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A NON-UNIFORM ESTIMATE IN THE LOCAL LIMIT THEOREM FOR DENSITIES, I

By

Makoto Maejima

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1. Introdnction and a result

Let $\{X_k, k=1, 2, \cdots\}$ be a sequence of independent and identically distributed random variables with $EX_1=0$, $EX_1^2=1$, and with distribution function $F(x)$. Write $Z_n=n^{-1/2}\sum_{k=1}^{n} X_k$, and let $p_n(x)$ denote the density function of Z_n, when Z_n has an absolutely continuous distribution. Furthermore, denote the standard normal density by $\phi(x)$.

In this paper, we shall deal with the convergence rate of a non-uniform estimate in the local limit theorem, and give the necessary and sufficient condition for the validity of $\sum n^{-1+\delta/2}\sup_x (1+|x|)^2|p_n(x)-\phi(x)|<\infty$, $0\leqq\delta<1$. Our theorem is an extension of a result of Galstjan [2] who studied the uniform convergence of $|p_n(x)-\phi(x)|$. Non-uniform estimates in the local limit theorem for densities have been investigated by Petrov [5], Basu [1] and others.

The theorem we are going to show is the following, which is a local version of a result due to Heyde ([4], Theorem 1 (i), (ii)).

Theorem. *Let* $0\leqq\delta<1$. *Suppose that*

(a) *there exists* N *such that* $\sup p_N(x)<\infty$.

Then, in order that

$$\sum_{n\geqq 2(N+1)} n^{-1+\delta/2}\sup_x (1+|x|)^2|p_n(x)-\phi(x)|<\infty ,$$

it is necessary and sufficient that

(b) $E|X_1|^{2+\delta}<\infty$, *if* $0<\delta<1$, $\qquad EX_1^2\log(1+|X_1|)<\infty$, *if* $\delta=0$.

Galstjan [2] proved that (b) is equivalent to $\sum n^{-1+\delta/2}\sup_x |p_n(x)-\phi(x)|<\infty$, under the condition (a). Therefore, it suffices only to show the sufficiency part of the theorem and to get the estimate for $\sup_x x^2|p_n(x)-\phi(x)|$.

2. Proof

Write $f(t)=Ee^{itX_1}$ and $\theta_n(t)=Ee^{itZ_n}=\{f(n^{-1/2}t)\}^n$. Since $p_N(x)$ is bounded, we have $p_N(x) \in L^2$, so that $\theta_N(t) \in L^2$ by Parseval identity. Thus, it follows that $\theta_n(t) \in L^1$ for all $n \geqq 2N$, so that we may write

$$p_n(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itx} \theta_n(t) dt .$$

Since the variance of X_1 exists by our assumption, we have

$$\theta_n''(t) = (n-1)\{f'(n^{-1/2}t)\}^2 \{f(n^{-1/2}t)\}^{n-2} + f''(n^{-1/2}t)\{f(n^{-1/2}t)\}^{n-1} . \tag{2.1}$$

Noting that $|f'(\cdot)| \leqq 1$ and $|f''(\cdot)| \leqq 1$, we have

$$|\theta_n''(t)| \leqq n |f(n^{-1/2}t)|^{n-2} .$$

Hence, we see that $\theta_n''(t) \in L^1$ for $n \geqq 2(N+1)$, so that

$$x^2 p_n(x) = -(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itx} \theta_n''(t) dt .$$

Therefore, making use of

$$x^2 \phi(x) = -(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-itx} (t^2-1) e^{-t^2/2} dt ,$$

we have

$$\sup_x x^2 |p_n(x) - \phi(x)| \leqq \int_{-\infty}^{\infty} |\theta_n''(t) - (t^2-1)e^{-t^2/2}| dt . \tag{2.2}$$

Now, under our assumption of finite variance, we can express $f(t)$, $f'(t)$, $f''(t)$ in the following respective form:

$$\begin{cases} f(t) = 1 - \frac{1}{2} t^2 (1+\gamma(t)) = \exp\left\{-\frac{1}{2} t^2 (1+\gamma_0(t))\right\} , \\ f'(t) = -t(1+\gamma_1(t)) , \\ f''(t) = -(1+\gamma_2(t)) , \end{cases} \tag{2.3}$$

where $\lim_{t\to 0} \gamma(t) = 0$ and $\lim_{t \to 0} \gamma_j(t) = 0$ $(j=0, 1, 2)$.

