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# A WEAK CONVERGENCE THEOREM FOR FUNCTIONALS OF SUMS OF MARTINGALE DIFFERENCES

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**1. Introduction.** Let $\{\xi_i\}$ be a sequence of random variables defined on a probability space. Let $\mathcal{M}_a^b$ denote the $\sigma$-algebra of events generated by $\xi_a, \cdots, \xi_b$. We call $\{\xi_i\}$ a sequence of martingale differences (*md*) if $E|\xi_i|<\infty$ and $E\{\xi_i|\mathcal{M}_{-\infty}^{i-1}\}=0$ for each $i$.

Let $D=D\,[0,1]$ be the space of functions on $[0,1]$ that are right-continuous and have left-hand limits. We assume that $D$ is endowed with Skorokhod $J_1$-topology. (cf. Billingsley [1]).

Set

$$S_k=\sum_{j=1}^{\infty}\xi_j \qquad \text{and} \qquad S_0=0\,.$$

Let

$$w=\{w(t):\ 0\leqq t\leqq 1\}$$

be a standard Wiener process.

A sufficient condition for the validity of a weak convergence theorem of

$$\sum_{i=1}^{n-1} f_n\left(\frac{S_i}{\sqrt{n}}\right)\frac{\xi_{i+1}}{\sqrt{n}}$$

is known when $\{\xi_i\}$ is a sequence of identically and independently distributed random variables. (Skorokhod and Slobodeneuk [4]).

In this paper, we shall consider a sufficient condition to ensure

$$\sum_{i=1}^{n-1} f_n\left(\frac{i}{n}\,,\ \frac{S_i}{\sqrt{n}}\right)\frac{\xi_{i+1}}{\sqrt{n}} \xrightarrow{D} \int_0^1 f(t,\,w(t))dw(t)$$

where $\{\xi_i\}$ is a sequence of *md*.

**2. Conditions and the main result.** In this and the following sections, we shall denote by the letter $K$, various absolute positive constants.

Let $F_M$ be the space of functions defined on $[0,1]\times R^1$ satisfying the following condition; there exists an absolute constant $M$ such that for $f\in F_M$, $f$ and its derivatives satisfy inequalities of the form

$$(2.1)\qquad |Df(s, x)| \leqq M(1+|x|^{\alpha})$$

where $D$ denotes either the identity operator or a first derivative and $\alpha$ is some positive constant.

**Remark 2.1.** If $f \in F_M$, then it is obvious that

$$(2.2)\qquad |f(s, x)-f(s', x')| \leqq M\{|s-s'|+|x-x'|\}\{1+\max(|x|^{\alpha}, |x'|^{\alpha})\}$$

Further, if $f \in F_M$ and $f(t, x)=0$ $(|x|>C)$ for each $t \in [0, 1]$ and for some $C$, then

$$(2.3)\qquad |f(s, x)-f(s', x')| \leqq M(1+C^{\alpha})\{|s-s'|+|x-x'|\} .$$

We shall consider the following conditions:

Condition (A). $\{\xi_i\}$ is a strictly stationary, ergodic sequence of *md* with $E\xi_1^2=1$.

Condition (B). $\{\xi_i\}$ is a sequence of *md* for which

(B1) $E\xi_i^2=1$ $(i=1, 2, \cdots)$,

(B2) $E|\xi_i|^{2+\delta} \leqq K$ $(i=1, 2, \cdots)$ for some $\delta>0$, and

(B3) $E|n^{-1}\sum_{i=a+1}^{a+n} E\{\xi_i^2|\mathscr{M}_{-\infty}^{a}\}-1|=B(n)\downarrow 0$ $(n\to\infty)$.

**Remark 2.2.** It is known that under Condition (A) or (B)

$$(2.4)\qquad \lim_{n\to\infty} P(n^{-1/2}S_n \leqq x)=\Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy$$

for all $x$. Furthermore, all finite dimensional distributions of the form $(n^{-1/2}S_{[nt_1]}, \cdots, n^{-1/2}S_{[nt_b]})$ converges weakly as $n\to\infty$, to those of a standard Wiener process $w=\{w(t)\colon 0\leqq t\leqq 1\}$ where $[s]$ denotes the largest integer $p$ such that $p\leqq s$. (cf. Brown [2] and Serfling [4]).

