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UNIQUE CONTINUATION FOR SYMMETRIC HYPERBOLIC SYSTEMS OF FIRST ORDER WITH DISCONTINUOUS COEFFICIENTS*

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ABSTRACT. The weak solutions of the Cauchy problem for symmetric hyperbolic systems of discontinuous coefficients with the following fact that the coefficients and their derivatives in distribution sense are bounded is vanishing almost everywhere in a strip domain if the weak solution is vanished at some time in the strip domain. The weak solutions of the same equation are also uniquely determined by their initial conditions.

§ 1. Introduction.

Gel'fand [3] proposed the study of existence and uniqueness of solutions to the Cauchy problem for linear hyperbolic equations whose coefficients are possibly discontinuous. Hurd and Sattinger [4] proved the existence theorem for the case of first order hyperbolic systems in several space variables and the uniquences theorem in the case of one space variable, they remarked that the uniqueness theorem may be extended to the equations in several variables, it requires some stronger assumption in place of the conditions given in the case of one space variable. Recently, Hurd [5] proved a theorem about uniqueness for weak solutions of symmetric quasilinear hyperbolic systems of one space variable and the author [7] proved the uniqueness property by exponential decay of weak solutions for hyperbolic systems of first order with discontinuous coefficients. In this note we present some conditions for the uniqueness property of the weak solutions of hyperbolic systems in several space variables in which the energy inequality mentioned in Lai [7] is available, we can also easily apply to show the existence theorem.

§ 2. Notations and preliminaries.

Most of the notations are the same as in Lai [7].

Let $x=(x_1, x_2, \cdots, x_n)$ be a point in the n-dimensional Eucliden space E^n and t a point on the real line $(-\infty, \infty)$. We denote by $\mathscr{D}$ the half space $\{(x, t);$

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$(x, t) \in E^n \times [0, \infty)\}$. The hyperplane $t=\tau$ is denoted by H_τ.

For two (vector-valued) functions $u=(u_1, u_2, \cdots, u_l)$ and $v=(v_1, v_2, \cdots, v_l)$ defined in $\mathscr{D}$, we put

$$\langle u, v\rangle = \sum_{i=1}^{l} u_i v_i \qquad \text{and} \qquad |u|^2 = \langle u, u\rangle .$$

Let $L^2(\mathscr{D})$ be the Hilbert space consisting of all measurable functions $u=(u_1, u_2, \cdots, u_l)$ for which

$$\iint_{\mathscr{D}} |u|^2 dxdt < \infty ,$$

and denote

$$(u, u) = \|u\|^2 = \iint_{\mathscr{D}} |u|^2 dxdt .$$

In the following $f \in L^2_{\text{loc}}(\Omega)$ for a function or a vector-valued function f defined on a region Ω in some Euclidean space means that f or every component of f is square integrable on every compact subset of Ω.

Consider a linear operator L of the first order of symmetric hyperbolic systems defined as follows

$$(1) \qquad Lu \equiv \frac{\partial u}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(A^i u) + Bu$$

where A^i $(i=1, 2, \cdots, n)$ are $l \times l$ symmetric matrices, B is an $l \times l$ matrix.

We say that a vector valued function $u \in L^2_{\text{loc}}(\mathscr{D})$ is a weak solution of the Cauchy problem

$$(2) \qquad Lu = 0 , \qquad u(x, 0) = \phi(x)$$

for the Cauchy data $\phi(x) \in L^2_{\text{loc}}(H_0)$ if it satisfies (2) and

$$(3) \qquad \int_0^t \int_{H_s} \left[\left\langle u, \frac{\partial \varphi}{\partial s} \right\rangle + \sum_{i=1}^{n} \left\langle A^i u, \frac{\partial \varphi}{\partial x_i} \right\rangle - \langle Bu, \varphi\rangle \right] dxds + \int_{H_0} \langle \phi(x), \varphi(x, 0)\rangle dx - \iint_{H_t} \langle u, \varphi\rangle dx = 0$$

for any $t>0$ and any $\varphi \in C^1[0, \infty; H_0^{1,2}(E^n)]$, where the space $H_0^{1,2}(E^n)$ is the closure of $C_0^\infty(E^n)$ by the norm

$$\|\varphi(\cdot, t)\|_1 = \left(\int_{E^n} \sum_{|\alpha| \le 1} |D_x^\alpha \varphi|^2 dx \right)^{1/2}$$

and the space $C^1[0, \infty; H_0^{1,2}(E^n)]$ consists of all functions φ with the following properties:

(i) φ is measurable in E^n

(ii) for almost all $t\in[0,\infty)$, the function $\varphi(x,t)$ in x belongs to $H_0^{1,2}(E^n)$ and

(iii) the norm $\|\varphi(\cdot,t)\|_1$ as a function of t belongs to $C^1([0,\infty))$.

