YOKOHAMA MATHEMATICAL
JOURNAL VOL. 26, 1978

GENERAL POSITIONING IN A MAPPING CYLINDER

By

H. W. BERKOWITZ and PRABIR ROY

(Received April 4, 1977)

1. Introduction. In [2] and [3] Homma developed some techniques for modifying piecewise linear (P.L.) mappings from one combinatorial manifold into another. In [1] a counter-example to Theorem 2 of [2] is described. The principal result of the present paper is a weaker version of Theorem 2 of [2].

For the Basic definitions of P.L. topology the reader is referred to Hudson [4]. Some other definitions follow.

E^n denotes n-dimensional Euclidean space. If $a, b \in E^n$, $[a, b]$ denotes the closed line in E^n between a and b.

If K is a complex and L is a subcomplex of K, then $St_K(L)=\bigcup\{A \in K\colon A\cap |L|\neq\phi\}$, where $|L|=\bigcup\{x\colon x\in A\in L\}$. $A<B$ for two simplices A and B means A is a *face* of B.

If P is a polyhedron, Q a subpolyhedron of P, and T a triangulation of P in which Q is triangulated, then $T|Q=\{A\in T\colon A\subset Q\}$.

All manifolds in this paper are compact combinatorial manifolds. If M is a manifold, the interior of M is denoted $\operatorname{Int}(M)$ or $\operatorname{Int} M$.

Let M be an m-manifold, L a triangulation of M. For each vertex $v_i\in L$, $St_L(v_i)$ is a combinatorial m-ball; let $\theta_i\colon St_L(v_i)\to B^m$ be a P.L. homeomorphism from $St_L(v_i)$ onto the standard m-simplex. If K is a complex and $g\colon |K|\to\operatorname{Int}(M)$, where g is a P.L. mapping, then g is *semi-simplicial* iff for each $A\in K$, there exists a vertex v_i of L such that $g(St_K(A))\subset\operatorname{Int}(St_L(v_i))$ and $\theta_i g|St_K(A)$ is a linear mapping of $St_K(A)$ into $\operatorname{Int}(B^m)$.

Let $g\colon K\to E^m$ be a semi-simplicial mapping of a complex K into E^m, i.e., for $A\in K$, $g|A$ is linear. g is in *general position* if for any collection of vertices $\{w_0, \cdots, w_r\}$, $r\leq m$, of K, $\{g(w_0), \cdots, g(w_r)\}$ spans an r-dimensional hyperplane in E_m. If g is in general position and $A_1, A_2\in K$, then $\dim g(\operatorname{Int}(A_1)\cap\operatorname{Int}(A_2))\leq\dim A_1+\dim A_2-m$.

Let $\{v_0, \cdots, v_s, \cdots, v_t\}\subset E^r$. Let the hyperplane spanned by $\{v_0, \cdots, v_s\}$ be of dimension p, $p\leq r$. Let $\{w_0, \cdots, w_p\}$ be $p+1$ linearly independent points in $\{v_0, \cdots, v_s\}$. Then $\{v_0, \cdots, v_s, \cdots, v_t\}$ is in *general position with respect to* (g.p.w.r.t.) $\{v_0, \cdots, v_s\}$, if $\{w_0, \cdots, w_p, v_{s+1}, \cdots, v_t\}$ is in general position in E^r. "With respect to" is abbreviated by "w.r.t."

If D is a hyperplane in E^r, $\{v_1, \cdots, v_t\}\subset E^r$, and $\{w_0, \cdots, w_s\}$ is any linearly independent set of points spanning D, then $\{v_1, \cdots, v_t, D\}$ is in g.p.w.r.t. D, if $\{v_1, \cdots, v_t, w_0, \cdots, w_s\}$ is in general position in E^r.

If $\{w_1, \cdots, w_s\}\subset E^r$, its convex closure is denoted by $\langle w_1, \cdots, w_s\rangle$. If $A_1, \cdots, A_t$ are convex subsets of E^r, the convex closure of $\{w_1, \cdots, w_s\}\cup A_1\cup\cdots\cup A_t$ is denoted by $\langle w_1, \cdots, w_s, A_1, \cdots, A_t\rangle$.

