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DEHN'S SURGERY ALONG 2-BRIDGE KNOTS

By

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1. Introduction.

Our main concern is the study of the 3-manifolds obtained by Denh's surgery along 2-bridge knots. In this paper, we prove that such 3-manifolds have Heegaard splittings of genus two and so are decomposed to two lens spaces when they are not irreducible. Furthermore we shall verify that such 3-manifolds are not 3-spheres S^3 even though they are homology 3-spheres. This result is proved by using Homma's theorem [1] and Volodin-Kuznetsov-Fomenko [6], which is recently found;

Homma's Theorem. *Heegaard splittings of genus two for* 3-*spheres* S^3 *are reducible except for the extended canonical one.*

We work in piecewise linear category throughout the paper. Furthermore, unless specified otherwise, by $N(X, Y)$ we shall denote a regular neighborhood of a subpolyhedron X in a polyhedron Y and by $\mathring{X}$ we shall denote the interior of X.

Definition. Let W_1, W_2 be solid tori of genus two and M a closed orientable 3-manifold and h: $\partial W_2 \rightarrow \partial W_1$ a homeomorphism of tori. Then the triple $(W_1, W_2; h)$ (or (W_1, W_2, F)) is called a Heegaard splitting of genus two for M when $M = W_1 \cup_h W_2$ and $W_1 \cap W_2 = \partial W_1 = \partial W_2 = F$, a closed 2-manifold.

Next let $\{D_{i1}, D_{i2}\}$ be a meridian-disk pair of W_i $(i=1, 2)$, that is, D_{ij} $(j=1, 2)$ is a properly embedded 2-disk in W_i such that D_{i1} and D_{i2} are disjoint and $W_i - D_{i1} \cup D_{i2}$ is connected. Such a 2-disk D_{ij} $(j=1, 2)$ is called a meridian disk of W_i and the circle $u_{ij} = \partial D_{ij}$ $(j=1, 2)$ a meridian of W_i.

Now let h be a attaching homeomorphism from ∂W_2 onto ∂W_1. Then the manifold $M = W_1 \cup_h W_2$ is determined up to homeomorphisms by the collection of circles v_1 and v_2 on ∂W_1 such that $v_k = h(u_{2k})$ $(k=1, 2)$. For example, let us illustrate the canonical Heegaard splitting of genus two for S^3 as the one in Fig. 1.

2. Dehn's surgery along 2-bridge knots.

A 3-manifold M is said to be obtained by a Dehn's surgery along a 2-bridge

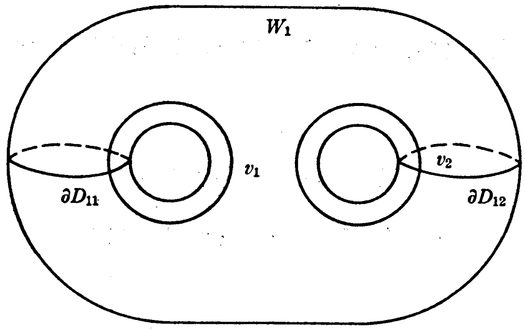

Fig. 1. The canonical Heegaard splitting for S^3.

knot K if followings hold; Let $N(K, S^3)$ be a regular neighborhood of K in S^3 and then $N(K, S^3)$ is a solid torus. Furthermore let E be $S^3-\mathring{N}(K, S^3)$, V a solid torus, i: $\partial N(K, S^3)\to\partial E$ be the trivial attaching which induces $S^3=E\overset{i}{\cup}N(K, S^3)$, and Ψ: $\partial V\to\partial E$ a homeomorphism of tori with the property that $i^{-1}\Psi$ does not extend to a homeomorphism from V onto $N(K, S^3)$. Now in the disjoint union,

$$(S^3-\mathring{N}(K, S^3))+V$$

identify points $x\in\partial V$ with points $\Psi(x)\in\partial E$. Then the resulting manifold is the 3-manifold M.

