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VECTOR TOPOLOGIES ON SPACES OF SEQUENTIALLY CONTINUOUS LINEAR TRANSFORMATIONS*

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Let $(X, \mathscr{T}_x)$ and $(Y, \mathscr{T}_y)$ be topological vector spaces over the real or complex field K. Let $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ be the set of all sequentially continuous linear transformations on X into Y. With addition of vectors and scalar multiplication defined pointwise, $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ is a linear subspace of the linear space $\mathscr{L}(X, Y)$ over K of all linear transformations on X into Y.

In this paper, we will consider: (1) a number of important vector topologies on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$; (2) bounded subsets of $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$; (3) equi-sequentially-continuous sets of functions; and (4) uniformly equi-sequentially-continuous sets of functions. Our most important result will be the derivation of a form of the Banach-Steinhaus theorem valid for S-barrelled topological vector spaces ([9]).

Quite obviously, our entire development closely parallels the theory of vector topologies on spaces $\mathscr{C}\mathscr{L}(X, Y)$ of continuous linear transformations, the theory of equicontinuous and uniformly equicontinuous sets of functions, and the derivation of the Banach-Steinhaus theorem for barrelled topological vector spaces (see [1], p. 216 and [6], pp. 79-87). There are, however, many known examples of linear transformations which are sequentially continuous but not continuous (see [5], p. 38). In fact, if a topological vector space $(X, \mathscr{T}_x)$ over K is *not* C-sequential (see [8], p. 275), there exists a locally convex topological vector space $(Y, \mathscr{T}_y)$ over K and a sequentially continuous linear transformation f: $X \rightarrow Y$ which is not continuous.

1. $\mathscr{S}$-Topologies on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$

In Theorem 1, we recall the well-known result that a sequentially continuous linear transformation is bounded on bounded sets. As an immediate consequence of this fact, we can state Theorem 2 on the existence of vector $\mathscr{S}$-topologies (topologies of uniform convergence on the sets S in $\mathscr{S}$) on the linear space $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ of sequentially continuous linear transformations.

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* Some of these results are contained in the author's Ph.D. thesis written at the University of Virginia under the direction of Professor E. J. McShane.

Theorem 1. *Let $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ be topological vector spaces over the real or complex field K. Let $f: X \to Y$ be a sequentially continuous linear transformation on X into Y. Then f is bounded on bounded subsets of X, i.e., if B is a bounded subset of X, the set $f[B]$ is a bounded subset of Y.*

Proof. Assume that B is a bounded subset of X. Let V be a neighborhood of the zero vector 0 in Y. There exists a balanced neighborhood W of 0 in Y such that $W \subseteq V$. Since f is linear and sequentially continuous, $f^{-1}[W]$ is a balanced sequential neighborhood of the zero vector 0 in X. Consequently $f^{-1}[W]$ is a bornivore. Thus $f^{-1}[W]$ absorbs B, i.e., there exists a real number λ where $\lambda > 0$ such that $\lambda B \subseteq f^{-1}[W]$. Thus, we have $f[\lambda B] = \lambda f[B] \subseteq W \subseteq V$. Clearly, $f[B]$ is bounded.///

Theorem 2. (*Vector $\mathcal{S}$-Topologies on $\mathcal{SCL}(X, Y)$*) *Let $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ be topological vector spaces over the real or complex field K. Let $\mathcal{SCL}(X, Y)$ be the linear space over K of all sequentially continuous linear transformations on X into Y. Let $\mathcal{S}$ be an upward direction in X (a non-empty collection of non-empty subsets of X upwardly directed by set inclusion), composed of $\mathcal{T}_x$-bounded subsets of X. Let $\mathcal{T}$ be the $\mathcal{S}$-topology on $\mathcal{SCL}(X, Y)$, i.e., let $\mathcal{T}$ be the relative topology on $\mathcal{SCL}(X, Y)$ induced by the $\mathcal{S}$-topology on the linear space $\mathcal{F}(X, Y)$, over K, of all functions on X into Y. Then $(\mathcal{SCL}(X, Y), \mathcal{T})$ is a topological vector space over K. If $\mathcal{M}$ is a local base for $\mathcal{T}_y$, a local base for $\mathcal{T}$ is $\{N(S, M) \cap \mathcal{SCL}(X, Y) \mid S \in \mathcal{S}$ and $M \in \mathcal{M}\}$ where $N(S, M) = \{f \in \mathcal{F}(X, Y) \mid f[S] \subseteq M\}$. If $(Y, \mathcal{T}_y)$ is a locally convex topological vector space over K, the vector topology $\mathcal{T}_y$ being generated by the non-empty family of semi-norms $\mathcal{P}$, then $(\mathcal{SCL}(X, Y), \mathcal{T})$ is a locally convex topological vector space over K with the vector topology $\mathcal{T}$ generated by the non-empty family of semi-norms $\{P_{S,p}: S \in \mathcal{S}$ and $p \in \mathcal{P}\}$ where $P_{S,p}: \mathcal{SCL}(X, Y) \to R$ is defined by the correspondence $P_{S,p}(f) = \sup_{x \in S} (p \circ f)(x) = \sup_{x \in S} p(f(x))$ for all f in $\mathcal{SCL}(X, Y)$. If $\mathcal{P}$ is filtrant (or directed), so is $\{P_{S,p}: S \in \mathcal{S}$ and $p \in \mathcal{P}\}$.*

