ON HYPERSURFACES IN A HOMOGENEOUS RIEMANNIAN MANIFOLD

By

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(Received May 4, 1977)

Introduction.

We obtained the motive of this paper from the following elementary property of a closed curve in the Euclidean 2-plane R^2.

Proposition. *Let* $c: [0, a] \to R^2$ *be a closed curve of class* C^2 *parametrized by arc-length which is cotained in a closed ball in* R^2 *of radius* r. *Then either* $|k(s)| > 1/r$ *holds for some* $s \in [0, a]$ *or* $c([0, a])$ *is contained in the circle of radius* r *where* k *denotes the curvature (defined up to a sign) of* c.

We want to extend this proposition to immersed hypersurfaces which are contained in a domain with regular smooth boundary in a Riemannian manifold in terms of mean curvature.

In this paper we investigate the problem stated above in the case where ambieant spaces are homogeneous Riemannian manifolds. Theorem 1.1 is the main theorem of this paper which is an extension of the proposition stated above. In order to prove Theorem 1.1 we need to study a quasilinear elliptic partial differential equation of second order. It will be carried out in Section 2. We think that the results obtained there is useful for studies of hypersurfaces in a Riemannian manifold. The complete proof of Theorem 1.1 will be given in Section 4.

In the latter half of Section 4 we study some properties of minimal hypersurfaces in a homogeneous Riemannian manifold. In the last section we generalize the results obtained in [6].

1. Hypersurfaces in a homogeneous Riemannian manifold.

Throughout this paper we assume that Riemannian manifolds and apparatus on them are of class C^∞ and that manifolds are connected unless otherwise stated.

Theorem 1.1. *Let* N *be a homogeneous Riemannian manifold of dimension* $n+1 (n \geqq 1)$, *and let* D *be a domain in* N *with regular smooth boundary* ∂D *and* $\mathscr{H}$ *the mean curvature of* ∂D *with respect to the inward unit normal vector to* ∂D. *Let* $f: M \to N$ *be an isometric immersion of an* n*-dimensional compact*

Riemannian manifold M into N such that $f(M)\subset \bar{D}$ where $\bar{D}=D\cup\partial D$. Suppose that for a non-negative constant H_0 $\mathscr{H}$ satisfies the condition

$$\mathscr{H}\geqq H_0 \tag{1.1}$$

at each point of ∂D. Then either $|H(m)|>H_0$ holds at some point m of M or there exists an isometry φ of N such that $\varphi(f(M))\subset\partial D$ where H denotes the mean curvature (defined up to a sign) of M for isometric immersion f.

The proof of theorem 1.1 will be given in Section 4. The following argument is used in the proof of Theorem 1.1 and of Theorem 4.2.

Let N be a homogeneous Riemannian manifold of dimension $n+1(n\geqq 1)$. Let D be a domain in N with regular smooth boundary ∂D and $\mathscr{H}$ the mean curvature of ∂D with respect to the inward unit normal vector to ∂D. Let m_0 be a point of ∂D. Since N is homogeneous, there exists a Killing vector field X on N such that X coincides with the inward unit normal vector to ∂D at m_0. Let $\{\varphi_t\}$, $|t|<\tau'$, be the local 1-parameter subgroup of local transformations generated by X. Then, since X is a Killing vector field, φ_t is isometric for each $t\in(-\tau',\tau')$. We can choose a local coordinate neighborhood U of m_0 in ∂D and a positive τ so that the mapping $\Phi: U\times(-\tau,\tau)\to N$ defined by $\Phi(m,t)=\varphi_t(m)$ for $(m,t)\in U\times(-\tau,\tau)$ is imbedding. Let $(x_1,\cdots,x_n)$ be the local coordinate system on U. We denote by $\langle\,,\rangle$ the Riemannian metric tensor on $U\times(-\tau,\tau)$ induced by Φ. We set $g_{\alpha\beta}=\langle\partial/\partial x_\alpha,\partial/\partial x_\beta\rangle$, $1\leqq\alpha,\beta\leqq n+1$, where we put $x_{n+1}=t$. Since φ_t is isometric for each $t\in(-\tau,\tau)$ and X coincides with the inward unit normal vector to ∂D at m_0, we see that $g_{\alpha\beta}(1\leqq\alpha,\beta\leqq n+1)$ are independent of t, $t\in(-\tau,\tau)$, and that $g_{in+1}(m_0,t)=0$, $1\leqq i\leqq n$. For simplicity, in what follows, we shall use the following notations:

$$g_i=g_{in+1}\,,\quad 1\leqq i\leqq n\,,\quad\text{and}\quad g=g_{n+1n+1}\,.$$

For an open subset V of ∂D we shall denote by $C^2(V)$ the set of real-valued functions of class C^2 on V and we shall put $u_i=\partial u/\partial x_i$, $u_{ij}=\partial^2u/\partial x_i\partial x_j$, $1\leqq i$, $j\leqq n$, for each $u\in C^2(V)$. Now for a domain Ω of U which contains m_0 we consider the subset $C^2_*(\Omega,\tau)$ of $C^2(\Omega)$ whose each element u satisfies the condition

$$|u(m)|<\tau\quad\text{for all}\quad m\in\Omega\quad\text{and}\quad 1+\sum_{i=1}^n u^ig_i>0\quad\text{on}\quad\Omega \tag{1.2}$$

where we put $u^i=\sum_{j=1}^n g^{ij}u_j$, $1\leqq i\leqq n$, and where g^{ij} is the (i,j)-component of the inverse matrix of the matrix (g_{ij}), $1\leqq i$, $j\leqq n$. This is possible. In fact, since for any $t\in(-\tau,\tau)$ $g_i(m_0,t)=0$, $1\leqq i\leqq n$, for a $u\in C^2(U)$ there exists a domain of U

containing m_0 on which $1+\sum_{i=1}^{n} u^i g_i>0$ holds. For a $u \in C^2_*(\Omega, \tau)$ let us consider a hypersurface $S(u)$ in $\Omega\times(-\tau, \tau)$ defined by

$$S(u)=\{(m, u(m)) \in \Omega\times(-\tau, \tau);\ m \in \Omega\} . \tag{1.3}$$

In particular, if u is constant, say $t \in (-\tau, \tau)$, we shall denote it by S_t. We put $X_i=\partial/\partial x_i+u_i\partial/\partial t$, $1\leqq i\leqq n$. Then $X_1, \cdots, X_n$ are linearly independent vector fields on $S(u)$. We set

$$\bar{g}_{ij}=\langle X_i, X_j\rangle=g_{ij}+g_iu_j+g_ju_i+gu_iu_j\ , \qquad 1\leqq i, j\leqq n\ . \tag{1.4}$$

We can give a unit normal vector field $\eta=\eta^\alpha(\partial/\partial x_\alpha)^*$ on $S(u)$ by

$$\eta^i=-\frac{1}{\sqrt{G}}a^{ij}a_j\ , \qquad 1\leqq i\leqq n\ , \qquad \eta^{n+1}=\frac{1}{\sqrt{G}} \tag{1.5}$$

where

$$\begin{aligned} &a^{ij}=g^{ij}-u^ig^j(1+u^kg_k)^{-1}\ , \qquad a_j=g_j+gu_j\ , \\ &u^i=g^{ik}u_k\ , \qquad g^j=g^{jk}g_k\ , \\ &G=g_{ij}a^{ik}a^{jl}a_ka_l-2a^{ik}g_ia_k+g>0\ . \end{aligned} \tag{1.6}$$

Remark. For a $u \in C^2_*(\Omega, \tau)$ we put $a_{ij}=g_{ij}+u_ig_j$, $1\leqq i, j\leqq n$. Then, it is easy to see that $a_{ik}a^{kj}=\delta_{ij}$ and $\det(a_{ij})=\det(g_{ij})(1+u^kg_k)>0$(by (1.2)).

For the moment we shall denote by D the Riemannian connection on $U\times(-\tau, \tau)$. Let $\Gamma^\alpha_{\beta\gamma}$, $1\leqq\alpha, \beta, \gamma\leqq n+1$, be the Riemannian connection coefficients on $U\times(-\tau, \tau)$.

Let H be the mean curvature of $S(u)$ with respect to η. It is defined by $H=(1/n)\bar{g}^{ij}\langle D_{X_i}X_j, \eta\rangle$ where $\bar{g}^{ij}$ is the (i, j)-component of the inverse matrix of the matrix $(\bar{g}_{ij})$, $1\leqq i, j\leqq n$. By (1.5) we have

$$nH\sqrt{G}=\bar{g}^{ij}\{(g-a^{kl}a_lg_k)u_{ij}+(g_{\alpha n+1}-a^{kl}a_lg_{\alpha k})(\Gamma^\alpha_{ij}+\Gamma^\alpha_{in+1}u_j)\}\ .$$

We put

$$G^{ij}=\det(\bar{g}_{ij})\bar{g}^{ij}\ , \qquad 1\leqq i, j\leqq n\ . \tag{1.7}$$

Then we can rewrite the equality above as follows:

$$\sum_{i,j=1}^{n} A_{ij}(x, \nabla u)u_{ij}=B(x, \nabla u, H) \tag{1.8}$$

where

* From now on, we suppose that Greek indices $\alpha, \beta, \gamma, \cdots$ run over the range $1, 2, \cdots, n, n+1$ and that Latin indices $i, j, k, \cdots$ run over the range $1, 2, \cdots, n$, and we shall use the Einstein convention for repeating indices.