**Theorem.** *Let $\{\xi_i\}$ be a sequence of random variables satisfying Condition* (A) *or* (B). *Let $f_n \in F_M$ $(n=1, 2, \cdots)$ and $f \in F_M$. Assume that for each $s \in [0, 1]$*

$$(2.5)\qquad Df_n(s, x)\to Df(s, x)\quad (n\to\infty)$$

*uniformly in $x$ on every finite interval. Then*

$$(2.6)\qquad \sum_{i=1}^{n-1} f_n\left(\frac{i}{n}, \frac{S_i}{\sqrt{n}}\right)\frac{\xi_{i+1}}{\sqrt{n}} \xrightarrow{D} \int_0^1 f(t, w(t))dw(t) .$$

*Here, the stochastic integral in* (2.6) *is taken in the $L^2$ sense.*

**3. Some lemmas.** In this section, we shall prove some lemmas.

**Lemma 3.1.** *Let $f_n \in F_M$ $(n=1, 2, \cdots)$. Let $u_n$ $(n=1, 2, \cdots)$ be functions such* *that for some* $C(C>0)$

$$u_n(s, x)=f_n^{(C)}(s, x)=\begin{cases} f_n(s, x) & \text{if } (s, x)\in[0, 1]\times[-C, C] \\ 0, & \text{otherwise,}\end{cases} \quad (n=1, 2, \cdots). \tag{3.1}$$

*Let* $\{\xi_i\}$ *be a sequence of md satisfying Condition* (A) *or* (B). *Let* $\{t_0, t_1, \cdots, t_n\}$ *be any collection of nonnegative numbers such that* $0=t_0<t_1<\cdots<t_b=1$. *For any* $\varepsilon>0$, *put*

$$P_1(\varepsilon, \gamma, n)=P\left(\left|\sum_{i=1}^{n-1} u_n\left(\frac{i}{n}, T_i\right)\frac{\xi_{i+1}}{\sqrt{n}}-\sum_{j=1}^{b-1} u_n(t_j, T_{[nt_j]})\cdot(T_{[nt_{j+1}]}-T_{[nt_j]})\right|>\varepsilon\right) \tag{3.2}$$

*where* $\gamma=\max_{0\leq j\leq b-1}(t_{j+1}-t_j)$ *and* $T_i=n^{-1/2}S_i$. *Then*

$$\lim_{\gamma\downarrow 0}\lim_{n\to\infty} P_1(\varepsilon, \gamma, n)=0. \tag{3.3}$$

**Proof.** For $N>0$, define

$$\bar{\xi}_i=\begin{cases}\xi_i & \text{if } |\xi_i|\leq N, \\ 0 & \text{if } |\xi_i|>N,\end{cases}$$

and put

$$\eta_i=n^{-1/2}[\bar{\xi}_i-E\{\bar{\xi}_i|\mathscr{M}_{-\infty}^{i-1}\}]$$

and

$$\zeta_i=n^{-1/2}\xi_i-\eta_i=n^{-1/2}[\xi_i-\bar{\xi}_i-E\{\xi_i-\bar{\xi}_i|\mathscr{M}_{-\infty}^{i-1}\}].$$

For brevity, we write $\Sigma_{(j)}$ instead of $\Sigma_{i=[nt_j]+1}^{[nt_{j+1}]}$. Then

$$\begin{aligned} P_1(\varepsilon, r, n)\leq & P\left(\left|\sum_{i=1}^{n-1} u_n\left(\frac{i}{n}, T_i\right)\eta_{i+1}-\sum_{j=0}^{b-1} u_n(t_j, T_{[nt_j]})\Sigma_{(j)}\eta_{i+1}\right|\geq\frac{\varepsilon}{2}\right) \\ & +P\left(\left|\sum_{i=1}^{n-1} u_n\left(\frac{i}{n}, T_i\right)\zeta_{i+1}-\sum_{j=0}^{b-1} u_n(t_j, T_{[nt_j]})\Sigma_{(j)}\zeta_{i+1}\right|\geq\frac{\varepsilon}{2}\right). \end{aligned} \tag{3.4}$$