Proof. Use Theorem 1 and a theorem given in the book by H. H. Schaefer ([6], Theorem (3.1), p. 79).///

Theorem 3. *Let $(X, \mathcal{T}_x)$ and $(Y, \mathcal{T}_y)$ be topological vector spaces over the real or complex field K. Let $\mathcal{S}$ be an upward direction in X, composed of $\mathcal{T}_x$-bounded subsets of X. Let $(\mathcal{SCL}(X, Y), \mathcal{T})$ be the topological vector space over K of all sequentially continuous linear transformations on X into Y with the $\mathcal{S}$-topology. If $(Y, \mathcal{T}_y)$ is Hausdorff, and if $\bigcup \mathcal{S}$ is sequentially total in X, i.e., if* *the linear hull* $[\bigcup \mathscr{S}]$ *of* $\bigcup \mathscr{S}$ *is sequentially dense in* X*, then* $(\mathscr{S}\mathscr{C}\mathscr{L}(X, Y), \mathscr{T})$ *is Hausdorff.*

Proof. Let $A=\bigcup \mathscr{S}=\bigcup \{S: S\in \mathscr{S}\}$. Let $f_0 \in \mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ such that $f_0\neq 0$. Since $\overline{[A]}^s=X$, there exists a point x_0 in A or x_0 in S_0 where $S_0\in \mathscr{S}$ such that $f_0(x_0)\neq 0$. Since $(Y, \mathscr{T}_y)$ is Hausdorff, there exists a $\mathscr{T}_y$-neighborhood M_0 of 0 in Y such that $f_0(x_0)\notin M_0$. Clearly, $f_0 \notin N(S_0, M_0)\cap \mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$. Thus the topological vector space $(\mathscr{S}\mathscr{C}\mathscr{L}(X, Y), \mathscr{T})$ is Hausdorff.///

We want to list those vector $\mathscr{S}$-topologies on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ which are especially important. But before we can do this, we need several definitions.

Let $(X, \mathscr{T})$ be a topological space, and let $(X, \overline{\cdot}^s)$ be the *sequential closure space* generated by $(X, \mathscr{T})$. See [7], p. 95. Given a set $A\subseteq X$, the *sequential interior* of A is the set $A^{\circledS}=X\backslash\overline{(X\backslash A)}^s$. Let B be a subset of X. Then the set B is said to be *S-compact* or $\overline{\cdot}^s$*-compact* if and only if every sequential interior cover of B has a finite subcover, i.e., if and only if: given any family of subsets of X, $\{G_i: i\in I\}$, such that $B\subseteq \bigcup_{i\in I} G_i^{\circledS}$; there exists a finite set of indices $\{i_1, \cdots, i_n\}$, where $\{i_1, \cdots, i_n\}\subseteq I$, such that $B\subseteq \bigcup_{j=1}^{n} G_{i_j}$. Every finite subset of X is S-compact. As another example, every convergent sequence together with its limit is S-compact, i.e., if $(x_n: n\in N)$ is a sequence in X, if $a\in X$, and if $\lim_{n\to+\infty} x_n=a$; then the set $\{a\}\cup\{x_n: n\in N\}$ is S-compact.

Let $(X, \mathscr{T})$ be a topological vector space over the real or complex field K. Let B be a subset of X. Then the set B is said to be *sequentially totally bounded* (*S-totally bounded*) or *sequentially precompact* (*S-precompact*) if and only if: given any sequential neighborhood N of the zero vector 0 in X, there exists a finite subset B_0 of X such that $B\subseteq B_0+N$.

The following theorem gives the relationships between finite, S-compact, S-precompact, compact, precompact, and bounded sets.

Theorem 4. *Let* $(X, \mathscr{T})$ *be a topological vector space over the real or complex field* K*. Let* B *be a subset of* X*. Then for the set* B*, the following implications hold*:

$$\begin{array}{ccccc}
B \text{ is finite.} & \Rightarrow & B \text{ is } S\text{-compact.} & \Rightarrow & B \text{ is } S\text{-precompact.} \\
 & & \Downarrow & & \Downarrow \\
 & & B \text{ is compact.} & \Rightarrow & B \text{ is precompact.} \\
 & & & & \Downarrow \\
 & & & & B \text{ is bounded.}
\end{array}$$

Proof. First, we will show that if B is S-compact, then B is compact.

Assume that B is S-compact. Let $\{G_i: i \in I\}$ be an open cover for B. Then $B \subseteq \bigcup_{i \in I} G_i$. Since $G_i = G_i^{\circledS} = G_i^0$, we have $B \subseteq \bigcup_{i \in I} G_i^{\circledS}$. Since B is S-compact, there exists a finite set of indices $\{i_1, \cdots, i_n\} \subseteq I$ such that $B \subseteq \bigcup_{j=1}^{n} G_{i_j}$. Clearly, B is compact.