Let $w_k(x,s)$ be the mollifier defined as in [7] and put

$$f_k(x,t)=f*w_k(x,t)=\iint_{\mathscr{D}} f(y,s)w_k(x-y,t-s)dyds\,.$$

Then one sees that f_k $(k=1,2,\cdots)$ is infinitely (many times) differentiable in $\mathscr{D}$ and f_k tends to f in the L^p-norm $(p\geq 1)$ on very compact subset of $\mathscr{D}$ as k tends to infinity. By this way, we put $A_k(s,t)=(a_{ij}^k(x,t))$ for a matrix $A(x,t)=(a_{ij}(x,t))$ with $a_{ij}\in L_{\mathrm{loc}}^2(\mathscr{D})$ and $a_{ij}^k(x,t)=a_{ij}*w_k(x,t)$. Therefore we can obtain the smoothed sequence of systems

$$L_k u\equiv\frac{\partial u}{\partial t}+\sum_{i=1}^n\frac{\partial}{\partial x_i}(A_k^i u)+B_k u=0\,. \tag{4}$$

In this note, we shall suppose that the coefficients in the system $Lu=0$ would statisfy the following conditions:

(i) The coefficients A^i, B and their first derivatives in the distribution sense are bounded in $\mathscr{D}$.

(ii) Let $\Delta t=t'-t$, $\Delta x_i=x_i'-x_i$ and $\Delta_i A^j=A^j(x_1,x_2,\cdots,x_i',\cdots,x_n,t)-A^j(x_1,x_2,\cdots,x_n,t)$,

then there exist nonnegative locally integrable functions $\nu(t)$, $\mu_i(t)$ $(i=1,2,\cdots,n)$ bounded in $[0,\infty)$ such that in $\mathscr{D}$

$$|\langle B\xi,\xi\rangle|\leq\nu(t)\langle\xi,\xi\rangle\,,\qquad\left|\left\langle\frac{\Delta_i A^i}{\Delta x_i}\xi,\xi\right\rangle\right|\leq\mu_i(t)\langle\xi,\xi\rangle$$

$(i=1,2,\cdots,n)$ for any vector ξ.

By the condition (ii) there exists a function $\mu(t)\in L_{\mathrm{loc}}^1([0,\infty))$ being bounded and satisfying

$$\left|\left\langle\left(B_k+\frac{1}{2}\sum_{i=1}^n\frac{\partial A_k^i}{\partial x_i}\right)\xi,\xi\right\rangle\right|\leq\mu(t)\langle\xi,\xi\rangle \tag{5}$$

for any ξ, where B_k, A_k^i are smoothed matrices respectively for the given matrices B and A^i (see [7; Lemma 1]).

Accordingly Lemma 3 and Lemma 4 in Lai [7], and from the Cauchy problem, we obtain

$$\begin{cases} L_{kw}\varphi\equiv\dfrac{\partial\varphi}{\partial t}+\displaystyle\sum_{i=1}^n A_k^i\frac{\partial\varphi}{\partial x_i}-B_k^*\varphi=F \\ \varphi(x,0)=\phi(x)\end{cases} \tag{6}$$

for any given $\phi \in C_0(E^n)$ and $F \in C^\infty(\mathscr{D})$ with compact support, one can obtain the energy inequalities

$$\int_{H_t} \langle\varphi, \varphi\rangle dx + \int_0^t \int_{H_s} \langle\varphi, \varphi\rangle dx ds \leq e^{2\lambda t}\left(\int_{H_0} \langle\phi, \phi\rangle dx + \int_0^t \int_{H_s} \langle F, F\rangle dx ds\right) \tag{7}$$