For convenience, the proof of the theorem will be divided into two lemmas.

Lemma 1. *Let* $0 \leqq \delta < 1$. *If*

$$\int_0^{\varepsilon} t^{-1-\delta} |\gamma_j(t)| dt < \infty , \qquad j=0, 1, 2 \tag{2.4}$$

for some $\varepsilon > 0$, *then we have*

$$\sum_{n \geqq 2(N+1)} n^{-1+\delta/2} \sup_x x^2 |p_n(x) - \phi(x)| < \infty .$$

Proof. We have from (2.2),

$$\sum_{n\geq 2(N+1)} n^{-1+\delta/2} \sup_x x^2|p_n(x)-\phi(x)|$$
$$\leq \sum n^{-1+\delta/2}\int_{|t|<\varepsilon n^{1/2}} |\theta_n''(t)-(t^2-1)e^{-t^2/2}|dt$$
$$+\sum n^{-1+\delta/2}\int_{|t|\geq\varepsilon n^{1/2}} |\theta_n''(t)|dt+\sum n^{-1+\delta/2}\int_{|t|\geq\varepsilon n^{1/2}}|t^2-1|e^{-t^2/2}dt$$
$$\equiv I_1+I_2+I_3\ ,$$

say, where ε is chosen so small that $0<\varepsilon<2$ and

$$\max_{0<|t|<\varepsilon} |\gamma_0(t)|\leq\frac{1}{10}\ . \tag{2.5}$$

We first show that $I_1<\infty$. From (2.1) and (2.3), we have

$$\theta_n''(t)=(n-1)n^{-1}t^2(1+\gamma_1(n^{-1/2}t))^2\exp\{-(n-2)(2n)^{-1}t^2(1+\gamma_0(n^{-1/2}t))\}$$
$$-(1+\gamma_2(n^{-1/2}t))(1-t^2(2n)^{-1}(1+\gamma(n^{-1/2}t)))\exp\{-(n-2)(2n)^{-1}t^2(1+\gamma_0(n^{-1/2}t))\}$$
$$=(t^2-1)c_n(t)\exp\left\{-\frac{1}{2}t^2(1+\gamma_0(n^{-1/2}t))\right\}+R_n(t)\ ,$$

where

$$c_n(t)=\exp\{n^{-1}t^2(1+\gamma_0(n^{-1/2}t))\}$$

and

$$R_n(t)=\{-n^{-1}t^2(1+\gamma_1(n^{-1/2}t))^2+t^2(2\gamma_1(n^{-1/2}t)+\{\gamma_1(n^{-1/2}t)\}^2)$$
$$+(2n)^{-1}t^2(1+\gamma(n^{-1/2}t))-\gamma_2(n^{-1/2}t)(1-(2n)^{-1}t^2(1+\gamma(n^{-1/2}t)))\}$$
$$\cdot\exp\{-(n-2)(2n)^{-1}t^2(1+\gamma_0(n^{-1/2}t))\}$$
$$\equiv I_{13}+I_{14}+I_{15}+I_{16}\ ,$$

say. Now we need the estimate of the following quantity:

$$|\theta_n''(t)-(t^2-1)e^{-t^2/2}|$$
$$\leq|t^2-1|e^{-t^2/2}\left|1-c_n(t)\exp\left\{-\frac{1}{2}t^2\gamma_0(n^{-1/2}t)\right\}\right|+|R_n(t)|$$
$$\leq(t^2+1)e^{-t^2/2}\left|1-\exp\left\{-\frac{1}{2}t^2\gamma_0(n^{-1/2}t)\right\}\right|$$
$$+(t^2+1)e^{-t^2/2}|1-c_n(t)||\exp\{-2^{-1}t^2\gamma_0(n^{-1/2}t)\}|+|R_n(t)|$$
$$\equiv I_{11}+I_{12}+\left|\sum_{j=3}^{6} I_{1j}\right|\ ,$$