Next, we will prove that if B is S-compact, then B is S-precompact. Assume that B is S-compact. Let N be a sequential neighborhood of 0 in X. If $x \in X$, then $x+N$ is a sequential neighborhood of x. Thus, $x \in (x+N)^{\circledS}$. Consequently, $B \subseteq \bigcup_{x \in B} (x+N)^{\circledS}$. Since B is S-compact, there exists a finite set of points $\{x_1, \cdots, x_n\}$ in B so that $B \subseteq \bigcup_{j=1}^{n} (x_j+N)$. Thus $B \subseteq B_0 + N$ where $B_0 = \{x_1, \cdots, x_n\} \subseteq B \subseteq X$. Clearly, B is S-precompact.

The remaining implications obviously hold.///

Now we can list the important vector $\mathscr{S}$-topologies on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$. They are the following:

(1) $\mathscr{T}_o$, the *topology of pointwise convergence* or the *topology of simple convergence* or the *weak topology*, which is the $\mathscr{S}$-topology on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ when $\mathscr{S}$ is the collection of all finite (non-empty) subsets of X.

(2) $\mathscr{T}_{sc}$, the *topology of S-compact convergence* or the *topology of uniform convergence on S-compact subsets of X*, which is the $\mathscr{S}$-topology on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ when $\mathscr{S}$ is the collection of all S-compact (non-empty) subsets of X.

(3) $\mathscr{T}_c$, the *topology of compact convergence* or the *topology of uniform convergence on compact subsets of X*, which is the $\mathscr{S}$-topology on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ when $\mathscr{S}$ is the collection of all compact (non-empty) subsets of X.

(4) $\mathscr{T}_{spc}$, the *topology of S-precompact convergence* or the *topology of uniform convergence on S-precompact subsets of X*, which is the $\mathscr{S}$-topology on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ when $\mathscr{S}$ is the collection of all S-precompact (non-empty) subsets of X.

(5) $\mathscr{T}_{pc}$, the *topology of precompact convergence* or the *topology of uniform convergence on precompact subsets of X*, which is the $\mathscr{S}$-topology on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ when $\mathscr{S}$ is the collection of all precompact (non-empty) subsets of X.

(6) $\mathscr{T}_b$, the *topology of bounded convergence* or the *topology of uniform convergence on bounded subsets of X* or the *strong topology*, which is the $\mathscr{S}$-topology on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ when $\mathscr{S}$ is the collection of all bounded (non-empty) subsets of X.

From Theorem 4, it is clear that these six topologies on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ are related in the following manner:

$$\begin{array}{ccccc} \mathscr{T}_\sigma & \subseteq & \mathscr{T}_{sc} & \subseteq & \mathscr{T}_{spc} \\ & & \cap\!\!| & & \cap\!\!| \\ & & \mathscr{T}_c & \subseteq & \mathscr{T}_{pc} \subseteq \mathscr{T}_b . \end{array}$$

In the next theorem, we give a number of different characterizations of bounded subsets of $\mathscr{SCL}(X, Y)$.

Theorem 5. *($\mathscr{S}$-Bounded Subsets of $(\mathscr{SCL}(X, Y), \mathscr{T})$) Let $(X, \mathscr{T}_x)$ and $(Y, \mathscr{T}_y)$ be topological vector spaces over the real or complex field K. Let $\mathscr{S}$ be an upward direction in X, composed of $\mathscr{T}_x$-bounded subsets of X. Let $(\mathscr{SCL}(X, Y), \mathscr{T})$ be the topological vector space over K of all sequentially continuous linear transformations on X into Y with the $\mathscr{S}$-topology. Let $\mathscr{M}$ be a local base for $\mathscr{T}_y$. Let $\mathscr{K} \subseteq \mathscr{SCL}(X, Y)$. Then $\mathscr{K}$ is bounded for the $\mathscr{S}$-topology or $\mathscr{K}$ is a $\mathscr{T}$-bounded subset of $\mathscr{SCL}(X, Y)$ or $\mathscr{K}$ is $\mathscr{S}$-bounded if and only if any of the following equivalent assertions hold:*

(1) Given any set M in $\mathscr{M}$ and given any set S in $\mathscr{S}$, there exists a real number λ where $\lambda > 0$ such that: $\lambda\mathscr{K} \subseteq N(S, M) \cap \mathscr{SCL}(X, Y)$.

(2) Given any set M in $\mathscr{M}$ and given any set S in $\mathscr{S}$, there exists a real number λ where $\lambda > 0$ such that: $f \in \mathscr{K} \Rightarrow \lambda f[S] \subseteq M$ (or $f[\lambda S] \subseteq M$; or $\lambda S \subseteq f^{-1}[M]$).

(3) Given any set S in $\mathscr{S}$, the set $\bigcup_{f \in \mathscr{K}} f[S]$ is $\mathscr{T}_y$-bounded in Y.

(4) Given any set M in $\mathscr{M}$ and given any set S in $\mathscr{S}$, there exists a real number λ where $\lambda > 0$ such that: $S \subseteq \lambda \bigcap_{f \in \mathscr{K}} f^{-1}[M]$.

(5) Given any set M in $\mathscr{M}$ and given any set S in $\mathscr{S}$, the set $\bigcap_{f \in \mathscr{K}} f^{-1}[M]$ absorbs S.