and

$$\begin{aligned} \int_{H_t} (|\varphi|^2 + |D^1\varphi|^2) dx + \int_0^t \int_{H_s} (|\varphi|^2 + |D^1\varphi|^2) dx ds \\ \leq C e^{4\lambda t}\left[\int_{H_0} (|\phi|^2 + |D^1\phi|^2) dx + \int_0^t \int_{H_s} (|F|^2 + |D^1 F|^2) dx ds\right] \end{aligned} \tag{8}$$

for some constant C and sufficient large number λ, where

$$D^1 u = \sum_{i=1}^n \frac{\partial u}{\partial x_i} \quad \text{and} \quad |D^1 u|^2 = \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^2 .$$

Hence for the solution φ_k of the Cauchy problem

$$\begin{cases} L_{kw}\varphi = F \\ \varphi(x, 0) = 0 \end{cases} \tag{9}$$

with the given function $F \in C^\infty(\mathscr{D})$ having the compact support must have the properties that the integrals

$$\int_0^t \int_{H_s} \langle\varphi_k, \varphi_k\rangle dx ds \quad \text{and} \quad \int_0^t \int_{H_s} \left\langle \frac{\partial\varphi_k}{\partial x_i}, \frac{\partial\varphi_k}{\partial x_i} \right\rangle dx ds \tag{10}$$

are uniformly bounded (independent of k). The solution φ_k of the Cauchy problem (9) sufficiently smooth and has compact support, one can choose such a function as the test function.

§ 3. The main theorems.

Theorem 1. *Let Γ be the differential operator defined in* (1) *and satisfying the conditions* (i) *and* (ii) *in* § 2. *Suppose that u is a weak solution of the Cauchy problem*

$$\begin{cases} Lu = 0 \\ u(x, 0) = \phi(x) \end{cases} \tag{11}$$

$\phi(x) \in L^2_{\mathrm{loc}}(H_0)$ and that

$$u(x, T) = 0 \tag{12}$$

in $\Omega = E^n \times [0, T]$. Then the weak solution $u \in L^2_{\mathrm{loc}}(\Omega)$ of (11) *vanishes almost everywhere in Ω.*

Proof. Let $u\in L^2_{loc}(\Omega)$ be a weak solution of (11) satisfying $u(x, T)=0$ for large T for which a given function F has compact support in $E^n\times(0, T)$.

Let φ_k $(k=1, 2, \cdots)$ be the solution of

$$L_{kw}\varphi\equiv\frac{\partial\varphi}{\partial t}+\sum_{i=1}^{n}A_k^i\frac{\partial\varphi}{\partial x_i}-B_k^*\varphi=F(x, t)$$

(13) $$\varphi(x, 0)=0\ ,$$

$0<t<T$, where A_k^i and B_k^* are the smoothed matrices associated with the matrices A^i and the transpose matrix B^* of B respectively. Evidently, $\varphi_k\in C_0^1((\Omega)$ and so

$$\begin{aligned}\iint_\Omega\langle u, F\rangle dxdt&=\iint_\Omega\left\langle u, \sum_{i=1}^{n}A_k^i\frac{\partial\varphi_k}{\partial x_i}-B_k^*\varphi_k\right\rangle dxdt+\iint_\Omega\left\langle u, \frac{\partial\varphi_k}{\partial t}\right\rangle dxdt\\&=\int_0^T\int_{H_t}\left\langle u, \sum_{i=1}^{n}A_k^i\frac{\partial\varphi_k}{\partial x_i}-B_k^*\varphi_k\right\rangle dxdt\\&\quad-\int_0^T\int_{H_t}\left\langle u, \sum_{i=1}^{n}A^i\frac{\partial\varphi_k}{\partial x_i}-B^*\varphi_k\right\rangle dxdt\\&\quad-\int_{H_0}\langle u(x, 0), \varphi_k(x, 0)\rangle dx+\int_{H_T}\langle u(x, T), \varphi_k(x, T)\rangle dx\end{aligned}$$

since $u\in L^2_{loc}(\Omega)$ is a weak solution of (11). Since $\varphi_k(x, 0)=0$ and $u(x, T)=0$ in Ω, we obtain

