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ON THE WEAK SOLUTIONS FOR THE CAUCHY PROBLEM OF PARABOLIC EQUATIONS WITH DISCONTINUOUS AND UNBOUNDED COEFFICIENTS*

(Dedicated to Professor Fan Ky on his 60th birthday)

By

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ABSTRACT. In this note we shall prove the uniqueness and existence of the weak solutions for the Cauchy problem:

$$\begin{cases} Lu=f & \text{for } (x,t)\in R^n\times(0,T] \\ u(x,0)=u_0(x)\in L^2_{loc}(R^n) & \text{for } x\in R^n \end{cases}$$

where the coefficients of L are measurable real valued functions and satisfy some assumptions and f is a given function in $R^n\times(0,T]$.

1. Let L be a uniformly parabolic partial differential operator of the form

$$(1)\qquad Lu\equiv\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right)+\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}+cu-\frac{\partial u}{\partial t}$$

in the $(n+1)$-dimensional Euclidean half space $\Omega=R^n\times(0,\infty)$, where $x=(x_1,\cdots,x_n)$ is the space variable, $t\in(0,\infty)$ is the time variable and the coefficients a_{ij} $(=a_{ji})$, b_i, c are measurable in Ω. Under the assumptions that a_{ij} are bounded and that b_i, c belong to some functions spaces in $|x|<R_0$ and b_i are bounded in $|x|>R_0$, Aronson [1] discussed the Cauchy problem for inhomogeneous equations $Lu=f$ and proved the existence and the uniqueness of the weak solutions of the Cauchy problem in some strip domain $\Omega_T=R^n\times(0,T)$. Bodanko [2] also discussed the question of a regular solution for the Cauchy problem of linear parabolic equations of the form

$$\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}+c(x,t)u-\frac{\partial u}{\partial t}=f(x,t)$$

with unbounded coefficients under some assumptions. Recently, Ikeda [3] using Aronson's idea, also proved the existence and the uniqueness of the weak solutions of the Cauchy problem for $Lu=f$ under somewhat weaker conditions on

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coefficients of L. In Ikeda's results, the coefficients a_{ij} and b_i need not be bounded in $|x|>R_0$. More recently the first author [4] discussed the asymptotic behavior of weak solutions of the Cauchy problem for some parabolic equations. In this note we shall prove the uniqueness and existence of the weak solutions for the same problem.

Our method to establish this result is a modification of a procedure suggested by Ikeda [3].

2. Let T be a fixed positive number and consider the domain $Q=R^n\times(0, T]$. For some fixed positive number R_0 we put $Q_0=\Sigma_{R_0}\times(0, T]$, where $\Sigma_{R_0}=\{x\in R^n; |x|<R_0\}$. Now let ω be a domain in R^n.

The space $L^q[0, T; L^p(\omega)]$ is the set of real functions $W(x, t)$ with the following properties;

(i) $W(x, t)$ is defined and measurable in $\tilde{\omega}=\omega\times(0, T]$,

(ii) $W\in L^p(\omega)$ for almost all $t\in(0, T]$,

(iii) $\|W\|_{L^p(\omega)}(t)\in L^q((0, T])$.

The space $L^q[0, T; L^p(\omega)]$ is denoted by $L^{p,q}(\tilde{\omega})$. For $W\in L^{p,q}(\tilde{\omega})$ with $1\leq p$, $q<\infty$ we define

$$\|W\|_{p,q,\tilde{\omega}}=\left(\int_0^T\left(\int_\omega|W|^p dx\right)^{q/p}dt\right)^{1/q}.$$

In case either p or q is infinite, $\|W\|_{p,q}$ is defined in a similar fashion using L^∞-norms rather than integers.

We assume that the coefficients of the operator L are measurable in Q and satisfy the following conditions:

(H.1) For all $\xi\in R^n$ and for almost all (x, t) there exist positive constants k_1, K_1, non-negative constant λ, and any real number ν such that

$$\begin{aligned}&k_1(|x|^2+1)^{1-\lambda}[\log(|x|^2+1)+1]^{-\nu}|\xi|^2\\&\qquad\leq\sum_{i,j=1}^n a_{ij}\xi_i\xi_j\leq K_1(|x|^2+1)^{1-\lambda}[\log(|x|^2+1)+1]^{-\nu}|\xi|^2\end{aligned}$$

for all real n-vectors $\xi=(\xi_1, \cdots, \xi_n)$.