If P, Q are polyhedra and f: $P\to Q$ is a P.L. mapping, then a point $x\in P$ such that $f^{-1}f(x)\neq x$ is a *singular point of* f; the closure of the set of singular points of f is the *singular set of* f and is denoted S_f.

Let K, L be triangulations of P, Q such that f: $K\to L$ is simplicial. Let L_b be the barycentric subdivision of L and K_f a subdivision of K such that K_f is isomorphic to K_b, and f: $K_f\to L_b$ is simplicial. The *mapping cylinder of* f, C_f, is formed as follows. For b a vertex in K_f, let $B(b)$ be the simplex of K such that $b\in \operatorname{Int}(B(b))$; similarly if a is a vertex in L_b, let $A(a)$ be the simplex of L_b such that $a\in\operatorname{Int}(A(a))$. The vertices of C_f are those of L_b plus those of K_f. $\langle b_0, b_1, \cdots, b_i, a_{i+1}, \cdots, a_j\rangle$ is a simplex of C_f iff $B(b_0)>B(b_1)>\cdots>B(b_i)$ and $f(B(b_i))>A(a_{i+1})>\cdots>A(a_j)$. By an obvious identification L_b, $K_f\subset C_f$ so that $Q, P\subset|C_f|$. Define the onto simplicial mapping p_f: $C_f\to L_b$ by $p_f(b)=f(b)$ for b a vertex of K_f, and $p_f(a)=a$ for a a vertex of L_b.

Given any triangulation T of $|C_f|$, there exists a refinement T' of T such that p_f: $T'\to T'||L|$ is simplicial. T' is called a *cylindrical subdivision* of T; $p_f(|C_f|)$ is the *base of* C_f. For $x\in$ the base of C_f, call $p_f^{-1}(x)$ the *fibre over* x. If $A\in T'$, then $p_f^{-1}(x)\cap A$ is the *fibre over* x *in* A.

Let Π be a P.L. mapping of a polyhedron F onto a polyhedron G such that for each $x\in G$, $\Pi^{-1}(x)$ is collapsible. The triple $\{F, G, \Pi\}$ is called a *semi-forest*. If for each $x\in G$, diameter $\Pi^{-1}(x)<\varepsilon$, then $\{F, G, \Pi\}$ is called an *ε-semi-forest*. These notions are due to Homma [3].

2. General Positioning in a Mapping Cylinder. Consider the P.L. mapping f: $M\to\operatorname{Int}(N)$ of a combinatorial closed m-manifold into the interior of a combinatorial n-manifold, where $m\leq 3n/4-5/4$, and $n\geq 4$. Assume M has a triangulation L and N has a triangulation K, and that f is in general position and is semi-simplicial with respect to these two triangulations. Let $P=S_f$, the singular set of f, and $Q=f(P)$. Let $|C_f|$ be the mapping cylinder associated with $f|P$: $P\to Q$.

The mapping g. *There exists a semi-simplicial mapping* g: $C_f\to M$, *where* C_f

is a cylindrical triangulation, satisfying:

(i) *if* $A, B \in C_f$ *with* $g(A) \cap g(B) \neq \phi$, *then there exists a vertex* $v \in L$ *with* $g(A) \cup g(B) \subset \operatorname{Int}(St_L(v))$;

(ii) g *is in general position*;

(iii) $g(x)=x$, *for each* $x \in P$.

It is easy to show that:

Lemma 1. $\dim C_f \leq n/2-2a+1$, *where* $a \geq 5/4$.

Corollary 1.1. $\dim S_g \leq n/4-3a+2$, *where* $a \geq 5/4$.

Lemma 2. *Let* $A \cup B$ *be a complex consisting of two principal simplices* A *and* B. *Let* A_1 *be a* 1-*dimensional face of* A, *with* $\dim(A_1 \cap B) \leq 0$. *Let* ϕ: $|A \cup B| \to E^m$ *be a mapping which is linear on* A *and* B, *and is in general position, where* $\dim A \leq m$, $\dim B \leq m-1$. *Then given* $\varepsilon > 0$, *there exists a simplicial mapping* ϕ': $A \cup B \to E^m$, *in general position, where* $d(\phi, \phi') < \varepsilon$, *such that, if* $x, y \in E^m$ *and the line segment between* x *and* y, $[x, y]$, *is parallel to* $\phi'(A_1)$, *then not both* x *and* y *lie in* $\phi'(B)$.

