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FLOWS AND FUNCTION ALGEBRAS OF GENERALIZED ANALYTIC FUNCTIONS

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Let X be a compact Hausdorff space and let (R, X) be a flow. This means that we are given a group homomorphism $t \rightarrow T_t$ from the real line R into the group of homeomorphisms of X with the property that the function $(t, p) \rightarrow T_t p$ from $R \times X$ to X is continuous. We denote by $C(X)$ the set of all continuous complex-valued functions on X, and by $H^\infty(R)$ the set of all continuous functions on R which admit bounded analytic extensions into the upper half-plane. We put $\mathfrak{A}_X = \{f \in C(X); f \circ T(t, x) \in H^\infty(R)$ for every $x \in X\}$, then $\mathfrak{A}_X$ is a uniformly closed subalgebra of $C(X)$. But $\mathfrak{A}_X$ need not be a function algebra.

In [2], Forelli showed that:

1) If the flow (R, X) is minimal, then $\mathfrak{A}_X$ is a maximal pervasive subalgebra of $C(X)$.

In [5] and [6], Muhly showed that:

2) If the flow (R, X) is strictely ergodic, then $\mathfrak{A}_X$ is a Dirichlet algebra and its maximal ideal space is completely determined.

In this paper, we defined the normalized flow $(R, \tilde{X})$ of the flow (R, X) so that $\mathfrak{A}_{\tilde{X}}$ is a function algebra and $\mathfrak{A}_X$ is isometric-isomorphic to $\mathfrak{A}_{\tilde{X}}$. In Sections 1 and 2, we study some properties of $\mathfrak{A}_X$. In Section 3, we give necessary and sufficient conditions of normalized flows (R, X) for which $\mathfrak{A}_X$ is essential, antisymmetric, analytic, pervasive and maximal, respectively.

§ 1. A normalized flow (R, X) and a maximal ideal space of $\mathfrak{A}_X$.

Recall that if A is a function algebra on a compact Hausdorff space Y, then a probability measure m on Y is called a representing measure for A if $\int_Y fgdm = \int_Y fdm \int_Y gdm$ for every $f, g \in A$. Let X be a compact Hausdorff space and let (R, X) be a flow. We write $M(X)$ the set of all bounded Baire measures on X. For $\mu \in M(X)$ and $t \in R$, we put $T_t\mu(E) = \mu(T_{-t}E)$ for every Baire set E of X. A measure $\mu \in M(X)$ is called quasi-invariant if $T_t\mu$ is absolutely continuous with respect to μ for each $t \in R$, and $\mu \in M(X)$ is called T_t-invariant if $T_t\mu(E) = \mu(E)$

for every $t\in R$ and for every Baire set E of X. For $\varphi\in C(X)$ and $f\in L^1(R)$, we put $T_t\varphi(x)=\varphi(T_{-t}x)$ for $x\in X$ and $t\in R$, and $\varphi*f=\int_{-\infty}^{\infty}(T_t\varphi)f(t)dt$. Then we have $\varphi*f\in C(X)$. For $\mu\in M(X)$ and $f\in L^1(R)$, $\mu*f$ is defined to be the measure such that $\int\varphi d\mu*f=\int\varphi*\tilde{f}d\mu$ for all $\varphi\in C(X)$, where $\tilde{f}$ is the function whose value at $t\in R$ is $f(-t)$. For $\mu\in M(X)$ and $\varphi\in C(X)$, we put $J(\varphi)=\{f\in L^1(R);\ \varphi*f=0\}$ and $J(\mu)=\{f\in L^1(R);\ \mu*f=0\}$. Then $J(\varphi)$ and $J(\mu)$ are closed ideals of $L^1(R)$. We denote by $sp(\varphi)$ ($sp(\mu)$) the intersection of the zero sets of the Fourier transforms of the functions in $J(\varphi)$ ($J(\mu)$).

Lemma 1. ([5], p. 114) *Let (R, X) be a flow, then $\mathfrak{A}_X=\{\varphi\in C(X);\ sp(\varphi)\subset[0,\infty)\}$.*

Lemma 2. ([5], p. 116) *If μ is a representing measure for $\mathfrak{A}_X$ which is not a point measure, then μ is quasi-invariant.*

Lemma 3. ([8], p. 57) *If a real measure $\mu\in M(X)$ satisfies $\int fd\mu=0$ for every $f\in\mathfrak{A}_X$ ($\mu\perp\mathfrak{A}_X$), then μ is T_t-invariant.*

We put $P=\{x\in X;\ T_tx=x$ for every $t\in R\}$, then P is a compact subset of X. At first, we note that $\mathfrak{A}_X$ need not separate the points of X. And by Lemma 3, we have that if $f(x)=f(y)$ for every $f\in\mathfrak{A}_X$ then $x, y\in P$.