(H.2) The restriction of every coefficient b_i to Q_0 belongs to some space $L^{p_i,q_i}(Q_0)$, where p_i and q_i satisfy

$$(*)\qquad 2<p_i,\ q_i\leq\infty\quad\text{and}\quad\frac{n}{2p_i}+\frac{1}{q_i}<\frac{1}{2}$$

and there exist a non-negative constant K_2 such that

$$|b_i|\leq K_2(|x|^2+1)^{1/2}\qquad\text{for}\quad(x, t)\in Q-Q_0\,.$$

(H.3) The restriction of c to Q belongs to $L^{p,q}(Q_0)$, where p and q satisfy

$$(**)\qquad 1<p,\ q\leq\infty \quad\text{and}\quad \frac{n}{2p}+\frac{1}{q}<1\,,$$

and $c\leq K_3(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu$ in $(x, t)\in Q-Q_0$ for a non-negative constant K_3.

The function $u(x)$ defined and measurable in ω is said to belong to $H^{1,p}(\omega)$ if $u(x)$ possesses a distribution derivative $(u_{x_1}, \cdots, u_{x_n})$ and $\|u\|_{L^p(\omega)}+\|u_x\|_{L^p(\omega)}<\infty$, where

$$\|u_x\|_{L^p}^p=\sum_{i=1}^n \|u_{x_i}\|_{L^p}^p\,.$$

The space $H_0^{1,p}(\omega)$ is the completion of the $C_0^\infty(\omega)$ functions in this norm. The space $H^{1,p}(R^n)$ is the completion of the $C_0^\infty(R^n)$ functions in the norm $\|\phi\|_{L^p(R^n)}+\|\phi_x\|_{L^p(R^n)}$.

A function $u(x, t)\in L^\infty[0, T; L_{\text{loc}}^2(R^n)]\cap L^2[0, T; H_{\text{loc}}^{1,2}(R^n)]$ is said to be a weak solution of the equation $Lu=f$ in Q and for the Cauchy data $u_0(x)\in L_{\text{loc}}^2(R^n)$, if the equality

$$(2)\qquad \iint_Q\left\{-\frac{\partial\phi}{\partial t}u+\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial\phi}{\partial x_j}-\sum_{i=1}^n b_i\frac{\partial u}{\partial x_i}\phi-cu\phi+f\phi\right\}dxdt=0$$

holds for any $T\in(0, \infty)$ and for any $\phi=C_0^1(Q)$ and if

$$(3)\qquad \lim_{t\to 0}\int_{R^n} u(x, t)\psi(x)dx=\int_{R^n} u_0(x)\psi(x)dx$$

is valid for any $\psi\in C_0^1(R^n)$.

3. Let D be a fixed bounded domain in R^n such that $D\supset\Sigma_{R_0}$, and set $Q_1=D\times(0, T]$. In Q_1 we consider the equation

$$(4)\qquad Lu=f$$

where $f\in L^{p,q}(Q_1)$ for any p and q satisfying $(**)$.

We prepare the useful lemmas derived from the results of Aronson and Ikeda, whose proofs will be omitted.

Lemma 1. (cf. [1], Lemma 2) *If* $W\in L^\infty[0, T; L^2(R^n)]\cap L^2[0, T; H_0^{1,2}(R^n)]$ *then* $W\in L^{2p',2q'}(Q)$ *for all values of* p' *and* q' *whose Hölder conjugates* p *and* q *satisfy*

$$\frac{n}{2p}+\frac{1}{q}\leq 1\,,$$

where if $n=2$, *then the strict inequality holds. Moreover*

$$\|W\|^2_{2p',2q',Q}\leq KT^{\Theta}\{\|W\|^2_{2,\infty,Q}+\|W_x\|^2_{2,2,Q}\}\ ,$$

where $|W_x|^2=\sum_{i=1}^n W^2_{x_i}$, $\Theta=1-1/q-n/(2p)$, *and K is a positive constant which depends only on n for $n\neq 2$ and only on p for $n=2$.*