Next let W be a standardly embedded solid torus of genus two in S^3, that is, there are two properly embedded 2-disks D_1, D_2 in $S^3-\mathring{W}$ such that D_1 and D_2 are disjoint and intersect two meridians of W transversely at points a, b respectively. Let $u_1=\partial D_1$ and $u_2=\partial D_2$. Then we have;

Lemma 1. *Let K be an arbitrary 2-bridge knot in S^3. Then K is embedded in ∂W such that K transversely intersects circles u_1, u_2 and $K\cap u_1=a$, $K\cap u_2=b$.*

Proof. The proof follows directly from the standard definition of 2-bridge knots [4].

Hereafter we may assume that in Lemma 1 the knot K is a Schubert's 2-bridge knot. In [4] Schubert described a normal form $K=(\alpha, \beta)$ for each 2-bridge type $\bar{K}$ and used it to classify the types completely. In the normal form, $\alpha=\det(\bar{K})$, the determinant of the knot, and β is an integer relatively prime to α satisfying $-\alpha<\beta<\alpha$, and furthermore we may assume that $\alpha\neq1$ is a positive odd integer because the knot K considered in the paper is exactly knotted and not a link.

Now let K be a 2-bridge knot and then the knot K is embedded in ∂W by Lemma 1 such that $a=u_1\cap K$ and $b=u_2\cap K$. Then $a\cup b$ separates K into two components c_1, c_2. Thus $\partial N(u_1\cup c_i\cup u_2, \partial W)$ $(i=1, 2)$ consists of three circles such that two of these three are isotopic to u_1 and u_2 in ∂W respectively and the last, which is called the knotting circle and denoted by $C(K, i)$, is not isotopic to u_1 and u_2 in ∂W. Then we have;

Lemma 2. *$C(K, 1)$ is isotopic to $C(K, 2)$ in ∂W.*

Proof. Since $N(u_i, \partial W)$ $(i=1, 2)$ is an annulus, $\partial W-\dot{N}(u_1, \partial W)-\dot{N}(u_2, \partial W)$ is homeomorphic to a 2-manifold L_4 which is obtained by removing the interior of four 2-disks from a 2-sphere. Thus $C(K, 1)$ and $C(K, 2)$ lie on L_4 and separate L_4 into three components such that two of these three components have three boundary circles and the remainder is an annulus which has $C(K, 1)$ and $C(K, 2)$ as its boundary. Hence $C(K, 1)$ is isotopic to $C(K, 2)$ in ∂W. The proof is complete.

Next M be a 3-mahifold obtained by a Dehn's surgery along the knot K. Then we have;

Theorem 1. *The manifold M has a Heegaard splitting of genus two.*

Proof. By Lemma 2, the knot K is embedded in ∂W and then W is standardly embedded in S^3. Let $C(K, 1)$ and $C(K, 2)$ be the knotting circles of K, and let C_K be one of these circles and then the choice is free by Lemma 2. Further let W' be a solid torus of genus two and $\{D_1', D_2'\}$ the meridian disk pair of W'. Then we have a Heegaard splitting $(W, W'; h)$ for S^3 such that h: $\partial W'\to\partial W$ is a homeomorphism defined as $h(\partial D_1')=u_1$ and $h(\partial D_2')=C_K$. $N(D_i', W')$ $(i=1, 2)$ is homoemorphic to $D^2\times I$ where D^2 is a 2-disk and I is an unit interval, and we may assume that $\partial N(D_i', W')\cap\partial W'=N(\partial D_i', \partial W')$ $(i=1, 2)$. Identifying points $x\in N(\partial D_1', \partial W')$ with points $h(x)\in N(u_1, \partial W)$ and points $y\in N(\partial D_2', \partial W')$ with points $h(y)\in N(C_K, \partial W)$, the resulting manifold $E'=W\underset{h}{\cup}N(D_1', W')\cup N(D_2', W')$ is a 3-manifold in S^3. Hence the Heegaard splitting $(W, W'; h)$ gives a 3-sphere S^3. Let $E=W\underset{h}{\cup}N(D_2', W')$ and $V'=S^3-\dot{E}$, which is a solid torus in S^3 and in which u_1 is a meridian such that u_1 is transverse to the knot K at only the point a. Then the knot K is isotopic to the center circle of the solid torus V' in S^3 and so we may assume that V' is a regular neighborhood of K in S^3. Let V be a solid torus, u a meridian of V, and Ψ: $\partial V\to\partial E$ a Dehn's surgery along K. Then removing the intersections between $\Psi(u)$ and $\partial E\cap N(D_2', W')$ by an isotopy in ∂E, we may assume that the intersections are empty. Thus the Dehn's surgery along K induces a Heedaard splitting $(W, W'; h')$ for M such that $h'(\partial D_1')=\Psi(u)$ and $h'(\partial D_2')=C_K$. The proof is complete.