The proof is straightforward.

By Lemma 2, g can be approximated by a semi-simplicial mapping g': $C_f \to M$, satisfying (i), (ii), (iii), and (iv$'$), where

(iv$'$) *if* $A, B \in C_f$, *where* B *is a* 1-*dimensional simplex*, $p_f(B)=$point, *and* $\dim(A \cap B) \leq 0$, *then any line parallel to* $g'(B)$ *intersects* $g'(A)$ *in at most one point.*

Note that in order to construct g', $\dim C_f$ must be $\leq m-1$; this follows from Lemma 1.

Assume g satisfies (iv$'$).

Lemma 3. g: $|C_f| \to M$ *has the following two properties, the sum of which are called* (iv).

(iv$_a$) *For any* $A, B \in C_f$, *if* $p_f|(A \cap B)$ *is a homeomorphism and if there is* $x \in A$ *such that* $g(x) \in (g(A) \cap g(B)) - g(A \cap B)$ *then there is no* $y \in A$, $y \neq x$, *with* $g(y) \in (g(A) \cap g(B)) - g(A \cap B)$ *and* $p_f(x)=p_f(y)$.

(iv$_b$) *For any* $A, B \in C_f$, *if there exists a* 1-*dimensional face* X *of* $A \cap B$ *with* $p_f(X)=$point, *and if* $x, y \in A-(A \cap B)$; $x', y' \in B-(A \cap B)$; *with* $g(x)=g(x')$, $g(y)=g(y')$, *and* $p_f(x)=p_f(y)$ *then* $p_f(x')=p_f(y')$.

Proof. (iv$_a$) follows from (iv$'$). To prove (iv$_b$) note that if $A \in C_f$ and $A_1 < A$, $\dim(A_1)=1$, with $p_f(A_1)=$point in base of C_f, then for any line segment $e \subset A$, where e is parallel to A_1, $p_f(e)$=point in base of C_f. Now $[x, y]$ is parallel to X in A, so $[g(x), g(y)]$ is parallel to $g(X)$. Since $[g(x), g(y)]=[g(x'), g(y')]$, $[g(x'), g(y')]$ is also parallel to $g(X)$, hence $[x', y']$ is parallel to X. Therefore $p_f(x')=p_f(y')$.

Thus if $A, B \in C_f$, and a fibre, $g(e)$, in $g(A)$ meets $g(B)$, then $g(e) \cap g(B)$ is contained in a fibre in $g(B)$ and consists of a point or a closed interval.

If $H \subset E^m$, then denote the hyperplane spanned by H, by $D(H)$.

Lemma 4. *Let G: $|K| \to E^m$ be a semi-simplicial mapping, in general position, from a complex K into E^m, where $m=3n/4-a$; $\dim K \leq n/2-2a+1$, where $a \geq 5/4$. Then G can be approximated by a semi-simplicial mapping G': $|K| \to E^m$, which is in general position and is at most 2-to-1.*

Proof. It is well known that the vertices of the image can be moved slightly and the resulting map will still be in general position. Let $A, B, C \in K$. $\dim((G(A) \cap G(B))-G(A \cap B)) \leq n/4-3a+1$. Thus $\dim D(G(A) \cap G(B)) \leq n/4-3a+1$. There are 2 cases.

(I) $C \cap (A \cup B)=\phi$. Then the image of C can be adjusted without moving the vertices of $A \cup B$. $\dim D(G(A) \cap G(B))+\dim C-m<-1$.

(II) $C \cap (A \cup B) \neq \phi$. It can be assumed that all the vertices of C are in $A \cup B$. Let $C_A=C \cap A$, $C_B=C \cap B$. Assume $\dim C_A \geq \dim C_B$, then $\dim C_B \leq (n/2-2a+1)/2$. Thus $G|(A \cup C_B)$ is a homeomorphism since $\dim A+\dim C_B-m \leq -1$.

