A REMARK ON THE NUMERICAL SOLUTION OF A SINGLE FIRST-ORDER ORDINARY DIFFERENTIAL EQUATION

By

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1. Abstract

In this paper we will discuss methods for the numerical solution of a single first-order ordinary differential equation

$$\frac{dy}{dx}=f(x, y) \equiv \frac{Q(x, y)}{P(x, y)}$$

with the initial condition $y(a)=b$.

To seek the solution, we often use Runge-Kutta method provided that $f(x, y)$ is continuous and satisfies Lipschitz condition

$$|f(x, y_1)-f(x, y_2)| \leqq M|y_1-y_2| .$$

If $P(x_0, y_0)=0$ and $Q(x_0, y_0) \neq 0$, then $dy/dx=\infty$, therefore we can not use Runge-Kutta method in the neighbourhood of (x_0, y_0). Nevertheless, there often exists the solution. (ex. 1) In this case, putting

$$g(x, y)=\frac{P(x, y)}{Q(x, y)}$$

$g(x, y)$ often satisfies Lipschitz condition

$$|g(x_1, y)-g(x_2, y)| \leqq M'|x_1-x_2|$$

in the neighbourhood of (x_0, y_0).

Therefore, when we use Runge-kutta method provided with

$$P(x, y)^2+Q(x, y)^2 \neq 0 ,$$

if $|P(x, y)| \geqq |Q(x, y)|$, we treat y as the function of the independent variable x, if $|P(x, y)|<|Q(x, y)|$, we treat x as the function of the independent variable y, that is,

$$\text{if } \left|\frac{dy}{dx}\right| \leqq 1 , \quad \text{then } \frac{dy}{dx}=\frac{Q(x, y)}{P(x, y)} ,$$

and

$$\text{if } \left|\frac{dy}{dx}\right| >1 , \quad \text{then } \frac{dx}{dy}=\frac{P(x, y)}{Q(x, y)} ,$$

then, as $|dy/dx|$ and $|dx/dy|$ are both smaller than 1, the error of the increment will be smaller.

If $P(x_0, y_0)^2+Q(x_0, y_0)^2=0$, we must stop pursuing the solution at (x_0, y_0) as (x_0, y_0) is a singular point. (ex. 6～ex. 11)

Furthermore, if we can find proper continuations between dy/dx and dx/dy, we can even pursue closed curves. (ex. 1)

