H_1(W, S^3)=pZ$. Thus, $H_3(W; Z_2)=\mathrm{Hom}\,(pZ, Z_2)=pZ_2$.

Corollary 4.6. *For the realization W_F of Theorem 4.2, we obtain*

$$r_0+r_1+\cdots+r_p=2p\,.$$

Proof. From Corollary 4.4, we have

$$\begin{aligned}H_2(F)&=H_2(F_0)+H_2(F_1)+\cdots+H_2(F_p)\\&=r_0Z+\sum(r_i-1)Z\\&=((r_0+r_1+\cdots+r_p)-p)Z\,.\end{aligned}$$

Hence $H_2(F; Z_2)=((r_0+r_1+\cdots+r_p)-p)Z_2$. On the othe hand, by Proposition 4.5, $H_2(F; Z_2)=H_3(W; Z_2)=pZ_2$. Thus, we obtain $r_0+\cdots+r_p=2p$.

Theorem 4.7. *For the realization W_F of Theorem 4.2, one of $F_1, \cdots, F_p$, say F_p, is a 3-ball.*

Proof. Suppose any of r_i, $i=1,\cdots,p$, is greater than 1. Since $r_0\geqq 1$, we have $r_0+\cdots+r_p\geqq 2p+1$. This contradicts to Corollary 4.6. Therefore, at least one of r_i, $i=1,\cdots,p$, say r_p, is exactly 1. Then the boundary of F_p consists of exactly one 2-sphere. Since F_p is a submanifold of a 3-sphere, F_p should be a 3-ball.

5. Main results.

In this section, our main aim is to construct a correspondence between the sets $\bar{C}(p-1, 1)$ and $\bar{C}(p, 1)$ for $q=1$ by surgery.

Let $W_F=(A, B; F)$ be a realization of $W\in\bar{C}(p, 1)$ such that F_p is a 3-ball. Put $X=\overline{W-N(F_p, W)}$, where $N(F_p, W)$ means a regular neighborhood of F_p in W

meeting the boundary regularly. Put $G=F_0+\cdots+F_{p-1}$, $A^*=\overline{A-N(F_p, A)}$ and $B^*=\overline{B-N(F_p, B)}$. Then, clearly, $X_G=(A^*, B^*; G)$ is a realization of X. Obviously, the following assertion holds.

Assertion 5.1. *X belongs to $C(p-1, 1)$.*

Note that X does not belong to $\bar{C}(p-1, 1)$.

Assertion 5.2. *$\dot{X}=S_1^3+S_2^3$, where S_i^3 $(i=1, 2)$ is a 3-sphere.*

Proof. From the construction of X, it is seen that W is obtained from X by attaching a 1-handle $D^1\times D^3$ $(=N(F_p, W))$ to X. That is,

$$W=X\cup D^1\times D^3\ ,$$
$$X\cap D^1\times D^3=\dot{X}\cap(D^1\times D^3)^\bullet=S^0\times D^3=\{-1\}\times F_p+\{1\}\times F_p\ .$$

Hence $\dot{X}=(\dot{W}-(-1, 1)\times\dot{F}_p)\cup\{-1\}\times F_p\cup\{1\}\times F_p$. Since $\dot{F}_p$ is a 2-sphere contained in the 3-sphere $\dot{W}$ and F_p is a 3-ball, it follows that $\dot{X}$ is the disjoint union of two 3-spheres S_1^3 and S_2^3.

In the following, we construct an element V of $\bar{C}(p-1, 1)$ from X by attaching a 4-ball D^4. We define $V=X\cup D^4$ and $X\cap D^4=S_2^3=\dot{D}^4$. Then, it is easy to see that $\dot{V}$ is a 3-sphere S_1^3 and $H_k(V)=H_k(X)$ for $k=1, 2$. Since the conditions on the boundary and homology groups are satisfied, it remains to check $b(V)=2$ in order to $V\in\bar{C}(p-1, 1)$.