Now we may assume that in the above Heegaard splitting $(W, W'; h')$ $\Psi(u)$

intersects each of u_1 and u_3 transversely at the intersections with the same orientations respectively.

Note that C_K gives the knot group $\pi_1(K, S^3)$, that is,

$$\pi_1(K, S^3)=\{s, t; C_K(s, t)=1\}$$

where s and t are two canonical generators for $\pi_1(W)=\{s, t; \text{free}\}$.

Corollary 1.1. *The manifold M is* (1) *irreducible* (*that is, any* 2-*spheres embedded in M bound* 3-*cells in M*) *or* (2) *the connected sum of two lens spaces. In particular, if M is a homology* 3-*sphere then M is irreducible.*

Proof. Suppose that M is not irreducible. By Theorem 1, M has a Heegaard splitting of genus two and let it denote $(W, W'; F)$ where $F=W\cap W'$. Then by Haken [2] there is a 2-sphere S^2 in M such that $S^2\cap F$ is a single simple closed curve c which is not homotopic to zero in F, since M is not irreducible. Thus M has a connected sum decomposition $M_1 \# M_2$ such that each of M_1 and M_2 have a Heegaard splitting of genus one (that is, a lens space), since the circle c is not homotopic to zero in F. The proof is complete.

Note that there happen curious cases, that is, the connected sum of two lens spaces really obtained by Dehn's surgery along torus knots, by Moser [3].

3. Reducible Heegaard splittings.

Let $(W_1, W_2; h)$ be a Heegaard splitting of genus two for a 3-manifold M and $\{D_{i1}, D_{i2}\}$ a meridian disk pair of W_i. Furthermore we take an additional 2-disk D_{i3} properly embedded in W_i such that D_{i3} is disjoint from D_{i1} and D_{i2} and that any pair of three disks, D_{i1}, D_{i2}, D_{i3} is a meridian disk pair of W_i. Let $w_k=\partial D_{1k}$ $(k=1, 2, 3)$ and $z_j=\partial D_{2j}$ $(j=1, 2, 3)$. The orientations of the circles $w_1, w_2, w_3, z_1, z_2, z_3$ and of $F=W_1\cap W_2$ are supposed to be given. Then the collection of the circles, which is called a net for the Heegaard splitting (compare with Definition 1.2.1 in [6]), gives rise to a partition of F into a set Γ of domains. The segments of the circles of the net that lie between the points where the circles intersect are called the edges of the net. A domain $U\in\Gamma$ is said to be distinguished if among the edges that form its boundary there are two a_1, a_2 belonging to a single circle. The edges a_1, a_2 are also said to be distinguished. Furthermore the Heegaard splitting (with the fixed meridian disk pair $\{D_{i1}, D_{i2}, D_{i3}\}$) is said to be W_1-reducible if it has distinguished edges belonging to w_1 or w_2 or w_3, also W_2-reducible if they belong to z_1 or z_2 or z_3, and also reducible if it is W_1-reducible or W_2-reducible.

Next let K be a 2-bridge knot and let W_1 be standardly embedded in S^3. Then K is able to be embedded in ∂W_1 by Lemma 1 such that $a=u_1\cap K$ and $b=u_2\cap K$ where u_1, u_2, a, and b are defined as in Lemma 1. The intersection between K and $w_1\cup w_2\cup w_3$ give rise to a partition of the knot K into a set of arcs. Among the arcs, there are four arcs c_{11}, c_{12}, c_{21} and c_{22} such that c_{1i} $(i=1, 2)$ contains the point a and c_{2j} $(j=1, 2)$ contains the point b. Then the next lemma follows from the assumption that the knot K is a Schubert's 2-bridge knot $\bar{K}=(\alpha, \beta)$ such that $\alpha>2$ is a positive odd integer.