Lemma 2. (cf. [1], Lemma 1 and [3], Lemma 3.1) *Let* $u\in L^\infty[0, T; L^2(D)]\cap L^2[0, T; H^{1,2}(D)]$ *be a weak solution of* (4) *with the Cauchy data* u_0, *that is, let u satisfy*

$$(5)\qquad \iint_{Q_1}\left\{-\frac{\partial\phi}{\partial t}u+\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial\phi}{\partial x_j}-\sum_{i=1}^n b_i\frac{\partial u}{\partial x_i}\phi-cu\phi+f\phi\right\}dxdt=0$$

for any $\phi\in C^1_0(Q_1)$ *and* (3), *and let* $\zeta=\zeta(x)$ *be a non-negative smooth functions such that* $\zeta u\in L^2[0, T; H^{1,2}_0(D)]$. *Then for any positive number* μ_0, *there exist positive constants* ε *and* μ *such that*

$$\begin{aligned}(6)\qquad &\|\zeta u\exp\{-\mu(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}\|^2_{2,\infty,Q_1'}\\ &\quad+\|\zeta u_x\exp\{-\mu(|x|^2+1)^\lambda(\log(|x|^2+1)+1]^\nu\}\|^2_{2,2,Q_1'}\\ &\leq\varepsilon\left\{\int_D\zeta^2\exp\{-2\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}u_0^2dx\right.\\ &\quad+\|\zeta_x\exp\{-\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}\\ &\quad\times(|x|^2+1)^{(1-\lambda)/2}[\log(|x|^2+1)+1]^{-\nu/2}u\|^2_{2,2,Q_1'}\\ &\quad+\|\zeta\exp\{-\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}\\ &\quad\times(|x|^2+1)^{(\lambda-1)/2}[\log(|x|^2+1)+1]^{\nu/2}f\|^2_{2p/(p+1),2q/(q+1),Q_1'}\\ &\quad+\|\zeta\exp\{-\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}\\ &\quad\left.\times(|x|^2+1)^{(1-\lambda)/2}[\log(|x|^2+1)+1]^{-\nu/2}u\|^2_{2,2,Q_1'}\right\}\end{aligned}$$

where $Q_1'=D\times(0, T')$ *with* $T'<\pi/2$. *The constants* T', ε *depend only on* k_1, K_1, K_2, K_3, λ, μ_0, ν, $\|b_i\|_{p_j,q_j,Q_0}$ *and* $\|c\|_{p,q,Q_0}$; *and* μ *depends on* μ_0 *and* T'.

4. We consider the following Cauchy problem:

$$(7)\qquad \begin{cases}Lu=f & \text{for}\quad (x,t)\in Q\\ u(x,0)=u_0(x) & \text{for}\quad x\in R^n\end{cases}$$

where the coefficients a_{ij}, b_i and c are measurable, real valued functions in Q and f is a given function in Q. If the measurable function $u(x,t)\in\mathscr{E}^{\lambda;\nu}_\mu(Q)$ then there exist constants $\lambda\geq 0$, and ν, $\mu>0$ such that $u(x,t)$ satisfies

$$\iint_D\exp\{-2\mu(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}u^2dxdt<+\infty\ .$$

Theorem 1. *If there are solutions u of the problem* (7) *in the class* $\mathscr{E}_{\mu_1}^{\lambda,\nu}(Q)$ *for some positive constant* $\mu_1<\mu_0$, *then u is uniquely determined in Q.*

Proof. If there were two solutions u_1 and u_2 of the problem (7) in the class $\mathscr{E}_{\mu_1}^{\lambda,\nu}(Q)$, then their difference $u=u_1-u_2$ would also in $\mathscr{E}_{\mu_1}^{\lambda,\mu}(Q)$ and be a weak solution of the problem

$$Lu=0 \quad \text{for} \quad (x, t) \in Q\,, \qquad u(x, 0)=0 \quad \text{for} \quad x \in R^n\,.$$

For each $R\geq R_0$, we set $\zeta=\zeta_R(x)$ in such a way that

$$|\zeta_R(x)|\leq 1 \quad \text{in} \quad (-\infty, \infty)\,,$$
$$\zeta_R(x)=\begin{cases}1 & |x|\leq R\\ 0 & |x|\geq R+1\end{cases}$$
$$|\zeta_R(x)|\leq c \quad \text{in} \quad (-\infty, \infty), \quad \text{where} \quad c \quad \text{is independent of} \quad R\,.$$