(14) $$\iint_\Omega\langle u, F\rangle dxdt=\int_0^T\int_{H_t}\left\langle u, \sum_{i=1}^{n}(A_k^i-A^i)\frac{\partial\varphi_k}{\partial x_i}\right\rangle dxdt+\int_0^T\int_{H_t}\langle u, (B^*-B_k^*)\varphi_k\rangle dxdt\ .$$

As $A_k^i\to A^i$ and $B_k^*\to B^*$ in L^2norm on any compact set when $k\to\infty$ and the integrals (10) are uniformly bounded (independent of k), the limit of identity (14) tends to zero when $k\to\infty$, i.e.

$$\iint_\Omega\langle u, F\rangle dxdt=0\ .$$

Since F is arbitrary function in $C^\infty(\mathscr{D})$ with compact support, we obtain

$$u\equiv0\ \text{ a.e. in any compact subset of }\ E^n\times[0, T]\ .$$

This proves the theorem. Q.E.D.

Under the assumptions (i) and (ii) in §2, one sees that the conditions I through out VI of Hurd and Sattinger [4] hold. Hence the existence theorem of weak solutions of (11) holds and then the weak solution in Theorem 1 is actually existed. One can show also sxistence theorem by applying the energy inequalities (7) and (8).

Theorem 1 is a uniqueness theorem from backward continuation. The analogue uniqueness theorem also can be determined by their initial conditions in several space variables under the same conditions on the discontinuous coefficients. It is an essential reason to have that the mean square norms of φ_k and $\sum_{i=1}^{n} \partial\varphi_k/\partial x_i$ are uniformly bounded on compact subsets of $\mathscr{D}$. We will show the following theorem.

Theorem 2. *Suppose that the coefficients of* (1) *satisfies the conditions* (i) *and* (ii) *in* § 2. *Then the weak solutions of the Cauchy problem of the symmetric hyperbolic systems*

$$
(11) \qquad \begin{cases} \dfrac{\partial u}{\partial t}+\sum_{i=1}^{n}\dfrac{\partial}{\partial x_i}(A^k u)+Bu=0 \\ u(x, 0)=\phi(x) \end{cases}
$$

$\phi(x)\in L^2_{loc}(H_0)$, *are uniquely determined by their initial condition.*

Proof. Let $u_1(x, t)$ and $u_2(x, t)$ be two weak solutions of (11) with the same initial condition $\phi(x)$. We will show that if the vector function $F(x, t)\in C^\infty(\mathscr{D})$ with compact support contained in $t>0$, then

$$
\iint_{\mathscr{D}} \langle u_1-u_2, F\rangle dxdt=0
$$

and hence proves that $u_1=u_2$ a.e. in $\mathscr{D}=E^n\times[0, \infty)$.

By the same arguments as in § 2, we will assume that w_k is the standard mollifier with support in $|x|^2+t^2\leq 1/k^2$ and hence, it can be constructed for each $k=1, 2, \cdots$, the vector valued functions $\varphi_k(x, t)=w_k*\varphi(x, t)$, with $\varphi(x, t)\in L^2_{loc}(\mathscr{D})$, satisfying the linear system

$$
(15) \qquad \begin{cases} \dfrac{\partial\varphi_k}{\partial t}+\sum_{i=1}^{n} A_k^i\dfrac{\partial\varphi_k}{\partial x_i}-B_k^*\varphi_k=F \\ \varphi_k(x, t)=0 \end{cases}
$$

where T can be chosen so that the support of F is assumed to be below $t=T$, i.e. $\operatorname{supp} F\subset E^n\times[0, T]$ for large T. This is achieved by solving the system

$$
\frac{\partial\tilde{\varphi}_k(x, t)}{\partial t}-\sum_{i=1}^{n} A_k^i(x, T-t)\frac{\partial\tilde{\varphi}_k}{\partial t}(x, t)+B_k^*(x, T-t)\tilde{\varphi}_k(x, t)=-F(x, T-t)
$$

for the vector $\tilde{\varphi}_k(x, t)$ in $\mathscr{D}$ with initial conditions