Lmma 3. *$\partial c_{1i}-a$ is contained in w_2 but not $w_1\cup w_3$ and $\partial c_{2j}-b$ is contained in w_1 but not $w_2\cup w_3$.*

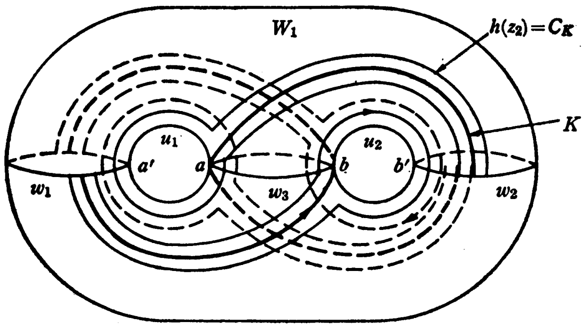

Fig. 2. The knotting circle along K.

Noting that there is an orientation preserving homeomorphism (involution) T: $W_1\rightarrow W_1$ such that $T(u_i)=u_i$, $T(w_j)=w_j$, $T(a)=a$, $T(b)=b$, and $T(C_K)=C_K$ and so we have Fig. 2 by Lemma 2 and Lemma 3.

Now let M be a 3-manifold obtained by a Dehn's surgery along the knot K and Ψ: $\partial V\rightarrow\partial E$ the matching homeomorphism. By Theorem 1, the manifold M has a Heegaard splittig $(W, W'; h')$. Then we have;

Main Theorem. *The Heegaard splitting $(W, W'; h')$ for M is not reducible.*

Proof. We change the notation as follows; let us denote $(W, W'; h')$ by $(W_1, W_2; h)$. Furthermore let $\{D_{i1}, D_{i2}\}$ be a meridian disk pair of W_i and D_{i3} the additional 2-disk in W_i and let $w_j=\partial D_{1j}$ and $z_j=\partial D_{2j}$ $(j=1, 2, 3)$. Then by Theorem 1 the Heegaard splitting $(W_1, W_2; h)$ is given by $h(z_1)=\Psi(z_1)$ and $h(z_2)=C_K$. We may assume that $E=W_1\cup N(D_{22}, W_2)$ and $V=W_2-\mathring{N}(D_{22}, W_2)$ and furthermore $\Psi(z_j)$ $(j=1, 3)$ intersects each of u_1 and u_2 transversely at the intersections with the same orientations respectively and is disjoint from $\partial E\cap N(D_{22}, W_2)$. Next let Γ be the set of domains associated with $\{D_{i1}, D_{i2}, D_{i3}\}$ given by the Heegaard splitting and let a domain $U\in\Gamma$. Then two cases happen by Lemma 2 and

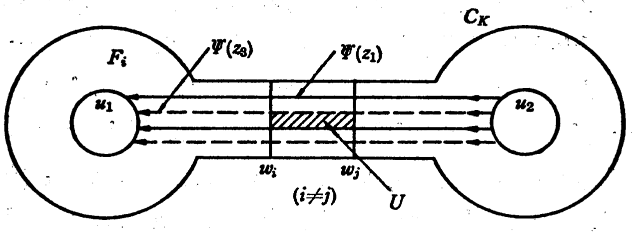

Fig. 3.