For a given flow (R, X), we will construct a new flow $(R,\tilde{X})$ as follows. For $x, y\in X$, we put $x\sim y$ iff $f(x)=f(y)$ for every $f\in\mathfrak{A}_X$. Then $\sim$ is an equivalence relation on X. For $x\in X$, we denote by $\tilde{x}$ the equivalence class of x. With the quotient topology, $\tilde{X}=X/\sim$ is a compact Hausdorff space and let ϕ: $X\to\tilde{X}$ be the natural map. For $t\in R$ and $\tilde{x}\in\tilde{X}$, we put $T_t\tilde{x}=\widetilde{T_tx}$, then T_t is a homeomorphism of $\tilde{X}$ to $\tilde{X}$.

Proposition 1. *$\{T_t\}_{t\in R}$ induces a flow on $\tilde{X}$.*

Proof. It is sufficient to show that $R\times\tilde{X}\ni(t,\tilde{x})\to T_t\tilde{x}\in\tilde{X}$ is continuous. Let $U(T_t\tilde{x})$ be an open neighborhood of $T_t\tilde{x}$ in $\tilde{X}$. We note that it $\phi^{-1}(\tilde{x})=\{x\}$, then $\phi^{-1}(\tilde{x})\subset P$. For each $y\in\phi^{-1}(\tilde{x})$, there are a neighborhood $V_y(t)$ of t in R and a neighborhood $V(y)$ of y in X such that $T_uz\in\phi^{-1}(U(T_t\tilde{x}))$ for every $u\in V_y(t)$ and $z\in V(y)$. Since $\phi^{-1}(\tilde{x})$ is compact, there are $y_1,\cdots,y_n\in\phi^{-1}(\tilde{x})$ such that $\bigcup_{i=1}^n V(y_i)\supset\phi^{-1}(\tilde{x})$. We put $V(t)=\bigcap_{i=1}^n V_{y_i}(t)$, then $T_uz\in\phi^{-1}(U(T_t\tilde{x}))$ for every $u\in V(t)$ and $z\in\bigcup_{i=1}^n V(y_i)$. Since $\bigcup_{i=1}^n V(y_i)$ is open and $\bigcup_{i=1}^n V(y_i)\supset\phi^{-1}(\tilde{x})$, we have that $(\bigcup_{i=1}^n V(y_i))^c$ is compact, $\phi((\bigcup_{i=1}^n V(y_i))^c)$ is compact in $\tilde{X}$ and $\tilde{x}\in(\phi((\bigcup_{i=1}^n V(y_i))^c))^c$. We put $V_1=(\phi((\bigcup_{i=1}^n V(y_i))^c))^c$, then V_1 is an open neighborhood of $\tilde{x}$. Since $T_uz\in\phi^{-1}(U(T_t\tilde{x}))$ for every $u\in V(t)$ and $z\in\bigcup_{i=1}^n V(y_i)$, we have $T_u\tilde{z}\in U(T_t\tilde{x})$ for $u\in V(t)$ and $\tilde{z}\in V_1$. This implies that $R\times\tilde{X}\ni(t,\tilde{x})\rightarrow T_t\tilde{x}\in\tilde{X}$ is continuous.

By Proposition 1, $\mathfrak{A}_X$ is isometric-isomorphic to $\mathfrak{A}_{\tilde{X}}$, and $\mathfrak{A}_{\tilde{X}}$ separates the points of $\tilde{X}$. So that $\mathfrak{A}_{\tilde{X}}$ is a function algebra on $\tilde{X}$. The flow $(R,\tilde{X})$ is called the normalization of the flow (R,X). In the rest of this paper, we assume that a flow (R,X) is normalized, that is, $\mathfrak{A}_X$ is a function algebra on X.

Next, for a normalized flow (R,X), we study the maximal ideal space of $\mathfrak{A}_X$. Let X_1 be the maximal ideal space of $\mathfrak{A}_X$. For $p\in X_1$, let m_p be one of the representing measures on X of p. For $t\in R$, we have $T_tf\in\mathfrak{A}_X$ for every $f\in\mathfrak{A}_X$. And we have that for $f,g\in\mathfrak{A}_X$,

$$\int fgdT_tm_p=\int T_{-t}fT_{-t}gdm_p=\int T_{-t}fdm_p\int T_{-t}gdm_p$$
$$=\int fdT_tm_p\int gdT_tm_p\ .$$

This implies that T_tm_p is a representing measure for $\mathfrak{A}_X$. If μ_1 and μ_2 are representing measures for $\mathfrak{A}_X$ of the same point $p\in X_1$, then $T_t\mu_1$ and $T_t\mu_2$ are representing measures for $\mathfrak{A}_X$ of the same point of X_1. Because,

$$\left\{f\in\mathfrak{A}_X;\ \int fdT_t\mu_1=0\right\}=\left\{f\in\mathfrak{A}_X;\ \int T_{-t}fd\mu_1=0\right\}=\left\{f\in\mathfrak{A}_X;\ \int T_{-t}fd\mu_2=0\right\}$$
$$=\left\{f\in\mathfrak{A}_X;\ \int fdT_t\mu_2=0\right\}\ .$$