$$
\tilde{\varphi}_k(x, 0)=0
$$

and then putting $\varphi_k(x, t)=\tilde{\varphi}(x, T-t)$. By classical existence theory guarantees that the solution $\tilde{\varphi}_k(x, t)$ exists, so does the solution $\varphi_k(x, t)$ of (15) and it is differentiable in any order, and has support contained in a compact set which is independent of k. Hence $\varphi_k(x, t)$ can be as a legitimate test function. By the definition of weak solution for the Cauchy problem, we obtain

$$\begin{aligned}\iint_{\mathscr{D}}\left\langle u_1-u_2, \frac{\partial\varphi_k}{\partial t}\right\rangle dxdt &= -\iint_{\mathscr{D}}\sum_{i=1}^{n}\left\langle A^i(u_1-u_2), \frac{\partial\varphi_k}{\partial x}\right\rangle dxdt+\iint_{\mathscr{D}}\langle B(u_1-u_2), \varphi_k\rangle dxdt\\ &\quad+\int_{H_T}\langle u_1-u_2, \varphi_k(x, T)\rangle dx-\int_{H_0}\langle\psi(x)-\psi(x), \varphi_k\rangle dx\\ &= -\iint_{\mathscr{D}}\left\langle u_1-u_2, \sum_{i=1}^{n}A^i\frac{\partial\varphi_k}{\partial x_i}\right\rangle dxdt+\iint_{\mathscr{D}}\langle u_1-u_2, B^*\varphi_k\rangle dxdt\,.\end{aligned}$$

Thus

$$\iint_{\mathscr{D}}\langle u_1-u_2, F\rangle dxdt=\iint_{\mathscr{D}}\left\langle u_1-u_2, \sum_{i=1}^{n}(A_k^i-A^i)\frac{\partial\varphi_k}{\partial x_i}+(B^*-B_k^*)\varphi_k\right\rangle dxdt\,. \tag{16}$$

Since for each k, $\operatorname{supp}\varphi_k$ lies in a fixed compact subset of $\mathscr{D}$, u_1 and u_2 are locally in $L^2(\mathscr{D})$ and the coefficients A_k^i and B_k^* (transpose of B_k) converge in L^2-norm on compact subsets of $\mathscr{D}$ to the coefficients A^i and B^* respectively, one sees immediately that the right hand side of (16) approaches zero as $k\to\infty$ if we can show that

$$\iint_{\mathscr{D}}\langle\varphi_k, \varphi_k\rangle dxdt \qquad\text{and}\qquad \iint_{\mathscr{D}}\sum_{i=1}^{n}\left\langle\frac{\partial\varphi_k}{\partial x_i}, \frac{\partial\varphi_k}{\partial x_i}\right\rangle dxdt$$

are uniformly bounded (independent of k) on compact subsets of $\mathscr{D}$. Fortunately, it follows from the energy inequalities mentioned in Lai [7; Lemmas 3 and 4]. Hence the proof is completed. Q.E.D.

References

[1] E. D. Conway: *Generalized solutions of linear equation with discontinuous coefficients and uniqueness question by multidimensional quasilinear conservative laws*, J. Math. Anal. Appl., **18**, (1967) 238-251.

[2] R. Courant and D. Hilbert: Methods of Mathematical Physics vol. II, Interscience, New York 1962.

[3] I. M. Gel'fand, *Some questions of analysis and differential equations.* Uspehi Mat. Nauk 14 (1959), **3** (87), 3-19; A.M.S. Transl., (2) **26** (1963), 201-219.

[4] A. E. Hurd and D. H. Sattinger: *Question of existence and uniqueness for hyperbolic equations with discontinuous coefficients.* Trans. Amer. Math. Soc., 132 (1968).

[5] A. E. Hurd, *A uniqueness theorem for weak solutions of symmetric quasilinear hyperbolic systems.* Pacific J. Math., **28** (1969), 555-559.

[6] T. Kuroda: *Note on the uniqueness property of weak solutions of parabolic equations.* Nagoya Math. J., **29** (1967), 45-49.

[7] H. C. Lai: *Exponential decay of weak solutions for hyperbolic systems of first order with discontinuous coefficients.* Trans. Amer. Math. Soc., **170** (1972), 425-436.

[8] S. L. Sobolev: *Application of Functional Analysis in Mathematical Physics*. Transl. of Math. Monograph Vol. 7, Amer. Math. Soc. Providence Phode Island 1963.

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