Proof. The above statements follow directly from the definition of a bounded set in a topological vector space. To obtain (5), one first considers the special case where $\mathscr{M}$ is a local base of balanced neighborhoods of the zero vector 0 in *Y*.///

Let $(X, \mathscr{T})$ be a topological vector space over *K*. A subset *B* of *X* is a *sequential barrel* if *B* is sequentially closed, convex, balanced, and absorbing (see [9]).

Theorem 6. *Let $(X, \mathscr{T}_x)$ be a topological vector space over the field of real or complex numbers K, and let $(Y, \mathscr{T}_y)$ be a locally convex topological vector space over K. Let $(\mathscr{SCL}(X, Y), \mathscr{T}_\sigma)$ be the locally convex topological vector space over K of all sequentially continuous linear transformations on X into Y with the topology of pointwise convergence $\mathscr{T}_\sigma$. Let $\mathscr{K} \subseteq \mathscr{SCL}(X, Y)$. Then $\mathscr{K}$ is $\mathscr{T}_\sigma$-bounded if and only if given any closed, convex, balanced neighborhood V of the zero vector 0 in Y; the set $\bigcap_{f \in \mathscr{K}} f^{-1}[V]$ is a sequential barrel in X.*

Proof. This result follows directly from (5) of Theorem 5.///

2. Equi-Sequentially-Continuous Sets of Functions

Let $(X, \mathscr{T}_x)$ be a topological space. Let $(Y, \mathscr{T}_y)$ be a topological vector space, over the real or complex field K, with associated uniform structure $\mathscr{V}$ and with local base $\mathscr{M}$. Let $\mathscr{F}(X, Y)$ be the linear space, over K, of all functions on X into Y. Let $\mathscr{K} \subseteq \mathscr{F}(X, Y)$, and let $x_0 \in X$. To say that $\mathscr{K}$ is *equi-sequentially-continuous at the point* x_0 means that either of the following equivalent statements holds:

(1) Given any entourage V in $\mathscr{V}$, there exists a sequential neighborhood $N(x_0)$ of the point x_0 in X such that: $f \in \mathscr{K}$ and $x \in N(x_0) \Rightarrow (f(x), f(x_0)) \in V$.

(2) Given any set M in $\mathscr{M}$, there exists a sequential neighborhood $N(x_0)$ of the point x_0 in X such that: $f \in \mathscr{K}$ and $x \in N(x_0) \Rightarrow f(x) - f(x_0) \in M$.

To say that $\mathscr{K}$ is *equi-sequentially-continuous* means that $\mathscr{K}$ is equi-sequentially-continuous at every point of X.

Let $(X, \mathscr{T}_x)$ and $(Y, \mathscr{T}_y)$ be topological vector spaces over the real or complex field K. Let $\mathscr{N}_s$ be a local base of $\mathscr{T}_x$-sequential neighborhoods of the zero vector 0 in X; and let $\mathscr{M}$ be a local base of $\mathscr{T}_y$-neighborhoods of the zero vector 0 in Y. Let $\mathscr{F}(X, Y)$ be the linear space, over K, of all functions on X into Y. Let $\mathscr{K} \subseteq \mathscr{F}(X, Y)$. To say that $\mathscr{K}$ is *uniformly equi-sequentially-continuous* means that given any set M in $\mathscr{M}$, there exists a set N in $\mathscr{N}_s$ such that:

$$f \in \mathscr{K} \text{ and } x_1, x_2 \in X \text{ and } x_1 - x_2 \in N \Rightarrow f(x_1) - f(x_2) \in M .$$

As a consequence of these definitions, we obtain the following theorem.

Theorem 7. *Let $(X, \mathscr{T}_x)$ and $(Y, \mathscr{T}_y)$ be topological vector spaces over the real or complex field K. Let $\mathscr{L}(X, Y)$ be the linear space, over K, of all linear transformations on X into Y. Let $\mathscr{K} \subseteq \mathscr{L}(X, Y)$. Then the following statements are equivalent*:

(1) *$\mathscr{K}$ is equi-sequentially-continuous.*

(2) *$\mathscr{K}$ is equi-sequentially-continuous at the zero vector 0 in X.*

(3) *$\mathscr{K}$ is equi-sequentially-continuous at some point x_0 in X.*

(4) *$\mathscr{K}$ is uniformly equi-sequentially-continuous.*

(5) *Given any neighborhood M of 0 in Y, there exists a sequential neighborhood N of 0 in X such that: $\bigcup_{f \in \mathscr{K}} f[N] \subseteq M$.*

(6) *Given any neighborhood M of 0 in Y, the set $\bigcap_{f \in \mathscr{K}} f^{-1}[M]$ is a sequential neighborhood of 0 in X.*

Theorem 8. *Let $(X, \mathscr{T}_x)$ be a topological vector space over the real or complex* *field K, and let* $(Y, \mathscr{T}_y)$ *be a Hausdorff topological vector space over K. Let* $(\mathscr{F}(X, Y), \mathscr{T}_p)$ *be the Hausdorff topological vector space, over K, of all functions on X into Y with the topology of pointwise convergence. Let* $\mathscr{L}(X, Y)$ *be the linear space, over K, of all linear transformations on X into Y. Let* $\mathscr{K} \subseteq \mathscr{L}(X, Y)$*. Then, if* $\mathscr{K}$ *is equi-sequentially-continuous;* $\bar{\mathscr{K}}^{\mathscr{T}_p}$*, the pointwise closure of* $\mathscr{K}$ *in* $\mathscr{F}(X, Y)$*, is an equi-sequentially-continuous subset of* $\mathscr{L}(X, Y)$*.*