For $t\in R$ and $p\in X_1$, we denote by T_tp the point of X_1 which is represented by T_tm_p. Then T_t: $X_1\rightarrow X_1$ is a one-to-one onto map. Moreover, we have

Proposition 2. *$\{T_t\}_{t\in R}$ induces the flow on X_1.*

Proof. (a) T_t: $X_1\rightarrow X_1$ is a homeomorphism. It is sufficient to show that T_t: $X_1\rightarrow X_1$ is continuous. Let $p_\alpha\rightarrow p$ in X_1 and $f\in\mathfrak{A}_X$. Then $\hat{f}(T_tp_\alpha)=\int fdT_tm_{p_\alpha}=\int T_{-t}fdm_{p_\alpha}\rightarrow\int T_{-t}fdm_p=\hat{f}(T_tp)$. This implies that T_t: $X_1\rightarrow X_1$ is continuous.

(b) For $f\in\mathfrak{A}_X$ and $\varepsilon>0$, there exists $\delta>0$ such that $\|T_tf-f\|_\infty<\varepsilon$ for every real number $|t|<\delta$. Because, suppose that there are $x_n\in X$ and $t_n\in R$ $(n=1,2,\cdots)$ such that $|f(T_{t_n}x_n)-f(x_n)|>\varepsilon$ for every positive integer n and $t_n\rightarrow 0$ $(n\rightarrow\infty)$. We may assume that $x_i\neq x_j$ if $i\neq j$. Then there is $x_0\in X$ which is contained in the cluster points of $\{x_i\}_{i=1}^{\infty}$ in X. Let U be a neighborhood of x_0 in X such that $|f(y)-f(x_0)|<\varepsilon/2$ for every $y\in U$. Since $R\times X\rightarrow X$ is continuous, there exist a neighborhood V_1 of x_0 in X and an interval $I=[-\eta,\eta]$ $(\eta>0)$ such that $T_ty\in U$ for every $t\in I$ and $y\in V_1$. Since x_0 is a cluster point of $\{x_i\}_{i=1}^{\infty}$, there is a sufficient large integer n_0 such that $x_{n_0}\in V_1$ and $t_{n_0}\in I$. Since $T_{t_{n_0}}x_{n_0}\in U$ and $x_{n_0}\in U$, we have

$$|f(T_{t_{n_0}}x_{n_0})-f(x_0)|<\frac{\varepsilon}{2} \quad \text{and} \quad |f(x_{n_0})-f(x_0)|<\frac{\varepsilon}{2}.$$

These imply that $|f(T_{t_{n_0}}x_{n_0})-f(x_{n_0})|<\varepsilon$. This is a contradiction and this completes the proof.

(c) $R\times X_1\ni(t,p)\to T_tp\in X_1$ is continuous. Let $t_n\to t$ in R and $p_\beta\to p$ in X_1. We show that $T_{t_n}p_\beta\to T_tp$. For $f\in\mathfrak{A}_X$, we have

$$\begin{aligned}\left|\int fdT_{t_n}m_{p_\beta}-\int fdT_tm_p\right| &\le \left|\int fdT_{t_n}m_{p_\beta}-\int fdT_tm_{p_\beta}\right|+|\widehat{T_{-t}f}(p_\beta)-\widehat{T_{-t}f}(p)| \\ &\le \|T_{-t_n}f-T_{-t}f\|_\infty+|\widehat{T_{-t}f}(p_\beta)-\widehat{T_{-t}f}(p)| \\ &\to 0 \quad (n\to\infty,\ \beta\to\infty) \quad \text{by (b)}.\end{aligned}$$

These imply that $R\times X_1\ni(t,p)\to T_tp\in X_1$ is continuous. This completes the proof.

Let $\hat{\mathfrak{A}}_X$ be the set of all Gelfand transforms of $f\in\mathfrak{A}_X$, then $\hat{\mathfrak{A}}_X\subset C(X_1)$. We have the following.

Proposition 3. $\hat{\mathfrak{A}}_X\subset\mathfrak{A}_{X_1}$.

Proof. For $p\in X_1$ and $f\in\mathfrak{A}_X$, we show that $\hat{f}(T_tp)\in H^\infty(R)$. We put $F(t)=\int fdT_tm_p=\hat{f}(T_tp)$ and $G(t)=F(-t)$. By Proposition 2 of [1], we have $sp(G)\subset -sp(f)\cap sp(m_p)\subset(-\infty,0]$. Since $sp(G)=-sp(F)$, we have $sp(F)\subset[0,\infty)$. Then $F\in H^\infty(R)$. This shows $\hat{\mathfrak{A}}_X\subset\mathfrak{A}_{X_1}$.