say. Using the inequality $|e^z-1|\leq|z|e^{|z|}$, we have for $|t|<\varepsilon n^{1/2}$,

$$I_{11}\leq\frac{1}{2}t^2(t^2+1)|\gamma_0(n^{-1/2}t)|\exp\left\{-\frac{1}{2}t^2(1-|\gamma_0(n^{-1/2}t)|)\right\}$$
$$\leq\frac{1}{2}t^2(t^2+1)|\gamma_0(n^{-1/2}t)|\exp\left\{-\frac{9}{20}t^2\right\}\ ,$$

because of (2.5). Hence,

$$(2.6)\qquad \sum n^{-1+\delta/2}\int_{|t|<\epsilon n^{1/2}} I_{11}dt \leqq \sum n^{-1+\delta/2}\int_0^{\epsilon n^{1/2}} t^2(t^2+1)|\gamma_0(n^{-1/2}t)|e^{-ct^2}dt$$
$$\leqq C\int_0^{\epsilon} u^{-1-\delta}(1+cu^2)^{-(5+\delta)/2}|\gamma_0(u)|du\,,$$

by the same arguments as in Heyde [3] and Galstjan [2]. Here and in what follows, C and c denote positive constants which may differ from one inequality to another. Therefore, we have, using (2.4) with $j=0$,

$$(2.7)\qquad \sum n^{-1+\delta/2}\int_{|t|<\epsilon n^{1/2}} I_{11}dt < \infty\,.$$

Using $|e^z|\leqq e^{|z|}$, we next see that for $|t|<\epsilon n^{1/2}$,

$$\begin{aligned} I_{12} &\leqq (t^2+1)|1-c_n(t)|\exp\left\{-\frac{1}{2}t^2(1-|\gamma_0(n^{-1/2}t)|)\right\} \\ &\leqq (t^2+1)\exp\left\{-\frac{9}{20}t^2\right\}|1-c_n(t)| \\ &\leqq (t^2+1)\exp\left\{-\frac{9}{20}t^2\right\}t^2n^{-1}(1+|\gamma_0(n^{-1/2}t)|)\exp\{n^{-1}t^2(1+|\gamma_0(n^{-1/2}t)|)\} \\ &\leqq \frac{11}{10}n^{-1}t^2(t^2+1)\exp\left\{-\left(\frac{9}{20}-\frac{11}{10}n^{-1}\right)t^2\right\}. \end{aligned}$$

Accordingly, it follows that for all $n\geqq 2(N+1)\geqq 4$,

$$\begin{aligned} I_{12} &\leqq \frac{11}{10}n^{-1}t^2(t^2+1)\exp\left\{-\frac{7}{40}t^2\right\} \\ &= Cn^{-1}t^2(t^2+1)e^{-ct^2}\,. \end{aligned}$$

Thus,

$$(2.8)\qquad \sum n^{-1+\delta/2}\int_{|t|<\epsilon n^{1/2}} I_{12}dt \leqq C\sum n^{-2+\delta/2} < \infty\,.$$

Furthermore, the following inequalities are readily seen:

$$(2.9)\qquad \sum n^{-1+\delta/2}\int_{|t|<\epsilon n^{1/2}} |I_{13}|dt \leqq C\sum n^{-1+\delta/2}\int_0^{\epsilon n^{1/2}} n^{-1}t^2e^{-ct^2}dt < \infty\,,$$

$$(2.10)\qquad \sum n^{-1+\delta/2}\int_{|t|<\epsilon n^{1/2}} |I_{14}|dt \leqq C\sum n^{-1+\delta/2}\int_0^{\epsilon n^{1/2}} t^2|\gamma_1(n^{-1/2}t)|e^{-ct^2}dt\,,$$

$$(2.11)\qquad \sum n^{-1+\delta/2}\int_{|t|<\epsilon n^{1/2}} |I_{15}|dt \leqq C\sum n^{-1+\delta/2}\int_0^{\epsilon n^{1/2}} n^{-1}t^2e^{-ct^2}dt < \infty\,,$$