Therefore G' can be constructed. Now assume g satisfies (i), (ii), (iii), (iv), and is 2-to-1.

Notation for the General Positioning in C_f. Let S_g be the singular set of g: $|C_f| \to M$. Let C'_f be a complex identical to C_f; let id: $|C'_f| \to |C_f|$ be the identity map. Let $x'=id^{-1}(x)$, for $x \in C_f$; and $A'=id^{-1}(A)$, for $A \subset |C_f|$. Let R_g be a polyhedron homeomorphic to S_g; let r: $R_g \to S_g$ be the homeomorphism. Let ϕ: $R_g \to S'_g$ be the homeomorphism satisfying $\phi(r^{-1}(x))=x'$. For $x \in$ base of $C_f(C'_f)$, denote the fibre over x by $F_x(F'_x)$.

Let $H=\{A \cap S_g\colon A \in C_f\}$. Let T_0 be a triangulation of S_g which defines H, is extendible to a subdivision of C_f, and such that $g|T_0$ is simplicial into some triangulation of M. Let T'_0 and RT_0 be the corresponding triangulations of S'_g and R_g.

Because g is at most 2-to-1 and $g|T_0$ is simplicial into some triangulation of M, for $A \in T_0$, either no other points of S_g are mapped by g into $g(\text{Int } A)$, or there exists a unique $A' \in T_0$ such that $g(A)=g(A')$.

Let T_1 be the barycentric subdivision of T_0, and T_2 the barycentric subdivision of T_1. Let T'_1, T'_2 and RT_1, RT_2 be the corresponding subdivisions of S'_g and R_g.

For K a complex, denote by K^i the i-th skeleton of K.

Let $Q_g=\{x\in|C_f|: g^{-1}g(x)\neq x\}$, $L_g=S_g-Q_g$. Let $p=\text{Max}\{i: L_g\cap\text{Int}(A)\neq\emptyset, A\in C_f^i\}$, $q=\text{Max}\{i: Q_g\cap\text{Int}(A)\neq\emptyset, A\in C_f^i\}$.

Lemma 5. $q>p$.

Proof. If $A\in C_f$ and $\text{Int}\,A\cap L_g\neq\phi$, then there exists $A_1, A_2\in C_f$ with $A\subset A_1\cap A_2$ and $g(A_1)\cap g(A_2)-g(A_1\cap A_2)\neq\phi$ iff $\dim A_1+\dim A_2-m-\dim(A_1\cap A_2)\geq 1$. In this case $A_1\cap A_2\subset L_g$, and $\dim(A_1\cap A_2)\leq\dim A_1+\dim A_2-m-1\leq 2(n/2-2a+1)-(3n/4-a)-1=n/4-3a+1$, or $p\leq n/4-3a+1$.

If $A_1\in C_f$ and $A_1\cap Q_g\neq\phi$, then there exists $A_2\in C_f$ such that $\dim A_1+\dim A_2-m\geq 0$, or $\dim A_1\geq m-\dim A_2$. Now $\dim A_1\geq m-(n/2-2a+1)=(3n/4-a)-(n/2-2a+1)=n/4+a-1$, or $q\geq n/4+a-1$. Thus $q-p\geq(n/4+a-1)-(n/4-3a+1)=4a-2>0$ since $a>5/4$.

Notation. Let $S_g^i=S_g\cap|C_f^i|$.

Lemma 6. *Let A be a simplex, G: $A\to E^r$ a linear homeomorphism, $A'<A$. Let K_b be the barycentric subdivision of A. Let G': $|K_b|\to E^r$ such that*

(i) *$B\in K_b$ implies $G'|B$ is linear*

(ii) *$G'|A'=G|A'$.*

Let $A\geq A_0>A_1>\cdots>A_s>A'$ with v_i the barycenter of A_i. The G' is a linear homemorphism on $\langle v_0, v_1, \cdots, v_s\rangle * A'$.