$$A_{ij}(x, \nabla u)=(g-a^{kl}a_l g_k)G^{ij}\ , \quad 1\leqq i, j\leqq n, \nabla u=(u_1, \cdots, u_n)\ ,$$

(1.9)
$$B(x, \nabla u, H)=nH\sqrt{G}\ \det(\bar{g}_{ij})$$
$$-G^{ij}(\Gamma^{\alpha}_{ij}+\Gamma^{\alpha}_{in+1}u_j)(g_{\alpha n+1}-g_{\alpha k}a^{kl}a_l)\ .$$

We note $g-a^{kl}a_l g_k=\langle\eta, \partial/\partial t\rangle$. It never vanish on $S(u)$. Since

$$(g-a^{kl}a_l g_k)(m_0, t)=g(m_0, t)>0$$

and $S(u)$ is connected, we see that $g-a^{kl}a_l g_k>0$ holds on $S(u)$. Therefore, when in (1.8) we regard H as a given continuous function on Ω, (1.8) is a quasilinear elliptic partial differential equation of second order on Ω. Then, if $u\in C^2_*(\Omega, \tau)$ is a solution of the equation (1.8), for this u the mean curvature of the hypersurface in $\Omega\times(-\tau, \tau)$ defined by (1.3) equals H. Since $g_{ij}(1\leqq i, j\leqq n)$ are independent of $t\in(-\tau, \tau)$, from (1.4) we see that for each $t\in(-\tau, \tau)$ the mean curvature of S_t is equal to $\mathscr{H}$. Hence from (1.6), (1.8) and (1.9) we have

Lemma 1.1. *For a fixed* $t\in(-\tau, \tau)$

$$\mathscr{H}=\frac{1}{n}(g-g^{kl}g_k g_l)^{-1/2}g^{ij}\Gamma^{\alpha}_{ij}(g_{\alpha n+1}-g_{\alpha k}g^{kl}g_l)\ .$$

Theorem 1.2. *Let N be a homogeneous Riemannian manifold of dimension $n+1$ ($n\geqq 1$), and let D be a domain in N with regular smooth boundary ∂D and $\mathscr{H}$ the mean curvature of ∂D with respect to the inward unit normal vector to ∂D. Let Ω be a domain in ∂D which is contained in some local coordinate neighborhood of ∂D satisfying the condition discussed above. Suppose that for a non-negative constant H_0 $\mathscr{H}$ satisfies the condition* (1.1). *Let H be a real-valued continuous function on Ω such that*

(1.10)
$$|H|\leqq H_0$$

holds on Ω. If $u\in C^2_*(\Omega, \tau)$ is a solution of the equation (1.8), *then u can not take its minimum value in Ω unless u is constant where $C^2_*(\Omega, \tau)$ is defined by* (1.2).

The proof of Theorem 1.2 will be given in Section 3.

2. Quasilinear elliptic partial differential equations of second order.

In this section we shall study quasilinear elliptic partial differential equations of second order with more general form than the equation (1.8).

Let Ω be a domain in the n-dimensional Euclidean space R^n and let $C^2(\Omega)$ be the set of real-valued functions of class C^2 on Ω. In the following, for a $u\in C^2(\Omega)$ we put

$u_i=\partial u/\partial x_i$, $1\leqq i\leqq n$, $\nabla u=(u_1,\cdots,u_n)$ and $u_{ij}=\partial^2 u/\partial x_i\partial x_j$, $1\leqq i, j\leqq n$, where $(x_1,\cdots,x_n)$ stands for the canonical coordinate system in R^n. We denote by $\|\ \|$ the Euclidean norm of R^n.

Let us consider on a domain Ω in R^n a quasilinear elliptic partial differential equation of second order:

$$\sum_{i,j=1}^{n} A_{ij}(x,u,\nabla u)u_{ij}=B(x,u,\nabla u) \tag{2.1}$$

where $A_{ij}(1\leqq i,j\leqq n)$ and B are real-valued continuous functions on $\Omega\times R\times R^n$ and $A_{ij}=A_{ji}(1\leqq i,j\leqq n)$. We shall denote by (x,u,p) a point of $\Omega\times R\times R^n$. Ellipticity of the equation (2.1) requires that the coefficient matrix satisfies the condition

$$\sum_{i,j=1}^{n} A_{ij}(x,u,p)X_iX_j>0 \quad \text{on} \quad \Omega\times R\times R^n \tag{2.2}$$

for arbitrary non-vanishing real vector $X=(X_1,\cdots,X_n)\in R^n$.

We set for a $u\in C^2(\Omega)$

$$L(u)=\sum_{i,j=1}^{n} A_{ij}(x,u,\nabla u)u_{ij}-B(x,u,\nabla u)\ .$$

It is called that u is a supersolution (subsolution) of the equation (2.1) if $L(u)\leqq 0$ $(L(u)\geqq 0)$.