2. Main Programme

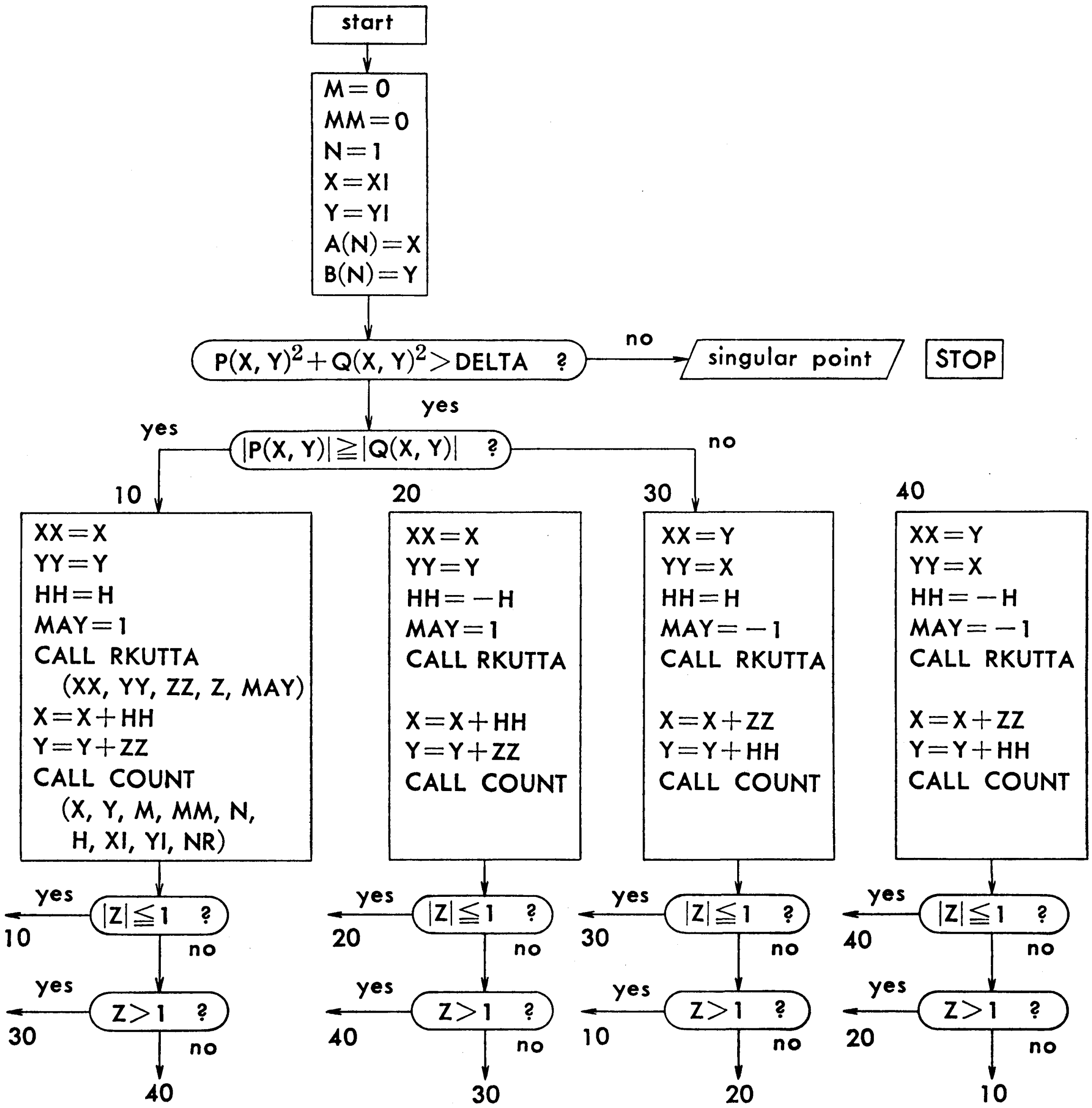
DIMENSION A(1000), B(1000)

(XI, YI); initial condition (initial point)

H; increment of the independent variable X (or Y) (it may be negative)

NR; repeated number

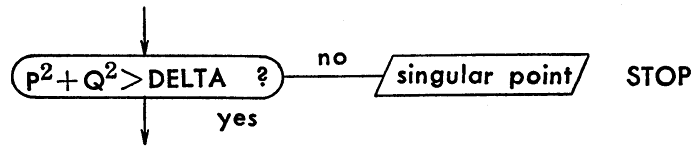
$DELTA$; detection of the singular points