Assertion 5.3. $b(V)=2$.

Proof. From the definition of V, it is obvious that $2\leqq b(V)\leqq 3$ because $V=A^*\cup B^*\cup D^4$. Let us consider $\hat{V}=V\cup E^4$ where E^4 is a 4-ball satisfying $V\cap E^4=\dot{V}=\dot{E}^4$. Then, $\hat{V}$ is a closed 4-manifold with the ball covering $\{A^*, B^*, D^4, E^4\}$. By [1], it follows that $b(V)=3$, because $D^4\cap E^4=\emptyset$. Hence, by the homogeneity of manifold, we have $b(V)=2$.

Therefore, we proved the following theorem.

Theorem 5.4. *Any manifold W of $\bar{C}(p, 1)$ is constructed from some V of $\bar{C}(p-1, 1)$ by the following way. First, remove the interior of a 4-ball D in the interior of V. Then, W is obtained by attaching a 1-handle $D^1\times D^3$ to $V-\mathring{D}$ putting one of the components of $\dot{D}^1\times D^3$ on $\dot{V}$ and the other on $\dot{D}$.*

Theorem 5.5. *Any manifold W of $\bar{C}(p, 1)$ has a spine homeomorphic to $S^2\vee(S_1^1\vee\cdots\vee S_p^1)\vee(S_1^3\vee\cdots\vee S_p^3)$ for $p\geqq 0$.*

Proof. First we deal with the case $p=0$. Let W be an element of $\bar{C}(0, 1)$. Then, W has a realization $(A, B; F)$ such that $W=A\cup B$ and $F=A\cap B$ is a solid torus $S^1 \times D^2$. Take two interior points a and b in A and B, respectively. Then, W collapses to $a*F \cup b*F$, where the symbol $*$ means the join. Since F collapses to the center line C, $a*F \cup b*F$ collapses to $a*C \cup b*C$. It is clear that $a*C \cup b*C$ is a 2-sphere S^2, because C is a circle.

Now, we prove the case $p=1$. Let $W \in \bar{C}(1, 1)$ and V an element of $\bar{C}(0, 1)$ corresponding to W obtained in Theorem 5.4. Then, we can write

$$W = \{V \# (S^3 \times I)\} \cup (D^1 \times D^3)$$

where I denotes the closed unit interval $[0, 1]$ and $\#$ means the connected sum. We can assume $V \cap D^1 \times D^3 = \emptyset$. Put the 3-ball $V \cap S^3 \times I = E$. Take a realization $(A, B; F)$ of V such that F is a solid torus. We can assume that $\dot{F} \cap E$ is a 2-ball X properly embedded in E, that is, $X \cap \dot{E} = \dot{X}$. Let J_1 denote a straight line in F joining the center of X and a point of the center line C of F, that is, $F \searrow C \cup J_1 \cup X$. Since V collapses to S^2 as shown above, we obtain

$$V \searrow S^2 \cup J_1 \cup X .$$

On the other hand, it is easy to see

$$S^3 \times I \cup D^1 \times D^3 \searrow S_1^1 \vee S_1^3 \cup J_2 \cup X$$

where J_2 means a straight line in $S^3 \times I$ joining the center of X and a point of $S_1^3 = S^3 \times 1/2$. Thus we have

$$W \searrow S^2 \cup J_1 \cup J_2 \cup X \cup S_1^1 \vee S_1^3 .$$

Since X is a 2-ball, $W \searrow S^2 \cup J \cup S_1^1 \vee S_1^3$ where $J = J_1 \cup J_2$. It is not hard to deform $S^2 \cup J \cup S_1^1 \vee S_1^3$ to $S^2 \vee S_1^1 \vee S_1^3$ in W, since J is a 1-ball. Therefore, $W \searrow S^2 \vee S_1^1 \vee S_1^3$.

The case $p \geqq 2$ can be proved similarly.

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