The proof is straightforward.

Lemma 7. *Let $A\in C_f^i$, $B\in T_1$ or T_2 such that $B\cap\text{Int}(A)\neq\phi$. Then there exists $A_t<A_{t-1}<\cdots<A_1=A$ such that $B\cap\text{Int}\,A_j\neq\phi$; and if $A'<A$ with $A'\neq A_j$ for $j=1,\cdots,t$, then $B\cap\text{Int}\,A'=\phi$.*

The proof follows from the way in which a barycentric subdivision divides a complex.

Corollary 7.1. *Let $A\in C_f^i$, $B\in T_1$ or T_2 such that $B\cap\text{Int}\,A\neq\phi$. Let $j<i$, then B meets at most one j-dimensional face of A.*

Let $A\subset|C_f|$. The *shadow of A in $|C_f|$* is the set $\{x\in|C_f|: p_f(x)\in p_f(A)\}$, and is denoted by $\text{Shad}(A)$. If $A\subset C_f'$, $\text{Shad}(A)$ is similarly defined in C_f'.

For each vertex $v_j\in T_2$, where $v_j\in\text{Int}(A_j)$, $A_j\in C_f'$, there exists a closed ball U_j centered at v_j, $U_j\subset\text{Int}(A_j)$, and this collection of U_j's satisfies: if F: $S_g'\to|C_f'|$ such that $F(v_j)\in U_j$, and F is the linear extension of this vertex map, then there exists an ambient isotopy H_t: $|C_f'|\to|C_f'|$, such that $H_0(x)=F(x)$ for all $x\in S_g'$, and $H_1(x)=x$ for all $x\in|C_f'|$. Furthermore $H_t(A)=A$ for all $t\in[0, 1]$ and $A\in C_f'$. The notation for v_j and U_j will be used in the following.

Proposition 8. *There exists ϕ': $R_g \to |C'_f|$, a* P.L. *homeomorphism of R_g into $|C'_f|$, ambient isotopic to ϕ in C'_f by an isotopy which moves no point more than ε, for any given $\varepsilon > 0$, and satisfying:*

(1) $\phi'|R^i_g = \phi|R^i_g$ *for* $0 \leq i \leq q-1$.

(2) *For each vertex* $v_j \in RT_2$, $\phi'(v_j) \in U_j$.

(3) *If* $B \in RT_2 - RT^p_2$, *and* $C \in RT_2$, *if* $v_1, \cdots, v_t$ *are the vertices of* B *in* $RT_2 - RT^p_2$ *such that* $\{v_1, \cdots, v_t\} \cap C = \phi$, *if* $A \in C'_f$ *such that* $\phi(B) \subset A$, *and if* B' *is the face of* B *opposite* $\{v_1, \cdots, v_t\}$, *then* $\{\phi'(v_1), \cdots, \phi'(v_t), \langle(\text{Shad}(\phi(C)) \cap A), \phi(B')\rangle\}$ *is in* g.p.w.r.t. $\langle(\text{Shad}(\phi(C)) \cap A), \phi(B')\rangle$.

The proof of this follows a lengthy construction.

If $B, C \in T_2$ and $v_1, \cdots, v_t$ are the vertices of B which are not in C, and $v_i \notin S^{q-1}_g$ for $1 \leq i \leq t$, then $v_1, \cdots, v_t$ are the *free vertices of B w.r.t. C.*

Construction of the ϕ_i. For $0 \leq i \leq q-1$, define ϕ_i: $R^i_g \to |C'_f|$ by $\phi_i = \phi|R^i_g$. For $q \leq i \leq \dim C_f$, ϕ_i will be constructed inductively. The final ϕ_i will be ϕ'.

For $q \leq i \leq \dim C_f$, ϕ_i will satisfy:

(1_i) $\phi_i|R^{i-1}_g = \phi_{i-1}$.

(2_i) For each vertex $v_j \in RT^i_2$, $\phi_i(v_j) \in U_j$.