3. Subroutine RKUTTA (X, Y, H, ZZ, Z, MAY)

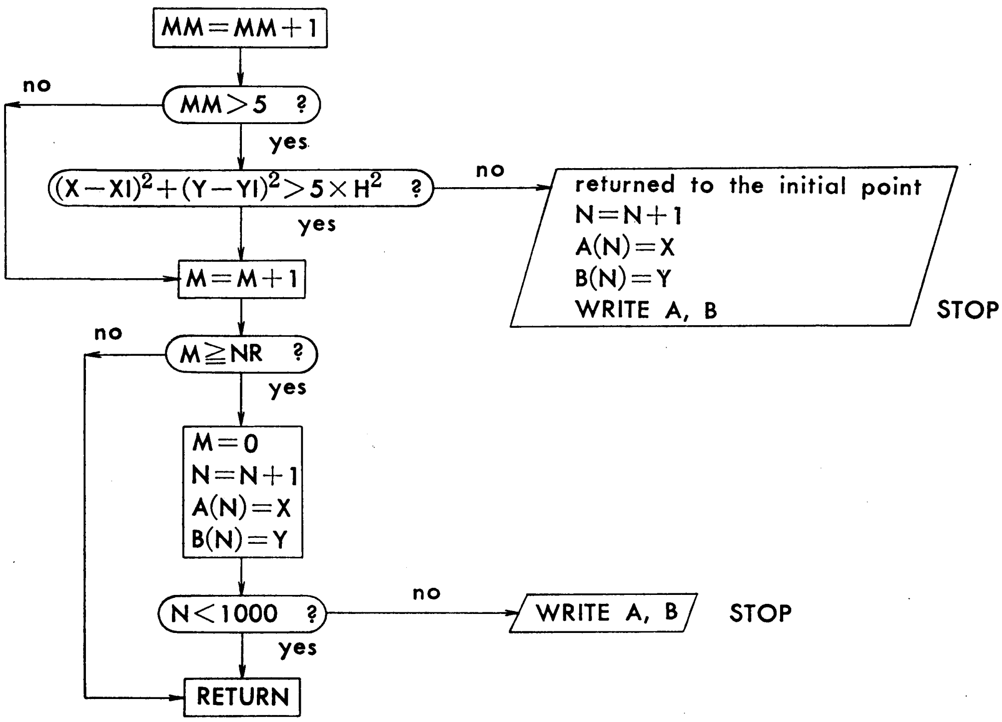
When $MAY=1$, putting $Z=Q(X, Y)/P(X, Y)$ $(=dy/dx)$, we calculate the increment ZZ of dependent variable Y by Runge-Kutta, and when $MAY=-1$, putting $Z=P(Y, X)/Q(Y, X)$ $(=dx/dy)$, we calculate the increment ZZ of dependent variable X by Runge-Kutta.

In the calculation procedure of $P(X, Y)$ and $Q(X, Y)$ (or $P(Y, X)$ and $Q(Y, X)$) we must check up "$P^2+Q^2>DELTA$", that is,



Here, $DELTA$ depends on the structure of the singular point. By putting $DELTA=5\times H^2 \sim 20\times H^2$, we shall achieve success in the detection of the singular point with the exception of it being a focus. When the singular point is a focus, $DELTA$ should be greater. (ex. 11)

4. Subroutine COUNT (X, Y, M, MM, N, H, XI, YI, NR)



In the programme, we use $5\times H^2$ for the detection of the return to the initial point (XI, YI). If N is greater than 1000, $5\times H^2$ might be replaced by a larger number $6\times H^2$, $7\times H^2$, $\cdots$. By $5\times H^2$ we completely succeeded in the detection of the return when $H=0.001$, $N\leqq 1000$ and $NR\leqq 20$.

5. A note on Lipscithz condition

We will here cite $dy/dx=\sqrt{|y|}$ as an illustrative instance. This ordinary differential equation does not satisfy Lipschitz condition at $y=0$, and the innumerable solutions.

The functions

$$y=f_1(x)\equiv 0 \qquad \text{(ex. 3)}$$

$$y=f_2(x)\equiv\begin{cases} \dfrac{x^2}{4} & x\geqq 0 \\ -\dfrac{x^2}{4} & x<0 \end{cases} \qquad \text{(ex. 2)}$$

are both differentiable in the infinite interval, satisfies the equation there and $f(0)=0$.

Therefore the function

$$y=f(x)\equiv\begin{cases} \dfrac{(x-a)^2}{4} & x>a \\ 0 & a\geqq x\geqq b \quad (a\geqq 0\geqq b) \\ -\dfrac{(x-b)^2}{4} & b>x \end{cases}$$

is a one-valued function which satisfies the equation with $f(0)=0$. Thus, there are the innumerable solutions with $f(0)=0$.