Remark 1. This fact implies that the flow (R,X_1) is normalized and $\mathfrak{A}_{X_1}|_X=\mathfrak{A}_X$, where $\mathfrak{A}_{X_1}|_X$ is the set of all functions of $\mathfrak{A}_{X_1}$ restricted to X.

Remark 2. If $\mathfrak{A}_{X_1}\supsetneqq\hat{\mathfrak{A}}_X$, then there is $f\in\mathfrak{A}_{X_1}\backslash\hat{\mathfrak{A}}_X$ such that $f\neq0$ and $f=0$ on X. For, let $g\in\mathfrak{A}_{X_1}\backslash\hat{\mathfrak{A}}_X$, then there is $h\in\mathfrak{A}_X$ such that $g|_X=h$. We put $f=g-\hat{h}$ then f satisfies the above conditions.

§ 2. Some properties of $\mathfrak{A}_X$.

In this section, we show some basic facts about the function algebra $\mathfrak{A}_X$ on the normalized flow (R,X). For $x\in X$, we put δ_x the unit point mass at x. We denote by P_z the Poisson kernel for evaluation at z in the upper half plane, that is, $P_z(t)=y/\pi(y^2+(x-t)^2)$, where $z=x+iy$, $y>0$.

Lemma 4. ([6], Proposition 3.2) If m is a representing measure for $\mathfrak{A}_X$ and if z is a point in the upper half plane, then $m*P_z$ is a represention measure for $\mathfrak{A}_X$.

Theorem 1. *For a given normalized flow (R, X), the following facts are equivalent.*

1) $\mathfrak{A}_X = C(X)$.
2) $X = P$, *where* $P = \{x \in X;\ T_t x = x \text{ for every } t \in R\}$.
3) $X_1 = X$, *where* (R, X_1) *is the normalized flow obtained in Proposition* 2.
4) $\hat{\mathfrak{A}}_X = \mathfrak{A}_{X_1}$.

Proof. 2)⇒1)⇒3) are trivial.

3)⇒2). Suppose that $X \supsetneqq P$, then there exists $x \in X \backslash P$. Then $\delta_x * P_{iy}$ $(y > 0)$ is a representing measure for $\mathfrak{A}_X$ by Lemma 4. Since $x \notin P$, $\delta_x * P_{iy}$ is quasi-invariant and not invariant by Lemma 2. Suppose that $\delta_x * P_{iy}$ represents the point $z \in X$. Then $\delta_z - \delta_x * P_{iy}$ is invariant by Lemma 3. If $z \in P$, then $\delta_x * P_{iy}$ is invariant, and this is a contradiction. If $z \notin P$, then $\delta_z - \delta_x * P_{iy}$ is not invariant, and this is a contradiction. Then $\delta_x * P_{iy}$ does not represent the point of X. This shows $X_1 \supsetneqq X$.

3)⇒4) is trivial. 4)⇒3). By the condition 4), the maximal ideal space of $\mathfrak{A}_{X_1}$ is X_1. So that we have $\mathfrak{A}_{X_1} = C(X_1)$ by the above argument. Then $\mathfrak{A}_{X_1}|_X = \mathfrak{A}_X = C(X)$, we have $X_1 = X$.

Remark 3. By the proof of Theorem 1, if μ is a representing measure for $\mathfrak{A}_X$ which represents a point x of $X \backslash P$, then we have $\mu = \delta_x$. And this implies that every point $x \in X \backslash P$ is a Choquet boundary point of X.

Proposition 4. *Let (R, X) be a normalized flow. Then the Shilov boundary of $\mathfrak{A}_X$ is X.*

Proof. Let ∂X be the Shilov boundary of $\mathfrak{A}_X$, then by Remark 3 we have $X \backslash \partial X \subset P$. Let $x_0 \in \operatorname{Int} P$, the interior of P in X, and $U(x_0)$ be an open neighborhood of x_0 such that $U(x_0) \subset \operatorname{Int} P$. For $f \in \mathfrak{A}_X$, there is a continuous function g on $U(x_0)$ such that $\tilde{f}(x) \in \mathfrak{A}_X$ and $|\tilde{f}(x_0)| > \sup_{x \notin U(x_0)} |\tilde{f}(x)|$, where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \notin U(x_0) \\ g(x) & \text{if } x \in U(x_0)\,. \end{cases}$$

This shows that $X = \partial X$.