Proof. Since $\mathscr{K} \subseteq \mathscr{L}(X, Y)$, we have $\bar{\mathscr{K}}^{\mathscr{T}_p} \subseteq \overline{\mathscr{L}(X, Y)}^{\mathscr{T}_p} = \mathscr{L}(X, Y)$ using the fact that $\mathscr{L}(X, Y)$ is a $\mathscr{T}_p$-closed subset of $\mathscr{F}(X, Y)$. We must show that if $\mathscr{K}$ is equi-sequentially-continuous, then $\bar{\mathscr{K}}^{\mathscr{T}_p}$ is equi-sequentially-continuous. Assume that $\mathscr{K}$ is equi-sequentially-continuous. Let M be a $\mathscr{T}_y$-neighborhood of 0 in Y. There exists a balanced $\mathscr{T}_y$-neighborhood W of 0 in Y such that $W + W \subseteq M$. Since $\mathscr{K}$ is equi-sequentially-continuous, there exists a $\mathscr{T}_x$-sequential neighborhood N of 0 in X such that $\bigcup_{f \in \mathscr{K}} f[N] \subseteq W$. We must show that $\bigcup_{f \in \bar{\mathscr{K}}^{\mathscr{T}_p}} f[N] \subseteq M$. Let $f \in \bar{\mathscr{K}}^{\mathscr{T}_p}$, and let $x_0 \in N$. Consider the set $N(\{x_0\}, W)$ which is a $\mathscr{T}_p$-neighborhood of the zero vector 0 in $\mathscr{F}(X, Y)$. Then $f + N(\{x_0\}, W)$ is a $\mathscr{T}_p$-neighborhood of f. There exists a function g in $\mathscr{K}$ such that $g \in f + N(\{x_0\}, W)$ hence $g(x_0) - f(x_0) \in W$. Since W is balanced, $f(x_0) - g(x_0) \in W$ hence $f(x_0) \in g(x_0) + W \subseteq W + W \subseteq M$. Clearly, $\bigcup_{f \in \bar{\mathscr{K}}^{\mathscr{T}_p}} f[N] \subseteq M$. This proves that $\bar{\mathscr{K}}^{\mathscr{T}_p}$ is equi-sequentially-continuous.///

Theorem 9. *Let* $(X, \mathscr{T}_x)$ *and* $(Y, \mathscr{T}_y)$ *be topological vector spaces over the real or complex field K. Let* $\mathscr{SCL}(X, Y)$ *be the linear space, over K, of all sequentially continuous linear transformations on X into Y. Let* $\mathscr{K} \subseteq \mathscr{SCL}(X, Y)$*, such that* $\mathscr{K}$ *is equi-sequentially-continuous. If* $\mathscr{T}_\sigma$ *is the topology of pointwise convergence on* $\mathscr{SCL}(X, Y)$*,* $\mathscr{T}_{sc}$ *is the topology of S-compact convergence on* $\mathscr{SCL}(X, Y)$*, and* $\mathscr{T}_{spc}$ *is the topology of S-precompact convergence on* $\mathscr{SCL}(X, Y)$*, then the corresponding relative topologies on* $\mathscr{K}$ *are the same, i.e.,* $\mathscr{T}_\sigma | \mathscr{K} = \mathscr{T}_{sc} | \mathscr{K} = \mathscr{T}_{spc} | \mathscr{K}$*.*

Proof. Clearly, $\mathscr{T}_\sigma | \mathscr{K} \subseteq \mathscr{T}_{sc} | \mathscr{K} \subseteq \mathscr{T}_{spc} | \mathscr{K}$. We must prove that $\mathscr{T}_{spc} | \mathscr{K} \subseteq \mathscr{T}_\sigma | \mathscr{K}$. Let $\mathscr{M}$ be a local base for $\mathscr{T}_y$. Then a local base for $\mathscr{T}_{spc} | \mathscr{K}$ is $\{N(B, M) \cap \mathscr{K} \mid B \subseteq X,\ B$ is S-precompact, and $M \in \mathscr{M}\}$. A local base for $\mathscr{T}_\sigma | \mathscr{K}$ is $\{N(\{x_1, \cdots, x_n\}, M) \cap \mathscr{K} \mid \{x_1, \cdots, x_n\} \subseteq X,\ \{x_1, \cdots, x_n\}$ is finite, and $M \in \mathscr{M}\}$. Consider the set $N(B, M) \cap \mathscr{K}$ where B is an S-precompact subset of X and where $M \in \mathscr{M}$. Let $W \in \mathscr{M}$ such that $W + W \subseteq M$. Since $\mathscr{K}$ is equi-sequentially-continuous, there exists a $\mathscr{T}_x$-sequential neighborhood N of 0 in X such that $\bigcup_{f \in \mathscr{K}} f[N] \subseteq W$. Since B is S-precompact, there exists a finite set $\{x_1, \cdots, x_n\} \subseteq X$ such that $B \subseteq \{x_1, \cdots, x_n\} + N$. We claim that $N(\{x_1, \cdots, x_n\}, W) \cap \mathscr{K} \subseteq N(B, M) \cap \mathscr{K}$. Indeed, if we have $f \in$