(3_i) If $B \in RT^i_2 - T^{i-1}_2$, and $C \in RT^i_2$, if $v_1, \cdots, v_t$ are the vertices of B in $RT^i_2 - RT^{i-1}_2$ such that $\{v_1, \cdots, v_t\} \cap C = \phi$, if $A \in (C'_f)^i$ such that $\phi(B) \subset A$, and if B' is the face of B opposite $\langle v_1, \cdots, v_t\rangle$, then $\{\phi_i(v_1), \cdots, \phi_i(v_t), \langle(\text{Shad}(\phi_i(C)) \cap A), \phi_i(B')\rangle\}$ is in g.p.w.r.t. $\langle \text{Shad}(\phi_i(C)) \cap A), \phi_i(B')\rangle$.

The main construction. For each i, $q \leq i \leq \dim C_f$, we prove:

Proposition 8$_i$. *Given ϕ_j: $R^j_g \to |C'_f|$, a* P.L. *homeomorphism, ambient isotopic to $\phi|R^j_g$ in $|C_f|$, and satisfying* (1_j), (2_j), *and* (3_j), *where $0 \leq j \leq i-1$, and given $\varepsilon > 0$, such that $d(\phi_j, \phi|R^j_g) < \varepsilon$, then there exists ϕ_i: $R^i_g \to |C_f|$ satisfying* (1_i), (2_i), (3_i), *and $d(\phi_i, \phi|R^i_g) < \varepsilon$.*

Proof. Let $\{v_k\}^J_1$ be the vertices of $RT^i_2 - RT^{i-1}_2$. Let $W_k = U_k - N(k, \varepsilon)$ where $N(k, \varepsilon)$ is the closure of the ε-ball centered at $\phi(v_k)$ for $1 \leq k \leq J$.

Consider the set of ordered pairs of simplices $\{(B_1, C_1), \cdots, B_R, C_R)\}$, where $B_s \in RT^i_2 - RT^{i-1}_2$, $C_s \in RT^i_2$, and where B_s has free vertices w.r.t. C_s which are contained in $RT^i_2 - RT^{i-1}_2$. For each such pair (B_s, C_s) r_s: $R^i_g \to |C'_f|$ is constructed which satisfies:

(1) r_s is simplicial on RT_2.

(2) For each vertex $v \in RT_2$, $r_s(v) = r_{s-1}(v)$, except when v is a free vertex of B_s w.r.t. C_s in $RT^i_2 - RT^{i-1}_2$. For such a free vertex, v_k, $r_s(v_k) \in B_k$. Define r_0 to be equal to ϕ_{i-1} on R^{i-1}_g, and to be equal to ϕ on the vertices of $RT^i_2 - RT^{i-1}_2$.

(3) If $v_1, \cdots, v_t$ are the free vertices of B_k w.r.t. C_k in $RT_2^i - RT_2^{i-1}$ and $\phi(B_k) \subset A$ where $A \in (C_f')^i$, then $\{r_s(v_1), \cdots, r_s(v_t), \langle \text{Shad}\,(r_s(C_k) \cap A), r_s(A') \rangle\}$ is in g.p.w.r.t. $\langle (\text{Shad}\,(r_s(C_k)) \cap A), r_s(B') \rangle$, where B' is the face of opposite $\langle v_1, \cdots, v_t \rangle$, and $1 \leq k \leq s$.

Let $\phi_i = r_R$.

The construction of r_1 follows, the construction of r_j, where $2 \leq j \leq s$ is similar.

Let $v_1, \cdots, v_t$ be the free vertices of B_1 w.r.t. C_1 which are in $RT_2^i - RT_2^{i-1}$. Let $\phi(C_1) \subset A$ where $A \in (C_f')^i$; thus $\{\phi(v_1), \cdots, \phi(v_t)\} \subset \text{Int}\, A$. Let B' be the face of B_1 opposite $\langle v_1, \cdots, v_t \rangle$.

Let $P_0 = \langle r_0(B'), \text{Shad}\,(r_0(C_1)) \cap A \rangle$. P_0 determines a hyperplane in A, denoted by $D(P_0)$.