(2.12)
$$\sum n^{-1+\delta/2}\int_{|t|<\varepsilon n^{1/2}} |I_{16}|dt$$
$$\leqq C\sum n^{-1+\delta/2}\int_0^{\varepsilon n^{1/2}} |\gamma_2(n^{-1/2}t)|e^{-ct^2}dt + C\sum n^{-1+\delta/2}\int_0^{\varepsilon n^{1/2}} n^{-1}t^2e^{-ct^2}dt\ ,$$

where the second series on the right hand side trivially converges. As to (2.10) and (2.12), using the same argument as we have obtained (2.6), we have

(2.13)
$$\sum n^{-1+\delta/2}\int_0^{\varepsilon n^{1/2}} t^2|\gamma_1(n^{-1/2}t)|e^{-ct^2}dt<\infty$$

and

(2.14)
$$\sum n^{-1+\delta/2}\int_0^{\varepsilon n^{1/2}} |\gamma_2(n^{-1/2}t)|e^{-ct^2}dt<\infty\ ,$$

under the condition (2.4). The estimates (2.7)-(2.14) thus imply $I_1<\infty$.

It remains to show that $I_2<\infty$ and $I_3<\infty$. Since Z_N has the density $p_N(x)$, we see that for any $\varepsilon>0$, $\sup_{|t|\geqq\varepsilon}|f(t)|\leqq e^{-c}$ for some $c>0$, so that for $|t|\geqq\varepsilon n^{1/2}$, we have $|f(n^{-1/2}t)|\leqq e^{-c}$. Then we find from (2.1) that for all $n\geqq 2(N+1)$,

$$|\theta_n''(t)|\leqq ne^{-c(n-2(N+1))}|f(n^{-1/2}t)|^{2N}\ ,$$

and hence

$$I_2=\sum n^{-1+\delta/2}\int_{|t|\geqq\varepsilon n^{1/2}}|\theta_n''(t)|dt\leqq\sum n^{\delta/2}e^{-c(n-2(N+1))}\int_{|t|\geqq\varepsilon n^{1/2}}|f(n^{-1/2}t)|^{2N}dt$$
$$\leqq\sum n^{(1+\delta)/2}e^{-c(n-2(N+1))}\int_{|u|\geqq\varepsilon}|f(u)|^{2N}du<\infty\ .$$

As to I_3, we have

$$I_3=\sum n^{-1+\delta/2}\int_{|t|\geqq\varepsilon n^{1/2}}|t^2-1|e^{-t^2/2}dt\leqq\sum n^{-(3-\delta)/2}\varepsilon^{-1}\int_{|t|\geqq\varepsilon n^{1/2}}|t||t^2-1|e^{-t^2/2}dt<\infty\ ,$$

which completes the proof of Lemma 1.

Lemma 2. *If the condition* (b) *is satisfied, then* (2.4) *holds for* $j=0, 1, 2$.

Proof. Heyde [3] proved the case $j=0$, and so it is sufficient to show that (2.4) holds for $j=1, 2$. We have

$$\int_0^\varepsilon t^{-1-\delta}|\gamma_1(t)|dt=\int_0^\varepsilon t^{-2-\delta}|t+f'(t)|dt$$
$$\leqq\int_0^\varepsilon t^{-2-\delta}|t+\mathrm{Re}\,f'(t)|dt+\int_0^\varepsilon t^{-2-\delta}|\mathrm{Im}\,f'(t)|dt\equiv J_1+J_2\ ,$$

say. Here we see that

$$J_1=\int_0^{\varepsilon} t^{-2-\delta}\left|\int_{-\infty}^{\infty} x(tx-\sin tx)dF(x)\right|dt \leqq \int_{-\infty}^{\infty} |x|dF(x)\int_0^{\varepsilon} t^{-2-\delta}|tx-\sin tx|dt$$

$$=\int_{-\infty}^{\infty} |x|^{2+\delta}dF(x)\int_0^{\varepsilon|x|} v^{-2-\delta}|v-\sin v|dv\ ,$$

where

$$\int_0^{\varepsilon|x|} v^{-2-\delta}|v-\sin v|dv \begin{cases} <\infty\ , & \text{if } 0<\delta<1\ , \\ \sim\log|x| \text{ as } |x|\to\infty\ , & \text{if } \delta=0\ . \end{cases}$$