$N(\{x_1, \cdots, x_n\}, W) \cap \mathscr{K}$, then $f \in \mathscr{K}$ and $f[B] \subseteq f[\{x_1, \cdots, x_n\} + N] = f[\{x_1, \cdots, x_n\}] + f[N] \subseteq W + W \subseteq M$ hence $f \in N(B, M) \cap \mathscr{K}$. This proves that $\mathscr{T}_{spc}|\mathscr{K} \subseteq \mathscr{T}_\sigma|\mathscr{K}$./ / /

Theorem 10. *Let $(X, \mathscr{T}_x)$ and $(Y, \mathscr{T}_y)$ be topological vector spaces over the real or complex field K. Let $\mathscr{S}$ be an upward direction in X, composed of $\mathscr{T}_x$-bounded subsets of X. Let $(\mathscr{S}\mathscr{C}\mathscr{L}(X, Y), \mathscr{T})$ be the topological vector space over K of all sequentially continuous linear transformations on X into Y with the $\mathscr{S}$-topology. Let $\mathscr{K} \subseteq \mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$. If $\mathscr{K}$ is equi-sequentially-continuous, then $\mathscr{K}$ is $\mathscr{T}$-bounded, i.e., $\mathscr{K}$ is bounded for the $\mathscr{S}$-topology.*

Proof. Let $\mathscr{M}$ be a local base for $\mathscr{T}_y$ consisting of balanced neighborhoods of 0 in Y. Assume that $\mathscr{K}$ is equi-sequentially-continuous. Let $M \in \mathscr{M}$. Then $\bigcap_{f \in \mathscr{K}} f^{-1}[M]$ is a balanced sequential neighborhood of 0 in X. Consequently, $\bigcap_{f \in \mathscr{K}} f^{-1}[M]$ is a bornivore, i.e., $\bigcap_{f \in \mathscr{K}} f^{-1}[M]$ absorbs every bounded subset of X. In particular, $\bigcap_{f \in \mathscr{K}} f^{-1}[M]$ absorbs every set S in $\mathscr{S}$. By (5) of Theorem 5, $\mathscr{K}$ is $\mathscr{T}$-bounded./ / /

3. Banach-Steinhaus Theorem for S-Barrelled Topological Vector Spaces

A topological vector space $(X, \mathscr{T})$ over K is *S-barrelled* if every sequential barrel in X is a sequential neighborhood of the zero vector 0 in X. A number of examples of spaces which are S-barrelled but not barrelled are given in [9].

Perhaps the most important of these examples is the sequence space $l^1 = \{(\alpha_n): \sum_{n=1}^{+\infty} |\alpha_n| < +\infty\}$ with the weak topology $\sigma(l^1, l^\infty)$. Incidently, $(l^1, \sigma(l^1, l^\infty))$ is *not* C-sequential. This example is important because a number of different classical Banach spaces have been found to be isomorphic to l^1 with the norm topology. See [4], p. 41, and [2], p. 182. As an example, $(l^1, \mathscr{T})$ is isomorphic (see [3], pp. 247–248) to the "natural space" of analytic functions $(H_1(D), \mathscr{T}_a)$. Here $H_1(D)$ is the linear space of analytic functions $f(z)$ on the unit disk $D = \{z: |z| < 1\}$ such that

$$\|f\|^1 = \iint_D |f(x+iy)| dx dy < +\infty$$

and $\mathscr{T}_a$ is the topology generated by the norm $\|\cdot\|^1$. Thus $H_1(D)$ with the weak topology $\sigma(H_1(D), H_1(D)')$ is S-barrelled but not barrelled.

We will now consider equi-sequentially-continuous and bounded sets of sequentially continuous linear transformations on S-barrelled topological vector spaces. Finally, we will derive a form of the Banach-Steinhaus theorem valid for S-barrelled topological vector spaces.

Theorem 11. *Let $(X, \mathscr{T}_x)$ be an S-barrelled topological vector space over the real or complex field K. Let $(Y, \mathscr{T}_y)$ be a locally convex topological vector space over K. Let $(\mathscr{S}\mathscr{C}\mathscr{L}(X, Y), \mathscr{T}_\sigma)$ be the locally convex topological vector space, over K, of all sequentially continuous linear transformations on X into Y with the topology of pointwise convergence $\mathscr{T}_\sigma$. Let $\mathscr{K} \subseteq \mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$. If $\mathscr{K}$ is $\mathscr{T}_\sigma$-bounded, i.e., if $\mathscr{K}$ is simply bounded, then $\mathscr{K}$ is equi-sequentially-continuous.*