From Lemma 2 and $\mu_1<\mu_0<\mu$,

$$\begin{aligned}(8)\qquad &\|\zeta_R u \exp\{-\mu(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}\|_{2,\infty}^2\\ &\leq\varepsilon\{\|\zeta_{R_x} \exp\{-\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}\\ &\quad\times(|x|^2+1)^{(1-\lambda)/2}[\log(|x|^2+1)+1]^{-\nu/2}u\|_{2,2,Q'}^2\\ &\quad+\|\zeta_R \exp\{-\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}\\ &\quad\times(|x|^2+1)^{(1-\lambda)/2}[\log(|x|^2+1)+1]^{-\nu/2}u\|_{2,2,Q'}^2\}\\ &\leq\tilde{\varepsilon}\int_T^{T'}\int_{|x|\geq R} \exp\{-2\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}\\ &\quad\times(|x|^2+1)^{1-\lambda}[\log(|x|^2+1)+1]^{-\nu}u^2dxdt\,,\end{aligned}$$

where $\tilde{\varepsilon}$ is independent of R. Since $\exp\{-\mu_1(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}u\in L^2[0, T'; L^2(R^n)]$ for $\mu_0>\mu_1$, we easily see

$$\begin{aligned}&\exp\{-\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}(|x|^2+1)^{(1-\lambda)/2}\\ &\qquad\times[\log(|x|^2+1)]^{-\nu/2}u\in L^2[0, T'; L^2(R^n)]\,.\end{aligned}$$

The integer on the right in (8) tends to zero as $R\to\infty$. Hence

$$\max_{[0,T']}\int_{|x|\leq\rho}\exp\{-2\mu(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}u^2dx=0$$

for arbitrary $\rho>0$. This means that $u\equiv 0$ in $R^n\times(0, T']$.

If $T'=T$ this completes the proof, otherwise the proof can be completed by a finite number of applications of the same argument on $R^n\times(T', 2T')$, $R^n\times(2T', 3T')$, etc, we conclude that $u_1\equiv u_2$ in $R^n\times(0, T]$.

By the same argument to that of [1], we can prove the following theorem whose proof will be omitted.

Theorem 2. *Assume that* $\exp\{-\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}f\in L^{p,q}(Q)$ *with* p *and* q *satisfying* $(**)$ *and* $\exp\{-\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}u_0\in L^2(R^n)$. *Then there exists a weak solution* u *of the problem* (7) *in* $Q'=R^n\times(0, T')$, *where* T' *depends on the constants in* (H.1), (H.2) *and* (H.3) *and* μ_0. *Moreover there exists a constant* μ *depending on* T' *and* μ_0 *such that*

$$\begin{aligned}
&\|u\exp\{-\mu(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}\|^2_{2,\infty,Q'}\\
&\qquad+\|u_x\exp\{-\mu(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}\|^2_{2,2,Q'}\\
&\qquad<\varepsilon\|\exp\{-\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}u_0\|^2_{L^2(R^n)}\\
&\qquad+\|\exp\{-\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}\\
&\qquad\times(|x|^2+1)^{(1-\lambda)/2}[\log(|x|^2+1)+1]^{-\nu/2}f\|^2_{p,q,Q'}\\
&\qquad+\|\exp\{-\mu_0(|x|^2+1)^\lambda[\log(|x|^2+1)+1]^\nu\}\\
&\qquad\times(|x|^2+1)^{(1-\lambda)/2}[\log(|x|^2+1)+1]^{-\nu/2}u\|^2_{2,2,Q'}\,.
\end{aligned}$$

Remark. (i) In Lemma 2, if $\lambda=\nu=0$, then we put $\alpha(t)=Bt+\mu_0$. Thus Lemma 2 is valid for $Q=R^n\times(0, T]$ if $\lambda=\nu=0$. Therefore if $\lambda=\nu=0$, a weak solution of the problem (7) exists in $Q=R^n\times(0, T]$.

(ii) The Theorem 1 and Theorem 2 of Ikeda [3] with $f_j=0$, $g=f$ are respectively two special cases of our Theorems 1 and 2 with $\nu=0$ and fixed $0\le\lambda\le 1$.

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