Proposition 5. *Let (R, X) be a normalized flow and $x \in X$. If there is a neighborhood $U(x)$ of x in X_1 such that $U(x) \subset X$, then $x \in \operatorname{Int} P$.*

Proof. Let $x \notin \operatorname{Int} P$ and $U(x)$ an open neighborhood of x in X_1. Then there is $y \in U(x) \cap X$ such that $y \notin P$. For $z > 0$, $\delta_y * P_{it}$ is a representing measure for $\mathfrak{A}_X$ which represents a point in $X_1 \backslash X$. We note that $\{P_{it}\}_{t>0}$ is an approximate identity for $L^1(R)$. Then we have $\lim_{t\to 0}\|f*\tilde{P}_{it}-f\|_\infty=0$ for $f\in\mathfrak{A}_X$ by Lemma 1 of [1]. For $f\in\mathfrak{A}_X$, we have $\left|\int fd\delta_y*P_{it}-\int fd\delta_y\right|=\left|\int(f*\tilde{P}_{it}-f)d\delta_y\right|\to 0\ (t\to 0)$. This is a contradiction.

§ 3. Function algebra $\mathfrak{A}_X$.

In this section, we give necessary and sufficient conditions of normalized flows (R, X) for which $\mathfrak{A}_X$ is essential, antisymmetric, analytic, pervasive and maximal, respectively. Let $P=\{x\in X;\ T_t x=x$ for every $t\in R\}$, then P is a comact subset of X. We put $H=\overline{X\backslash P}$. For a function algebra A on Y, there exists a unique minimal closed subset $E\subset Y$, called the essential set for A, such that if $f\in C(Y)$ and f vanishes on E then $f\in A$. We say that the function algebra A on Y is essential if the essential set for A coincides with Y.

Proposition 6. *For a normalized flow (R, X), H is the essential set for $\mathfrak{A}_X$.*

Proof. If $f\in C(X)$ and $f=0$ on H, then easily we have $f\in\mathfrak{A}_X$. Let E be a closed set with $E\subsetneqq H$. There exists $x\in H\backslash E$ such that $x\notin P$. Then there is an interval $I=[-\varepsilon, \varepsilon]$ such that $\{T_t x;\ t\in I\}\cap E=\phi$. Then there exists $g\in C(X)$ such that $g=0$ on E and $g|_{\{T_t x; t\in I\}}\neq h|_{\{T_t x; t\in I\}}$ for every $h\in\mathfrak{A}_X$. Because $\{T_t x;\ t\in I\}$ is not an interpolation set for $\mathfrak{A}_X$. This implies that E is not an essential set for $\mathfrak{A}_X$, so that H is an essential set for $\mathfrak{A}_X$.

By Proposition 6, we have

Theorem 2. *Let (R, X) be a normalized flow. Then $\mathfrak{A}_X$ is an essential algebra iff* Int $P=\phi$.

For a function algebra A on Y, a closed subset E of Y is called a set of antisymmetric for A if $f\in A$ and $f|_E$ real implies $f|_E$ is constant. If Y is a set of antisymmetric for A, A is called antisymmetric. For $x\in X$, we put $O(x)=\{T_t x;\ t\in R)$.

Lemma 5. *Let $\varphi\in C(X)$, then $sp(\varphi)=\{0\}$ iff $\varphi|_{O(x)}$ is constant for each $x\in X$.*

Proof. $(\Leftarrow)$ Let $\varphi\in C(X)$ and $\varphi|_{O(x)}$ is constant for each $x\in X$. Then we have $J(\varphi)=\{f\in L^1(R);\ \hat{f}(0)=0\}$, where $\hat{f}$ is the Fourier transform of f. Then $sp(\varphi)=\{0\}$.

$(\Rightarrow)$ Suppose that $\varphi|_{O(x)}$ is not constant for some $x\in X$. Then there is $g\in L^1(R)$ such that $\hat{g}(0)=0$ and $\varphi*g(x)\neq 0$. So that we have $J(\varphi)\neq\{f\in L^1(R);\ \hat{f}(0)=0\}$. Since $\{0\}\subset R$ is the set of spectral synthesis, we have $sp(\varphi)\neq\{0\}$.

Lemma 6. *If $\varphi\in C(X)$ and $\varphi|_{O(x)}$ is constant for each $x\in X$, then $\varphi\in\mathfrak{A}_X$.*

Proof. This is a trivial fact.

Theorem 3. *Let (R, X) be a normalized flow. Then $\mathfrak{A}_X$ is antisymmetric iff $\varphi \in C(X)$ and $sp(\varphi)=\{0\}$ imply φ is constant.*

Proof. $(\Leftarrow)$ Let $f \in \mathfrak{A}_X$ be a real function, then $f|_{O(x)}$ is constant for every $x \in X$. By Lemma 6, we have $sp(\varphi)=\{0\}$. By the condition, φ is constant.