Proof. Let $\mathscr{M}$ be a local base for $\mathscr{T}_y$ composed of closed, convex, balanced $\mathscr{T}_y$-neighborhoods of the zero vector 0 in Y. Let $M \in \mathscr{M}$. Then $\bigcap_{f \in \mathscr{K}} f^{-1}[M]$ is convex, balanced, and sequentially closed. Since $\mathscr{K}$ is $\mathscr{T}_\sigma$-bounded, the set $\bigcap_{f \in \mathscr{K}} f^{-1}[M]$ is absorbing (by (5) of Theorem 5). Thus $\bigcap_{f \in \mathscr{K}} f^{-1}[M]$ is a sequential barrel in X. Since $(X, \mathscr{T}_x)$ is S-barrelled, $\bigcap_{f \in \mathscr{K}} f^{-1}[M]$ is a $\mathscr{T}_x$-sequential neighborhood of 0 in X. By Theorem 7, $\mathscr{K}$ is equi-sequentially-continuous.///

As a corollary to Theorem 11, we have the following theorem.

Theorem 12. *Let $(X, \mathscr{T}_x)$ be an S-barrelled topological vector space over the real or complex field K. Let $(Y, \mathscr{T}_y)$ be a locally convex topological vector space over K. Let $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ be the linear space, over K, of all sequentially continuous linear transformations on X into Y. Let $\mathscr{T}_\sigma$ be the (locally convex vector) topology of pointwise convergence on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$; and let $\mathscr{T}_b$ be the (locally convex vector) topology of bounded convergence on $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$. Let $\mathscr{K} \subseteq \mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$. Then the following statements are equivalent:*

(1) *$\mathscr{K}$ is equi-sequentially-continuous.*

(2) *$\mathscr{K}$ is $\mathscr{T}_\sigma$-bounded, i.e., $\mathscr{K}$ is simply (weakly) bounded.*

(3) *$\mathscr{K}$ is $\mathscr{T}_b$-bounded, i.e., $\mathscr{K}$ is strongly bounded.*

Theorem 13. *Let $(X, \mathscr{T}_x)$ be a topological vector space over the real or complex field K, and let $(Y, \mathscr{T}_y)$ be a Hausdorff topological vector space over K. Let $(\mathscr{F}(X, Y), \mathscr{T}_p)$ be the Hausdorff topological vector space, over K, of all functions on X into Y with the topology of pointwise convergence. Let $(\mathscr{S}\mathscr{C}\mathscr{L}(X, Y), \mathscr{T}_{spc})$ be the Hausdorff topological vector space, over K, of all sequentially continuous linear transformations on X into Y with the topology of S-precompact convergence. Let $\mathscr{K}$ be an equi-sequentially-continuous subset of $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$. Let $\mathscr{N}$ be a direction (filter base) in $\mathscr{K}$, and let f be a function in $\mathscr{F}(X, Y)$. If $\mathscr{N}$ converges to f pointwise, i.e., if $\mathscr{N} \xrightarrow{\mathscr{T}p} f$; then $f \in \mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ and $\mathscr{N}$ converges to f uniformly on the S-precompact subsets of X, i.e., $\mathscr{N} \xrightarrow{\mathscr{T}spc} f$.*

Proof. Since $\mathscr{N}$ is a direction in $\mathscr{K}$ and $\mathscr{N} \xrightarrow{\mathscr{T}p} f$, we have $f \in \bar{\mathscr{K}}^{\mathscr{T}p}$. Since

$\mathscr{K}$ is equi-sequentially-continuous, $\bar{\mathscr{K}}^{\mathscr{T}_p}$ is equi-sequentially-continuous and $\bar{\mathscr{K}}^{\mathscr{T}_p} \subseteq \mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$. Thus we have $f \in \mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$. Since $\mathscr{T}_\sigma|\bar{\mathscr{K}}^{\mathscr{T}_p} = \mathscr{T}_{spc}|\bar{\mathscr{K}}^{\mathscr{T}_p}$, and since $\mathscr{N} \xrightarrow{\mathscr{T}_p} f$; we have $\mathscr{N} \xrightarrow{\mathscr{T}_{spc}} f$.///

Theorem 14. (*Banach-Steinhaus Theorem for S-Barrelled Spaces*) *Let* $(X, \mathscr{T}_x)$ *be an S-barrelled topological vector space over the real or complex field K. Let* $(Y, \mathscr{T}_y)$ *be a Hausdorff locally convex topological vector space over K. Let* $(\mathscr{F}(X, Y), \mathscr{T}_p)$ *be the Hausdorff locally convex topological vector space, over K, of all functions on X into Y with the topology of pointwise convergence. Let* $(\mathscr{S}\mathscr{C}\mathscr{L}(X, Y), \mathscr{T}_{spc})$ *be the Hausdorff locally convex topological vector space, over K, of all sequentially continuous linear transformations on X into Y with the topology of S-precompact convergence. Let* $\mathscr{N}$ *be a direction (filter base) in* $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$, *and let f be a function in* $\mathscr{F}(X, Y)$ *such that* $\mathscr{N}$ *converges to f pointwise, i.e.,* $\mathscr{N} \xrightarrow{\mathscr{T}_p} f$. *Suppose that* $\mathscr{N}$ *has either one of the following two properties:*

(1) $\mathscr{N}$ *is* $\mathscr{T}_p$*-bounded in the sense that there exists a set N in* $\mathscr{N}$ *such that N is a* $\mathscr{T}_p$*-bounded subset of* $\mathscr{F}(X, Y)$.