For every $i$ ($[nt_j]\leq i<[nt_{j+1}]$) let

$$V_{ij}=u_n\left(\frac{i}{n}, T_i\right)-u_n(t_j, T_{[nt_j]}).$$

Since for $i$ ($[nt_j]\leq i<[nt_{j+1}]$) and $i'$ ($[nt_{j'}]\leq i'<[nt_{j'+1}]$) $(i<i')$

$$E\{V_{ij}\eta_{i+1}V_{i'j'}\eta_{i'+1}\}=E\{V_{ij}\eta_{i+1}V_{i'j'}E\{\eta_{i'+1}|\mathscr{M}_{-\infty}^{i'}\}\}=0$$

and similarly

$$E\{V_{ij}\zeta_{i+1}V_{i'j'}\zeta_{i'+1}\}=0,$$

so from (2.3) it follows that for all $n$ sufficiently large

$$\begin{aligned} & E\left|\sum_{i=1}^{n-1} u_n\left(\frac{i}{n}, T_i\right)\eta_{i+1}-\sum_{j=0}^{b-1} u_n(t_j, T_{[nt_j]})\Sigma_{(j)}\eta_{i+1}\right|^2 \\ & =\sum_{j=0}^{b-1}\Sigma_{(j)}E|V_{ij}^2\eta_{i+1}^2|\leq N^2n^{-1}\sum_{j=0}^{b-1}\Sigma_{(j)}E|V_{ij}^2| \end{aligned} \tag{3.5}$$

$$\leqq KN^2 n^{-1}\sum_{j=0}^{b-1}\Sigma_{(j)}\left\{\left|\frac{i}{n}-t_j\right|^2+E|T_i-T_{[nt_j]}|^2\right\}$$

$$\leqq KN^2 n^{-1}\sum_{j=0}^{b-1}\{(t_{j+1}-t_j)^2([nt_{j+1}]-[nt_j])+\Sigma_{(j)}n^{-1}(i-[nt_j])\}$$

$$\leqq KN^2 n^{-1}\gamma\sum_{j=0}^{b-1}([nt_{j+1}]-[nt_j])\leqq KN^2\gamma\ .$$

On the other hand, since $u_n$ is bounded, so

$$E\left|\sum_{i=1}^{n-1}u_n\left(\frac{i}{n},\ T_i\right)\zeta_{i+1}-\sum_{j=0}^{b-1}u_n(t_j,\ T_{[nt_j]})\Sigma_{(j)}\zeta_{i+1}\right|^2 \tag{3.6}$$

$$=\sum_{j=0}^{b-1}\Sigma_{(j)}E|V_{ij}^2\zeta_{i+1}^2|\leqq Kn^{-1}\sum_{i=1}^{n-1}E\bar{\bar{\xi}}_i^2$$

where $\bar{\bar{\xi}}_i=\xi_i-\bar{\xi}_i$. Thus, from (3.5) and (3.6)

$$P_1(\varepsilon,\gamma,n)\leqq K\varepsilon^{-2}\{N^2\gamma+\max_{1\leqq i\leqq n}E\bar{\bar{\xi}}_i^2\}. \tag{3.7}$$

If Condition (A) is satisfied, then

$$\max_{1\leqq i\leqq n}E\bar{\bar{\xi}}_i^2=E\bar{\bar{\xi}}_i^2=o(1)\quad (\text{as } N\to\infty)$$

and if Condition (B) is satisfied, then $E|\xi_i|^{2+\delta}$ is uniformly bounded and so

$$\max_{1\leqq i\leqq n}E\bar{\bar{\xi}}_i^2\leqq KN^{-\delta}\ .$$

Hence, letting firstly $n\to\infty$, secondly $\gamma\downarrow 0$ and then $N\to\infty$, we have the desired result, when either Condition (A) or (B) is satisfied.

**Lemma 3.2.** *Let $f$ and $f_n$ $(n=1, 2, \cdots)$ be functions defined in Theorem. Let $u=f^{(C)}$ be the function defined by*

$$u(s,x)=\begin{cases}f(s,x) & \textit{if}\quad (s,x)\in[0,1]\times[-C,C]\ ,\\ 0 & \textit{otherwise}\ ,\end{cases}$$

*where $C>0$ is some constant. For any $\varepsilon>0$, put*

$$P_2(\varepsilon,\gamma,n)=P(|\sum_{j=0}^{b-1}u_n(t_j,\ T_{[nt_j]})(T_{[nt_{j+1}]}-T_{[nt_j]}) \tag{3.8}$$

$$-\sum_{j=0}^{b-1}u(t_j,\ T_{[nt_j]})(T_{[nt_{j+1}]}-T_{[nt_j]})|>\varepsilon)\ .$$