$(\Rightarrow)$ Suppose that there exists $\varphi \in C(X)$ such that φ is not constant and $sp(\varphi)=\{0\}$. By Lemma 6, $\varphi|_{O(x)}$ is constant for each $x \in X$. If $|\varphi|$ is not constant, then $|\varphi| \in \mathfrak{A}_X$ by Lemma 6 and $\mathfrak{A}_X$ is not antisymmetric. If $|\varphi|$ is constant, there is a continuous function h on the complex plane such that $|h \circ \varphi|$ is not constant. Since $|h \circ \varphi| \in \mathfrak{A}_X$, $\mathfrak{A}_X$ is not antisymmetric. This completes the proof.

A function algebra A on Y is called analytic if $f \in A$ vanishes on an open set of Y then $f=0$.

Theorem 4. *Let (R, X) be a normalized flow, then the following assertions are equivalent.*

1) *$\mathfrak{A}_X$ is analytic.*
2) *$\mathfrak{A}_X$ is an integral domain.*
3) *Each T_t-invariant open set of X is dense in X.*

Proof. 1)$\Rightarrow$2) is a trivial fact.

2)$\Rightarrow$3). Suppose that Q is a T_t-invariant open subset of X such that Q is not dense in X. We put $Q_1=X \backslash \bar{Q}$, then Q_1 is a non-void T_t-invariant open sbset of X. Here we can consider that (R, Q) and (R, Q_1) are flows on locally compact spaces Q and Q_1, respectively. By Lemma 2 of [1], there are non-zero $f \in C_0(Q)$ and $g \in C_0(Q_1)$ such that $sp(f) \subset [0, \infty)$ and $sp(g) \subset [0, \infty)$, where $C_0(\cdot)$ is the set of all continuous functions which vanish at infinity. We put

$$\varphi_1(x)=\begin{cases} f(x) & \text{if } x \in Q \\ 0 & \text{if } x \in X \backslash Q, \end{cases} \qquad \text{and}$$

$$\varphi_2(x)=\begin{cases} g(x) & \text{if } x \in Q_1 \\ 0 & \text{if } x \in X \backslash Q_1, \end{cases}$$

then $\varphi_1, \varphi_2 \in \mathfrak{A}_X$, $\varphi_1 \neq 0$, $\varphi_2 \neq 0$ and $\varphi_1 \varphi_2=0$. This is a contradiction.

3)$\Rightarrow$1) Let $f \in \mathfrak{A}_X$ and $f=0$ on an open set Q of X. Then $f=0$ on $\{T_t q,\ t \in R\}$ for every $q \in Q$, and $f=0$ on $\bigcup_{t \in R} T_t Q$. Since $\bigcup_{t \in R} T_t Q$ is dense in X, we have $f=0$.

A function algebra A on Y is called pervasive if for every closed subset $E \subsetneqq X$, $\mathfrak{A}_X|_E$ is sup-norm dense in $C(E)$.

Theorem 5. *If (R, X) is a normalized flow, then $\mathfrak{A}_X$ is pervasive iff $\mathfrak{A}_X|_P$ is dense in $C(P)$ and a proper T_t-invariant closed subset of X is contained in P.*

Proof. ($\Rightarrow$) Suppose that there is a closed T_t-invariant subset $F\subsetneqq X$ such that $F\not\subset P$. We note that (R, F) is a flow. Since $\mathfrak{A}_X|_F\subset\mathfrak{A}_F$ and $F\not\subset P$, we have $\mathfrak{A}_F\subsetneqq C(F)$ by Theorem 1. Then $\mathfrak{A}_X|_F$ is not dense in $C(F)$.

($\Leftarrow$) Suppose that $\mathfrak{A}_X$ is not pervasive. Then there is a closed subset $F\subsetneqq X$ such that $\overline{\mathfrak{A}_X|_F}=C(F)$. Then there exists a non-zero $\mu\in M(F)$ such that $\mu\perp\mathfrak{A}_X$. Since μ is quasi-invariant by Forelli [1], $\bigcup_{t\in Q} T_tF^c$ is an open μ-measure zero set, where Q is the set of rational numbers. We put $G=\bigcup_{t\in Q} T_tF^c$. Then $G=\bigcup_{t\in R} T_tF^c$. Since $\mu\in M(G^c)$, G^c is non-void, closed and T_t-invariant. By the condition, we have $G^c\subset P$. Since $\mathfrak{A}_X|_P$ is dense in $C(P)$, we have $\mu=0$. This is a contradiction.

A function algebra A on Y is called maximal if B is a closed subalgebra of $C(Y)$ such that $A\subset B\subset C(Y)$ then $B=A$ or $B=C(Y)$. A flow (R, Y) is called minimal if $O(x)$ is dense in X for every $x\in X$. In [2], Forelli showed that if (R, X) is minimal then $\mathfrak{A}_X$ is maximal. Here we give a necessary and sufficient condition for which $\mathfrak{A}_X$ is maximal. The proof is a precise modification of the proof of Forelli [2].