(2) $\mathscr{N}$ *is countable.*

Then $f \in \mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ *and* $\mathscr{N}$ *converges to f uniformly on the S-precompact subsets of X, i.e.,* $\mathscr{N} \xrightarrow{\mathscr{T}_{spc}} f$.

Proof. Assume $\mathscr{N}$ has property (1). Let $\mathscr{K} \in \mathscr{N}$ be such that $\mathscr{K}$ is a $\mathscr{T}_p$-bounded subset of $\mathscr{F}(X, Y)$. Then $\mathscr{K}$ is a $\mathscr{T}_\sigma$-bounded subset of $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$. Thus $\mathscr{K}$ is equi-sequentially-continuous. Consider the direction $\mathscr{N}_{\mathscr{K}} = \{\mathscr{K} \cap N: N \in \mathscr{N}\}$ in $\mathscr{K}$. Since $\mathscr{N} \xrightarrow{\mathscr{T}_p} f$ and $\mathscr{N}_{\mathscr{K}}$ is a subdirection of $\mathscr{N}$, we have $\mathscr{N}_{\mathscr{K}} \xrightarrow{\mathscr{T}_p} f$. Thus $f \in \mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ and $\mathscr{N}_{\mathscr{K}} \xrightarrow{\mathscr{T}_{spc}} f$. Since $\mathscr{K} \in \mathscr{N}$, $\mathscr{N}$ is a subdirection of $\mathscr{N}_{\mathscr{K}}$. Thus $\mathscr{N} \xrightarrow{\mathscr{T}_{spc}} f$.

Assume $\mathscr{N}$ has property (2). Then $\mathscr{N}$ is equivalent to a direction $\mathscr{M}$ in $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ such that $\mathscr{M} = \{M_j: j \in N\}$ with $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$. Let $(f_n: n \in N)$ be a sequence in $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ such that $f_n \in M_n$ for each natural number n in N. Since $\mathscr{N} \xrightarrow{\mathscr{T}_p} f$, we have $\mathscr{M} \xrightarrow{\mathscr{T}_p} f$. Given any point x in X, $\mathscr{M}(x) \xrightarrow{\mathscr{T}_y} f(x)$. (Note: $\mathscr{M}(x) = \{M(x): M \in \mathscr{M}\}$, where $M(x) = \{f(x): f \in M\}$.) Thus, given any point x in X, the sequence $(f_n(x): n \in N)$ is $\mathscr{T}_y$-convergent to $f(x)$. Given any point x in X, the set $\{f_n(x): n \in N\}$ is $\mathscr{T}_y$-bounded. Thus the set $\{f_n: n \in N\}$ is a $\mathscr{T}_\sigma$-bounded subset of $\mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$. Also, the sequence $(f_n: n \in N)$ is $\mathscr{T}_p$-convergent to f. Consider the direction $\{\{f_m: m > n\}: n \in N\}$ associated with the sequence $(f_n: n \in N)$. This direction satisfies condition (1). Thus $f \in \mathscr{S}\mathscr{C}\mathscr{L}(X, Y)$ and the sequence $(f_n: n \in N)$ is $\mathscr{T}_{spc}$-convergent to f.

Suppose $\mathscr{N}$ is not $\mathscr{T}_{spc}$-convergent to f. Then $\mathscr{M}$ is not $\mathscr{T}_{spc}$-convergent to f. There exists a $\mathscr{T}_{spc}$-neighborhood U of f in $\mathscr{SCL}(X, Y)$ such that none of the sets of $\mathscr{M}$ is contained in U. Given any index j in N, there exists a function f_j in $M_j \backslash U$. Clearly, the sequence $(f_n: n \in N)$ so chosen is not $\mathscr{T}_{spc}$-convergent to f. This is a contradiction! We must have $\mathscr{N} \xrightarrow{\mathscr{T}_{spc}} f$.///

As a corollary to Theorem 14, we have the following theorem.

Theorem 15. (*Banach-Steinhaus Theorem for S-Barrelled Spaces*) *Let* $(X, \mathscr{T}_x)$ *be an S-barrelled topological vector space over the real or complex field K. Let* $(Y, \mathscr{T}_y)$ *be a Hausdorff locally convex topological vector space over K. Let* $(\mathscr{F}(X, Y), \mathscr{T}_p)$ *be the Hausdorff locally convex topological vector space, over K, of all functions on X into Y with the topology of pointwise convergence. Let* $(\mathscr{SCL}(X, Y), \mathscr{T}_{spc})$ *be the Hausdorff locally convex topological vector space, over K, of all sequentially continuous linear transformations on X into Y with the topology of S-precompact convergence. Let* $(f_n: n \in N)$ *be a sequence in* $\mathscr{SCL}(X, Y)$, *and let f be a function in* $\mathscr{F}(X, Y)$. *If* $(f_n: n \in N)$ *converges to f pointwise* ($\mathscr{T}_p$-*convergence*), *then* $f \in \mathscr{SCL}(X, Y)$ *and* $(f_n: n \in N)$ *converges to f uniformly on S-precompact subsets of X* ($\mathscr{T}_{spc}$-*convergence*).

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