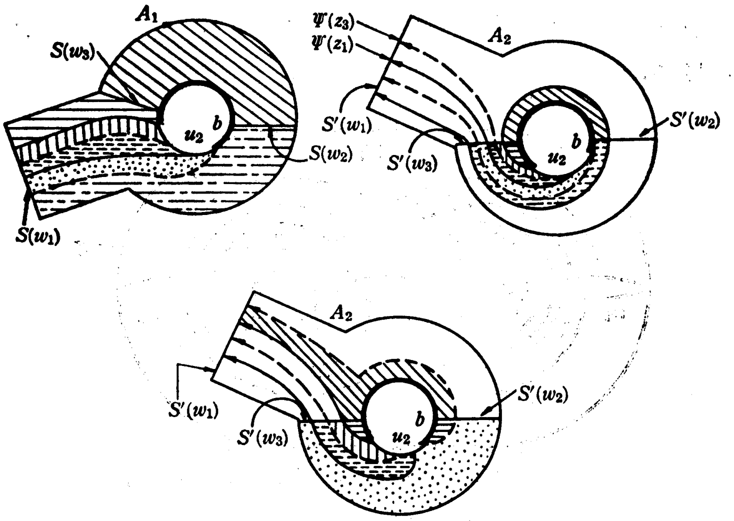

Fig. 4.

Lemma 3. Let F_1, F_2 be two 2-manifolds in ∂W_1 such that $F_1 \cup F_2 = \partial W_1$ and $F_1 \cap F_2 = C_K \cup u_1 \cup u_2$.

Case (1). The domain U is contained in one of F_1 and F_2; Then U is a rectangular domain (see Fig. 3 and Fig. 4), or a hexagonal domain (see Fig. 4). The circle u_1 (and u_2) intersects $\Psi(z_1)$ and $\Psi(z_3)$ alternatively and so U has no distinguished edges in z_1 or z_2 or z_3. By Lemma 3, it has also no those in w_1 or w_2 or w_3 (see Fig 3 and Fig. 4).

Case (2). The domain U has intersections with both of F_1 and F_2; Such the domain U is contained in one of two "knob" areas $F(u_1), F(u_2)$. We may assume that U is contained in $F(u_2)$ without loss of generality. Let $F(u_2) = A_1 \cup A_2$ where $A_1 \cap A_2 = u_2$ and A_i $(i=1,2)$ is an annulus. Then $\partial A_1 = u_2 \cup S(w_1) \cup S(C_K)$ and $\partial A_2 = u_2 \cup S'(w_1) \cup S'(C_K)$ where $S(w_1)$ and $S'(w_1)$ are arcs in w_1 and $S(C_K)$ and $S'(C_K)$ are arcs in C_K. Let $A_1 \cap w_2 = S(w_2)$, $A_1 \cap w_3 = S(w_3)$, $A_2 \cap w_2 = S'(w_2)$, and $A_2 \cap w_3 = S'(w_3)$. Now, by Takahashi [5], there is an involution T on W_1 such that $T(C_K) = C_K$, $T(\Psi(z_1)) = \Psi(z_1)$, $T(\Psi(z_3)) = \Psi(z_3)$, $T(w_1) = w_1$, $T(w_2) = w_2$, $T(w_3) = w_3$,

$T(u_1)=u_1$, $T(u_2)=u_2$, $T(a')=a'$, $T(b')=b'$, and T has six fixed points which all belong to C_K or $\Psi(z_1)$ or $\Psi(z_3)$. By Lemma 3, C_K can not contain the fixed point b' and so $(\Psi(z_1)\cup\Psi(z_3))\cap(S(w_3)\cup S'(w_3))\neq\phi$. Similarly $(\Psi(z_1)\cup\Psi(z_3))\cup(S(w_2)\cup S'(w_2))\neq\phi$. But we can assume that $\Psi(z_j)$ $(j=1,3)$ is disjoint from $S(w_2)$ and $S(w_3)$ (see Fig. 4). Hence the domain $U\in A_1\cup A_2$ is not distinguished since ∂U does not contain two edges belonging to a single circle in $\{w_1, w_2, w_3, z_1, z_2, z_3\}$ (see Fig 4). Thus the Heegaard splitting $(W_1, W_2; h)$ is not reducible. The proof is complete.

By Homma's Theorem, we have the following corollary;

Corollary 1.1. *All 3-manifolds obtained by Dehn's surgery along non-trivial 2-bridge knots are not 3-spheres.*

Note that Main Theorem is proved by the same manner as the above proof, when the net is defined without the additional 2-disks D_{13} and D_{23} (see Definition 1.2.1 in [6]).

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