*If for each $s\in[0,1]$ $u_n(s,\pm C)\to u(s,\pm C)$ $(n\to\infty)$, then*

$$\lim_{\gamma\downarrow 0}\lim_{n\to\infty}P_2(\varepsilon,\gamma,n)=0\ . \tag{3.9}$$

**Proof.** We use the same notations as the ones in the proof of Lemma 3.1. Let

$$v_j(x)=u_n(t_j, x)-u(t_j, x) \quad (j=0, 1, \cdots, b) .$$

Since

$$E\{(T_{[nt_{j+1}]}-T_{[nt_j]})^2 | \mathscr{M}_{-\infty}^{[nt_j]}\}=n^{-1}\Sigma_{(j)}E\{\xi_i^2 | \mathscr{M}_{-\infty}^{[nt_j]}\} ,$$

so, noting that $u_n$ and $u$ are uniformly bounded, we have

$$\begin{aligned}
(3.9)\qquad & E\Big|\sum_{j=0}^{b-1} u_n(t_j, T_{[nt_j]})(T_{[nt_{j+1}]}-T_{[nt_j]})-\sum_{j=0}^{b-1} u(t_j, T_{[nt_j]})(T_{[nt_{j+1}]}-T_{[nt_j]})\Big|^2 \\
&=\sum_{j=0}^{b-1} E\{|v_j(T_{[nt_j]})|^2 n^{-1}\Sigma_{(j)}E\{\xi_i^2|\mathscr{M}_{-\infty}^{[nt_j]}\}\} \\
&\leqq K\sum_{j=0}^{b-1}\Big(t_{j+1}-t_j+\frac{1}{n}\Big)E|\{[nt_{j+1}]-[nt_j]\}^{-1}\Sigma_{(j)}E\{\xi_i^2|\mathscr{M}_{-\infty}^{[nt_j]}\}-1| \\
&\quad +\sum_{j=0}^{b-1}\Big(t_{j+1}-t_j+\frac{1}{n}\Big)E|v_j(T_{[nt_j]})|^2 \\
&=P_2^{(1)}+P_2^{(2)} \quad \text{(say).}
\end{aligned}$$

Now, we shall show that under Condition (A) or (B)

$$(3.10)\qquad \lim_{\gamma\downarrow 0}\lim_{n\to\infty} P_2^{(1)}=0 .$$

We first note that

$$(3.11)\qquad P_2^{(1)}\leqq K\Big(1+\frac{b}{n}\Big)\max_{0\leqq j\leqq b-1} E|\{[nt_{j+1}]-[nt_j]\}^{-1}\Sigma_{(j)}E\{\xi_i^2|\mathscr{M}_{-\infty}^{[nt_j]}\}-1| .$$

If Condition (B) is satisfied, then, from (B3) and (3.11), (3.10) follows. On the other hand, by Jensen's inequality

$$E|m^{-1}\sum_{i=1}^{m}\{\xi_i^2|\mathscr{M}_{-\infty}^{0}\}-1|\leqq E|m^{-1}\sum_{i=1}^{m}\xi_i^2-1| .$$

So if Condition (A) is satisfied, then by the mean ergodic theorem the right-hand side of (3.11) tends to zero as $n\to\infty$. Thus, (3.10) is obtained.

Next, since by (2.5) $v_j(x)\to 0$ uniformly in $x\in[-C, C]$ for each $j$ $(0\leqq j\leqq b-1)$, so

$$E|v_j(T_{[nt_j]})|^2=\int_{-C}^{C}|v_j(x)|^2 dF_j(x)\to 0$$

where $F_j$ is the $df$ of $T_{[nt_j]}$. Hence

$$(3.12)\qquad P_2^{(2)}\leqq K\varepsilon^{-2}\sum_{j=0}^{b-1}E|v_j(T_{[nt_j]})|^2\to 0 \quad (n\to\infty) .$$

Combining (3.9), (3.10) and (3.12), we have the desired result.

**Lemma 3.3.** *Under the conditions of Lemma* 3.2

$$(3.13)\qquad \sum_{j=0}^{b-1} u(t_j, T_{[nt_j]})(T_{[nt_{j+1}]}-T_{[nt_j]})\xrightarrow{D}\sum_{j=0}^{b-1} u(t_j, w(t_j))(w(t_{j+1})-w(t_j))$$

*as* $n\to\infty$.