Note $\dim P_0 < \dim B_1 + \dim\,(\text{Shad}\,(r_0(C_1)) \cap A) + 1$. In order to show that $\dim B_1 + \dim\,(\text{Shad}\,(r_0(C_1)) \cap A) - \dim A \leq -1$ note that $\dim A = n/2 + 2a + 1 - b$, where $a \geq 5/4$, $b \geq 0$. $\dim B_1 \leq \dim\,(S_g' \cap A) \leq \dim A + \dim C_f' - m \leq n/4 - 3a + 2 - b$. Thus $\dim\,(\text{Shad}\,(r_0(C_1)) \cap A) \leq n/4 - 3a + 3 - b$, and $\dim A_1 + \dim\,(\text{Shad}\,(r_0(C_1)) \cap A) - \dim A \leq -4a + 4 - b \leq -4a + 4 \leq -1$.

Thus $D(P_0)$ does not fill up W_1, so there exists $y_1 \in W_1 - D(P_0)$. Let $P_1 = \langle y_1, P_0 \rangle$. If $t > 1$, there exists $y_2 \in W_2 - D(P_1)$. By induction $y_k \in W_k - D(P_{k-1})$ and $P_k = \langle y_k, P_{k-1} \rangle$ can be obtained for $2 \leq k \leq t$.

Now $\{y_1, \cdots, y_t, P_0\}$ is in g.p.w.r.t. P_0. Let $r_1(v_k) = y_k$ for $1 \leq k \leq t$; $r_1(v) = r_0(v)$ for each v in RT_2^i where $v \neq v_k$, $1 \leq k \leq t$. Extend r_1 linearly to all of R_g^i.

The following corollaries now follow.

Corollary 8.1. *If* $q \leq i \leq \dim C_f$, *and* $B, C \in RT_2^i$, $\phi'(B) \cap A \neq \phi$ *with* $A \in (C_f')^i$; *if* $v_1, \cdots, v_t$ *are the free vertices of* B *w.r.t.* C, *then* $x \in \text{Int}\,(B)$ *implies that* $\phi'(x) \in \text{Shad}\,(\phi'(C))$.

Corollary 8.2. *If* $B \in RT_2^i$ *and* $C \in RT_2^j - RT_2^{j-1}$ *where* $q \leq i < j \leq \dim C_f$; *if* $A \in (C_f')^i$ *with* $\phi'(B) \in A$; *if* $v_1, \cdots, v_t$ *are the free vertices of* B *w.r.t.* C, *then* $x \in \text{Int}\,(B)$ *implies that* $\phi'(x) \notin \text{Shad}\,(C)$.

Corollary 8.3. *If* $x \in \text{Int}\,(A)$, $y \in \text{Int}\,(B)$, *where* $B, C \in RT_2 - RT_2^{q-1}$; *and there exists a free vertex of* B *w.r.t.* C, *then there is no* $z \in$ *base of* C_f' *such that* $\phi'(x)$ *and* $\phi'(y)$ *are in* F_z.

Since ϕ' is ambient isotopic to ϕ in $|C_f'|$, there is an ambient isotopy $H_t\colon |C_f'| \to |C_f'|$ such that $H_0(x) = x$ for all $x \in |C_f|$, $H_1(\phi'(x)) = \phi(x)$ for all $x \in R_g$, and $H_t|A$ is a P.L. homeomorphism of A onto A for each $t \in [0, 1]$ and $A \in C_f'$.

Define $h\colon |C_f'| \to |C_f'|$ by $h(x) = idH_1(x)$.

Theorem 9. *Let $z \in$ base of C'_f. Suppose $gh(F_z) \cap gh(|C'_f| - F_z) \neq \phi$, then there exists a unique $z' \in$ base of C'_f, $z' \neq z$, such that $gh(F_z) \cap gh(F_{z'}) \neq \phi$. Furthermore $gh(F_z) \cap gh(F_{z'})$ is connected.*

The proof of Theorem 9 follows from Corollary 8.3.