When $f(x, y)$ (or $g(x, y)$) is continuous and does not satisfy Lipschitz condition at (x_0, y_0), it may be natural to regard the point (x_0, y_0) as a singular point especially in the numerical solution. In the case of $dy/dx=\sqrt{|y|}$, $\sqrt{|y|}$ should be replaced by $|y|/\sqrt{|y|}$ to make the point a singular one. Thereby the equation $dy/dx=|y|/\sqrt{|y|}$ is identified with $dy/dx=\sqrt{|y|}$ and satisfies Lipschitz condition if $y\neq 0$, and has singular points on $y=0$. (ex. 4, ex. 5)

6. Examples

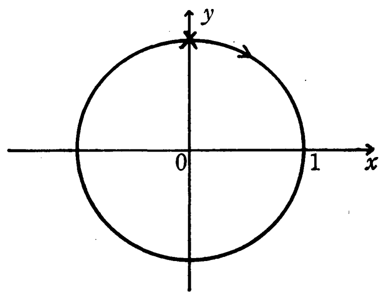
We will show several consequences of our experiments as examples. (* shows the initial point, and $\circ$ the singular point.)

ex. 1 $\dfrac{dy}{dx}=-\dfrac{x}{y}$

$XI=0.0$
$YI=1.0$
$H=0.001$
$NR=10$
$DELTA=0.000005$

$x^2+y^2=1$

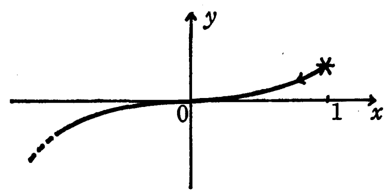
(Returned to the initial point)



ex. 2 $\dfrac{dy}{dx}=\sqrt{|y|}$

$XI=1.0$
$YI=0.25$
$H=-0.01$
$NR=1$
$DELTA=0.002$

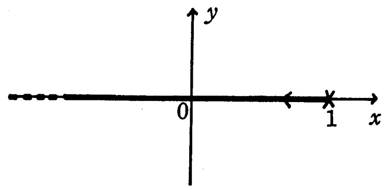
$$y=\begin{cases} \dfrac{x^2}{4} & x\geqq 0 \\ -\dfrac{x^2}{4} & x<0 \end{cases}$$