**Proof.** Since $u$ is continuous, so from Remark 2.2 (3.13) easily follows.

**Lemma 3.4.** *Under the conditions of Lemma* 3.2

$$P\left(\left|\sum_{j=0}^{b-1} u(t_j, w(t_j))(w(t_{j+1})-w(t_j))-\int_0^1 u(t, w(t))dw(t)\right|>\varepsilon\right)\to 0 \tag{3.14}$$

*for every* $\varepsilon>0$ *as* $\gamma\downarrow 0$.

**Proof.** We note that

$$\begin{aligned}J&=E\left|\sum_{j=0}^{b-1} u(t_j, w(t_j))(w(t_{j+1})-w(t_j))-\int_0^1 u(t, w(t))dw(t)\right|^2\\&=\sum_{j=0}^{b-1} E\left|\int_{t_j}^{t_{j+1}}\{u(t, w(t))-u(t_j, w(t_j))\}dw(t)\right|^2\\&=\sum_{j=0}^{b-1}\int_{t_j}^{t_{j+1}} E|u(t, w(t))-u(t_j, w(t_j))|^2 dt .\end{aligned}$$

From (2.2) it follows that for each $t$ $(t_j\leqq t\leqq t_{j+1})$

$$\begin{aligned}&E|u(t, w(t))-u(t_j, w(t_j))|^2\\&\quad\leqq KE\{(t-t_j)^2+|w(t)-w(t_j)|^2\}\{1+|w(t)-w(t_j)|^{2\alpha}+|w(t_j)|^{2\alpha}\}\\&\quad\leqq K[E\{(t-t_j)^2+|w(t)-w(t_j)|^2\}E\{|w(t_j)|^{2\alpha}+1\}\\&\qquad+E\{(t-t_j)^2|w(t)-w(t_j)|^{2\alpha}+|w(t)-w(t_j)|^{2+2\alpha}\}\\&\quad\leqq K(t-t_j)\quad (j=0, 1, \cdots, b) .\end{aligned}$$

Hence, we have

$$J\leqq K\sum_{j=1}^{b-1}\int_{t_j}^{t_{j+1}}(t-t_j)dt\leqq K\gamma ,$$

which implies (3.14) and the proof is completed.

**4. Proof of Theorem.** For any $C>0$, let $f^{(C)}$ and $f_n^{(C)}$ $(n=1, 2, \cdots)$ be functions as in the preceding section. Then, for any $\varepsilon>0$

$$\begin{aligned}&P\left(\left|\sum_{i=1}^{n-1} f_n\left(\frac{i}{n}, T_i\right)\frac{\xi_{i+1}}{\sqrt{n}}-\int_0^1 f(t, w(t))dw(t)\right|>\varepsilon\right)\\&\quad\leqq P\left(\left|\sum_{i=1}^{n-1} f_n^{(C)}\left(\frac{i}{n}, T_i\right)\frac{\xi_{i+1}}{\sqrt{n}}-\int_0^1 f^{(C)}(t, w(t))dw(t)\right|>\varepsilon\right)\\&\qquad+P(\sup_{1\leqq i\leqq n}|T_i|>C)+P(\sup_{0\leqq t\leqq 1}|w(t)|>C) .\end{aligned}\tag{4.1}$$

Let $\{C_m\}$ be an increasing sequence of positive numbers such that $C_m\uparrow\infty$. If Condition (A) or (B) is satisfied, then $S_0, S_1, \cdots, S_n$ are martingale, and so

$$\begin{aligned}\lim_{C_m\to\infty} P(\max_{1\leqq i\leqq n}|T_i|>C_m)&\leqq\lim_{C_m\to\infty}(C_m n^{1/2})^{-1}\int_{\{|S_n|\geqq C_m n^{1/2}\}}|S_n|dP\\&\leqq\lim_{C_m\to\infty} C_m^{-2}n^{-1}ES_n^2=0 .\end{aligned}\tag{4.2}$$

On the other hand, we have

(4.3) $$P(\sup_{0\leq t\leq 1} |w(t)|>C_m)\to 0 \quad (C_m\to\infty)\ .$$

Hence, the proof is obtained by Lemmas 3.1-3.4 and (4.1)-(4.3).

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