Lemma 7. ([2], Lemma 2.2) *If* $F\in L^\infty(R)$, *then* $F\in H^\infty(R)$ *iff* $\int F(t)G(t)dt=0$ *for every* G *in* $H^1(R)$.

Lemma 8. ([2], Lemma 2.4) *If* $\varphi\in C(X)$ *and* $f\in H^1(R)$, *then* $\int f(T_tx)F(-t)dt$ *belongs to* $\mathfrak{A}_X$.

We put $U(t, x)=(t, T_tx)$ for $(t, x)\in R\times X$, then U is a homeomorphism of $R\times X$.

Lemma 9. ([2], Lemma 2.6) *Let* $\lambda\in M(R)$ *and* $\mu\in M(X)$. *If* μ *is quasi-invariant, then there is a finite non-negative Baire measurable function* φ *on* $R\times X$ *such that* $\int fd(\lambda\times\mu)=\int(f\circ U)\varphi d(\lambda\times\mu)$ *for all* $f\in L^1(\lambda\times\mu)$.

Theorem 6. *Let* (R, X) *be a normalized flow. Then* $\mathfrak{A}_X$ *is maximal iff* $\mathfrak{A}_X|_P$ *is dense in* $C(P)$ *and* $O(x)$ *is dense in* $X\backslash P$ *for each* $x\in X\backslash P$.

Proof. ($\Leftarrow$) Let B be any closed subalgebra of $C(X)$ that contains $\mathfrak{A}_X$. Suppose that $B\neq C(X)$. Then there is a non-zero measure $\beta\in M(X)$ such that $\beta\perp B$. By Forelli [1], β is quasi-invariant, so that $|\beta|$ is quasi-invariant. For $G\in H^1(R)$ with $G\neq0$, we put $d\lambda=G(t)dt$, and $\mu=|\beta|$, and let φ be a finite non-negative Baire measurable function on $R\times X$ which satisfies Lemma 9. Furthermore let Y and χ be bounded complex measurable functions on R and X, such that $G=Y|G|$ and $\beta=\chi\mu$, respectively. We denote by Z_+ the class of all non-negative integers and Q_+ the class of all non-negative rational numbers. First we note that, since $\mathfrak{A}_X|_P$ is dense in $C(P)$, we have

$$(1) \qquad \beta \notin M(P)\,.$$

We will show that $B \subset \mathfrak{A}_X$. Let $g \in B$. We put $F(t, x) = e^{irt} Y(t)\chi(x)g(x)^k f(T_{-t}x)$, where $r \in Q_+$, $k \in Z_+$ and $f \in C(X)$. Then we have

$$(2) \qquad \int Fd(\lambda \times \mu) = \iint e^{irt} Y(t)\chi(x)g(x)^k f(T_{-t}x)d\lambda(t)d\mu(x)$$
$$= \int\left(\int f(T_{-t}x)e^{irt}G(t)dt\right)g(x)^k d\beta \qquad \text{and}$$

$$(3) \qquad \int (FU)\varphi d(\lambda \times \mu) = \iint e^{irt} Y(t)\chi(T_t x)g(T_t x)^k f(x)\varphi(t, x)d\lambda(t)d\mu(x)$$
$$= \int\left(\int e^{irt}G(t)\chi(T_t x)\varphi(t, x)g(T_t x)^k dt\right)f(x)d\mu(x)\,.$$

Since $e^{irt}G(t) \in H^1(R)$, we have $\int Fd(\lambda \times \mu) = 0$ by Lemma 8. By Lemma 9, we have $\int (F \circ U)\varphi d(\lambda \times \mu) = 0$ for every $f \in C(X)$. Therefore

$$\int e^{irt}G(t)\chi(T_t x)\varphi(t, x)g(T_t x)^k dt = 0$$

for μ almost all x. Since Q_+ and Z_+ are countable sets, there is a Baire set $N \subset X$ of μ measure 0 such that if $x \in N^c$, then

$$(4) \qquad \int e^{irt}G(t)\chi(T_t x)\varphi(t, x)g(T_t x)^k dt = 0$$

for all $r \in Q_+$ and all $k \in Z_+$. We remark that if $x \in N^c$, then

$$(5) \qquad \int |G(t)\chi(T_t x)|\varphi(t, x)dt < \infty\,.$$

By Lemma 9 (with $f(t, x) = |\chi(x)|$), we have

$$\|\lambda\|\,\|\mu\| = \int |\chi(x)|d(\lambda \times \mu) = \int |\chi(T_t x)|\varphi(t, x)d(\lambda \times \mu)\,.$$

We put $\mu = \mu_1 + \mu_2$, where $\mu_1 \in M(P)$ and $\mu_2 \perp M(P)$. Then we have

$$\int |\chi(T_t x)|\varphi(t, x)d(\lambda \times \mu) = \iint |\chi(T_t x)|\varphi(t, x)d\lambda(t)d\mu_1(x) + \iint |\chi(T_t x)|\varphi(t, x)d\lambda(t)d\mu_2(x)$$
$$= \iint \varphi(t, x)d\lambda(t)d\mu_1(x) + (\cdot)$$
$$= \iint \chi' d\lambda(t)d\mu_1(x) + (\cdot)$$
$$= \|\lambda\|\,\|\mu_1\| + \iint |\chi(T_t x)|\varphi(t, x)d\lambda(t)d\mu_2(x)\,,$$

where χ' is a Baire function such that $\mu_1=\chi'\mu$.