ex. 3 $\dfrac{dy}{dx}=\sqrt{|y|}$

$XI=1.0$
$YI=0.0$
$H=-0.01$
$NR=1$
$DELTA=0.002$

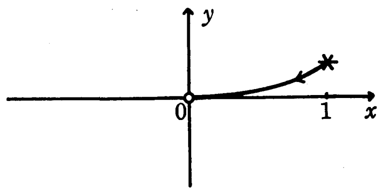
$y=0$



ex. 4 $\dfrac{dy}{dx}=\dfrac{|y|}{\sqrt{|y|}}$

$XI=1.0$
$YI=0.25$
$H=-0.01$
$NR=1$
$DELTA=0.002$

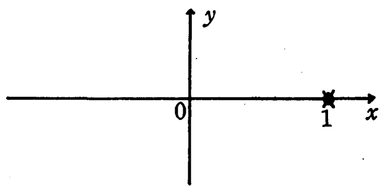
$y=\dfrac{x^2}{4} \quad x>0$



ex. 5 $\dfrac{dy}{dx}=\dfrac{|y|}{\sqrt{|y|}}$

$XI=1.0$
$YI=0.0$
$H=-0.01$
$NR=1$
$DELTA=0.002$

$y=0$; singular points

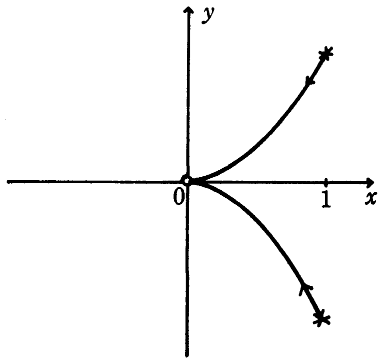


ex. 6 $\dfrac{dy}{dx}=\dfrac{3x^2}{2y}$

$y^2=x^3$

(0, 0); cusp

XI $=1.0$
YI $=\pm 1.0$
H $=-0.001$
NR $=10$
$DELTA$ $=0.000005$



ex. 7 $\dfrac{dy}{dx}=\dfrac{y+x}{x}$

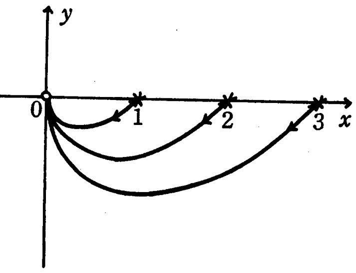
$y=x\log x$

$y=x\log\dfrac{x}{2}$

$y=x\log\dfrac{x}{3}$

(0, 0); end point

XI $=1.0,\ 2.0,\ 3.0$
YI $=0.0$
H $=-0.001$
NR $=10$
$DELTA$ $=0.000005$

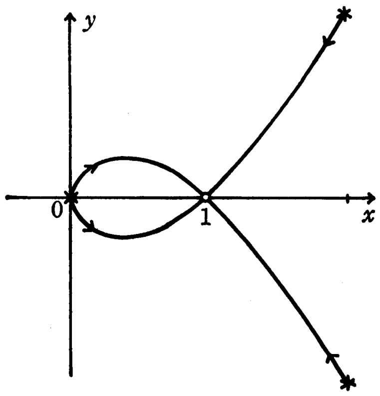


ex. 8 $\dfrac{dy}{dx}=\dfrac{3x^2-4x+1}{2y}$

$y^2=x(x-1)^2$

(1, 0); node

$XI=2.0$ $XI=0.0$
$YI=\pm\sqrt{2}$ $YI=0.0$
$H=\mp 0.001$ $H=\pm 0.001$
NR $=10$
$DELTA$ $=0.000005$

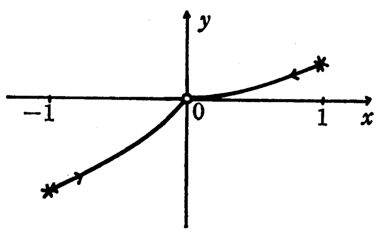


ex. 9 $\dfrac{dy}{dx}=\dfrac{y(x-y)+yx^2}{x^3}$

$y=\dfrac{x}{1+e^{1/x}}$

(0, 0); salient point

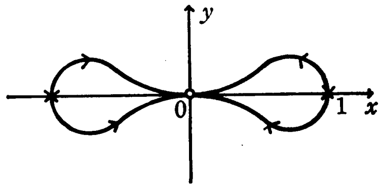
$XI=1.0$ $YI=-1.0$
$YI=\dfrac{1}{1+e}$ $YI=\dfrac{-e}{1+e}$
$H=-0.001$ $H=0.001$
NR $=10$
$DELTA$ $=0.000005$



ex. 10 $\dfrac{dy}{dx}=\dfrac{x^3(2-3x^2)}{y}$

$y^2=x^4(1-x^2)$
(0, 0); tac node

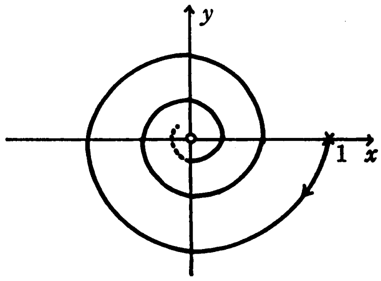
XI	$=\pm 1.0$
YI	$=0.0$
H	$=\pm 0.001$
NR	$=10$
$DELTA$	$=0.00005$



ex. 11 $\dfrac{dy}{dx}=\dfrac{x+10y}{y-10x}$

$r=e^{(1/10)\theta}$
(0, 0); focus

XI	$=1.0$
YI	$=0.0$
H	$=-0.001$
NR	$=10$
$DELTA$	$=0.0005$



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References

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