Let B'=base of C'_f. For $z \in B'$, define $B'(z) = \{z' \in B' : gh(F_z) \cap gh(F_{z'}) \neq \phi\}$. $B'(z)$ contains at most 2 points. Let $G' = \{B'(z) : z \in B'\}$. Let $G = B'|G'$, i.e. G is the decomposition space formed by identifying z with z' in B' if $B'(z) = B'(z')$. Let $\Theta: B' \to G$ be defined by $\Theta(z) = B'(z)$. A subset U of G is open iff $\Theta^{-1}(U)$ is open in B'. G is a polyhedron and Θ is a P.L. map, Let $\delta_1 := \mathrm{Max}\,\{\text{diameter } gp_f^{-1}(x) : x \in \text{base of } C_f\}$.

Theorem 10. *Given $\delta > 0$, there exists a $(2\delta_1 + \delta)$ semi-forest $\Gamma = \{F, G, \pi\}$ and a P.L. mapping Θ of Q onto G satisfying:*

(a) $P \subset F \subset \mathrm{Int}\,(M)$

(b) $\Theta f = \pi | P$

(c) $\dim S_\Theta < \dim Q$

(d) $\dim \pi^{-1}(x) \leq \underset{y \in Q}{\mathrm{Max}} \dim f^{-1}(y) + 1$, *for any $x \in G$.*

Proof. Given $\delta' > 0$, there exists $\varepsilon > 0$ such that if ϕ' is constructed as in the proof of Proposition 8 so that $d(\phi, \phi') < \varepsilon$, then $d(gh(x), gid(x)) < \delta'$ for any $x \in |C'_f|$.

Let K' be a triangulation of $|C'_f|$ which extends T'_2. Let $K(P)'$ denote the subcomplex of K' which triangulates $id^{-1}g^{-1}(P)$.

h can be chosen so that there is an ambient isotopy $H_t: M \to M$, such that $H_0(x) = x$ for $x \in M$, $H_1(gh(x)) = gid(x)$ for $x \in |K(P)'|$, and $d(x, H_t(x)) < \delta/8$ for any $t \in [0, 1]$, $x \in M$. Thus diameter $(gh(F_z)) < \delta_1 + \delta/4$ for any $z \in$ base of C'_f; and diameter $(H_1 gh(F_z)) < \delta_1 + \delta/2$. If $H_1 gh(F_z) \cap H_1 gh(F_{z'}) \neq \phi$ for $z, z' \in$ base of C'_f, then diameter $(H_1 gh(F_z) \cup H_1 gh(F_{z'})) < 2\delta_1 + \delta$.

Let $F = H_1 gh(|C'_f|)$. Identify Q with B', thus $\Theta: Q \to G$. For $x \in F$, $x \in H_1 gh(F_z)$ for some $z \in B'$. Let $\pi(z) = B'(z) \in G$. Since $x \in H_1 gh(F_{z'})$ implies $B'(z) = B'(z')$, π is well-defined.

For $x \in G$, $\pi^{-1}(x)$ can be shown to be collapsible by Theorem 9. It is clear that $\dim \pi^{-1}(x) \leq \underset{y \in Q}{\mathrm{Max}} \dim f^{-1}(y) + 1$, for any $x \in G$.

To show that $\dim S_\Theta < \dim Q$, note that $n \geq 4$ and: $\dim Q = \dim S_f \leq n/2 - 2a$, where $a \geq 5/4$. Thus $\dim Q = n/2 - 2a - b$ where $b \geq 0$. $\dim S_\Theta \leq \dim S_g \leq 2(\dim C_f) - m = n/4 - 3a - 2b + 2$. Therefore $\dim Q - \dim S_\Theta > 0$, and the theorem follows.

References

[1] H. W. Berkowitz: *A counterexample to a proof of Homma.* Yokohama Math. J., **19** (1971), 1–3.

[2] T. Homma: *A theorem of piecewise linear approximations.* Yokohama Math. J., **14** (1966), 47–54.

[3] T. Homma: *Piecewise linear approximations of embeddings of manifolds.* Florida State University mimeographed notes, (1964).

[4] J. F. P. Hudson: *Piecewise linear topology.* Benjamin, New York (1969).

State University of New York,
College at New Paltz, New Paltz, N.Y.,
12561
State University of New York
at Binghamton, Binghamton, N.Y. 13901