Therefore, the condition (b) implies $J_1<\infty$. For J_2, we find that

$$J_2=\int_0^{\varepsilon} t^{-2-\delta}\left|\int_{-\infty}^{\infty} x\cos tx dF(x)\right|dt=\int_0^{\varepsilon} t^{-2-\delta}\left|\int_{-\infty}^{\infty} x(\cos tx-1)dF(x)\right|dt$$

$$\leqq\int_{-\infty}^{\infty} |x|dF(x)\int_0^{\varepsilon} t^{-2-\delta}|\cos tx-1|dt\leqq\int_{-\infty}^{\infty} |x|^{2+\delta}dF(x)\int_0^{\infty} v^{-2-\delta}|\cos v-1|dv<\infty\ ,$$

since $\int_0^{\infty} v^{-2-\delta}|\cos v-1|dv<\infty$ if $0\leqq\delta<1$.

We finally consider $\gamma_2(t)$. We have

$$\int_0^{\varepsilon} t^{-1-\delta}|\gamma_2(t)|dt=\int_0^{\varepsilon} t^{-1-\delta}|1+f''(t)|dt$$

$$\leqq\int_0^{\varepsilon} t^{-1-\delta}|1+\mathrm{Re}\,f''(t)|dt+\int_0^{\varepsilon} t^{-1-\delta}|\mathrm{Im}\,f''(t)|dt\equiv J_3+J_4\ ,$$

say. Here we have

$$J_3=\int_0^{\varepsilon} t^{-1-\delta}\left|\int_{-\infty}^{\infty} x^2(1-\cos tx)dF(x)\right|dt\leqq\int_{-\infty}^{\infty} x^2dF(x)\int_0^{\varepsilon} t^{-1-\delta}|1-\cos tx|dt$$

$$=\int_{-\infty}^{\infty} |x|^{2+\delta}dF(x)\int_0^{\varepsilon|x|} v^{-1-\delta}|1-\cos v|dv\ ,$$

where

$$\int_0^{\varepsilon|x|} v^{-1-\delta}|1-\cos v|dv \begin{cases} <\infty\ , & \text{if } 0<\delta<1\ , \\ \sim\log|x| \text{ as } |x|\to\infty\ , & \text{if } \delta=0\ . \end{cases}$$

Thus we get $J_3<\infty$ by the condition (b). Furthermore, we see

$$J_4=\int_0^{\varepsilon} t^{-1-\delta}\left|\int_{-\infty}^{\infty} x^2\sin tx dF(x)\right|dt\leqq\int_{-\infty}^{\infty} x^2dF(x)\int_0^{\varepsilon} t^{-1-\delta}|\sin tx|dt$$

$$=\int_{-\infty}^{\infty} |x|^{2+\delta}dF(x)\int_0^{\varepsilon|x|} v^{-1-\delta}|\sin v|dv\ ,$$

where

$$\int_0^{\varepsilon|x|} v^{-1-\delta}|\sin v|dv \begin{cases} <\infty\ , & \text{if } 0<\delta<1\ , \\ \sim\log|x| \text{ as } |x|\to\infty\ , & \text{if } \delta=0\ , \end{cases}$$

which implies $J_4 < \infty$ because of the condition (b). This completes the proof of the lemma, and that of the theorem is thus completed.

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Department of Mathematics
Faculty of Engineering
Keio University
3-14-1, Hiyoshi, Kohoku-ku
Yokohama 223, Japan