By (1), we have $\iint|\chi(T_t x)|\varphi(t, x)d\lambda(t)d\mu_2(x)>0$. This implies that there is $x\in N^c$ with $x\in X\backslash P$ such that

$$\int|G(t)\chi(T_t x)|\varphi(t, x)dt>0 . \tag{6}$$

Fix such an x and we put $H(t)=G(t)\chi(T_t x)$, then $H(t)\in L^1(R)$ with $H(t)\neq 0$ by (5) and (6). By (4), we have $H(t)g(T_t x)^k\in H^1(R)$ for every $k\in Z_+$. Then we have $g(T_t x)\in H^\infty(R)$ by Gamelin ([3], p. 177). Since $H^\infty(R)$ is a translation invariant space, we have $g(T_t y)\in H^\infty(R)$ for every $y\in O(x)$, for each $x\in X$. Let $G\in H^1(R)$. Then $\int g(T_t z)G(t)dt\in C(X)$. By Lemma 7, we have $\int g(T_t y)G(t)dt=0$ for every $y\in O(x)$. Since $O(x)$ is dense in $X\backslash P$, we have $\int g(T_t z)G(t)dt=0$ for every $z\in X\backslash P$ and every $G\in H^1(R)$. By Lemma 7, we have $g(T_t z)\in H^\infty(R)$ for every $z\in X\backslash P$. Therefore $g(T_t z)\in H^\infty(R)$ for every $z\in X$ and we have $g\in\mathfrak{A}_X$.

($\Rightarrow$) *Case* I. Suppose that there is $x\in X\backslash P$ such that $\overline{O(x)}\cup P\subsetneqq X$. Since $\overline{O(x)}$ is a T_t-invariant set, $(R, \overline{O(x)})$ is a normalized flow. We put $B=\{f\in C(X); f|_{\overline{O(x)}}\in\mathfrak{A}_{\overline{O(x)}}\}$, then B is a closed subalgebra of $C(X)$. Since $\mathfrak{A}_{\overline{O(x)}}=C(\overline{O(x)})$, we have $B\subsetneqq C(X)$. Since there exists $y\in X\backslash\overline{O(x)}$ with $y\notin P$, we have $\mathfrak{A}_X\subsetneqq B$.

Case II. Suppose that $\mathfrak{A}_X|_P$ is not dense in $C(P)$. We put $B'=\{f\in C(X); f|_P=\overline{\mathfrak{A}_X|_P}\}$, then easily we have $\mathfrak{A}_X\subsetneqq B'\subsetneqq C(X)$.

Corollary 2. *Let (R, X) be a normalized flow. Suppose that $\mathfrak{A}_X$ is essential, then we have that $\mathfrak{A}_X$ is pervasive iff $\mathfrak{A}_X$ is maximal.*

Proof. ($\Rightarrow$) Suppose that $\mathfrak{A}_X$ is not maximal. By Theorem 6, there is a T_t-invariant closed subset $F\subsetneqq X$ such that $F\not\subset P$. Then there is $x\in F\backslash P$, and $\overline{O(x)}\subset F$. If $O(x)$ is dense in $X\backslash P$, then $X\backslash P\subset F$ and this contradicts with $\mathrm{Int}\, P=\phi$. Consequently, $O(x)$ is not dense in $X\backslash P$, and $\overline{O(x)}\supset X\backslash P$. By Theorem 5, $\mathfrak{A}_X$ is not pervasive.

($\Leftarrow$) For $x\in X\backslash P$, we have $\overline{O(x)}=X$. This implies that $O(x)$ is dense in X. By Theorem 5, this completes the proof.

Note added in proof. Professor Tomiyama points out that $\mathrm{int}\, P$, the interior of P in X, is open in X_1 by his paper: Some remarks on antisymmetric decompositions of function algebras, Tôhoku Math. J., 16 (1964), 340–344. Because let Q be the collection of all antisymmetric sets of $\mathfrak{A}_X$ in X which consist of a single point, and let Q^i be the essential set of $\mathfrak{A}_X$ in X by Theorem 3 of Tomiyama's paper. By Proposition 6, we have $\overline{X\backslash P}=X\backslash Q^i$ so that $\mathrm{int}\, P=Q^i$.

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