Theorem 2.1. *Assume that for the equation* (2.1) *the following conditions hold:*

$$B(x,u,0)\leqq 0 \quad \text{on} \quad \Omega\times R\ , \tag{2.3}$$

$$|B(x,u,p)-B(x,u,0)|\leqq h(x,u,p)F(\|p\|) \quad \text{on} \quad \Omega\times R\times R^n \tag{2.4}$$

where h is a non-negative valued continuous function on $\Omega\times R\times R^n$ and F is a non-negative valued function of one-variable such that

$$\lim_{t\to+0}\frac{F(t)}{t}=c \tag{2.5}$$

where c is a non-negative constant. Suppose that $u\in C^2(\Omega)$ is a supersolution of the equation (2.1). *Then u can not take its minimum value in Ω unless u is constant.*

Proof. Suppose for contradiction that u takes the minimum value m in Ω and that u is not constant. We set $\Omega'=\{x\in\Omega;\, u(x)=m\}$. Since Ω is connected and Ω' is closed in Ω, there exists a point y of Ω' such that for any $r>0$ Ω' can not contain the open ball in R^n of radius r and of center y. Then, taking r sufficiently small, we can choose a point $x_0 \in \Omega - \Omega'$ and a positive r_0 so that

$$\Omega_0 = \{x \in R^n;\ \|x - x_0\| \leqq r_0\} \subset \Omega\ , \qquad \Omega_0 \cap \Omega' = \{y_0\}\ . \tag{2.6}$$

Let Ω_1 be the closed ball in R^n of radius r_1 and of center y_0 such that $0 < r_1 < r_0$ and $\Omega_1 \subset \Omega$. Then from (2.6) we see that there exists a constant δ, $0 < \delta < 1$, satisfying the condition

$$u \geqq m + \delta \quad \text{on} \quad \Omega_0 \cap \partial\Omega_1 \tag{2.7}$$

where $\partial\Omega_1 = \{x \in R^n;\ \|x - y_0\| = r_1\}$. We also see that

$$r_2 \leqq \|x - x_0\| \leqq r_3 \quad \text{for all} \quad x \in \Omega_1 \tag{2.8}$$

where $r_2 = r_0 - r_1$ and $r_3 = r_0 + r_1$.

By the condition (2.2) there exist positive constants λ_1 and λ_2 such that

$$\lambda_1 \|X\|^2 \leqq \sum_{i,j=1}^{n} A_{ij}(x, u(x), p(x)) X_i X_j \leqq \lambda_2 \|X\|^2 \tag{2.9}$$

for any vector $X = (X_1, \cdots, X_n) \in R^n$ and any $x \in \Omega_1$ where we put $\nabla u(x) = p(x)$ (In the following we also use the same notation). We put

$$c_1 = \sup_{x \in \Omega_1} \{h(x, u(x), p(x))\}\ . \tag{2.10}$$

Let $\{\varepsilon_k\}$, $k = 1, 2, \cdots$, be a sequence such that $0 < \varepsilon_k < 1 (k = 1, 2, \cdots)$ and $\lim_{k \to \infty} \varepsilon_k = 0$. For each $\varepsilon_k (k = 1, 2, \cdots)$ we now consider the auxiliary function $w^{(k)}$ on Ω_1 defined by

$$w^{(k)}(x) = u(x) - \varepsilon_k \phi(x) \quad \text{for} \quad x \in \Omega_1 \qquad (k = 1, 2, \cdots) \tag{2.11}$$

where

$$\phi(x) = \exp(-\alpha \|x - x_0\|^2) - \exp(-\alpha r_0^2)\ , \tag{2.12}$$

α being a positive constant such that

$$\alpha > \max\{(r_2)^{-2} \log(1/\delta),\ (2\lambda_1 r_2^2)^{-1}(n\lambda_2 + c c_1 r_3)\}\ . \tag{2.13}$$

Since $|x - x_0| > r_0$ on $\partial\Omega_1 - \Omega_0$, $\phi < 0$ on $\partial\Omega_1 - \Omega_0$. Thus we have

$$w^{(k)} > m \quad \text{on} \quad \partial\Omega_1 - \Omega_0\ . \tag{2.14}$$

From (2.7), (2.8) and (2.13), on $\partial\Omega_1 \cap \Omega_0$ we have

$$w^{(k)} \geqq m + \delta - \varepsilon_k \phi > m + \delta - \varepsilon_k \exp(-\alpha r_2^2) > m\ . \tag{2.15}$$

On the other hand, at y_0 we have

$$w^{(k)}(y_0) = u(y_0) = m\ . \tag{2.16}$$

Thus it follows from (2.14), (2.15) and (2.16) that $w^{(k)}$ takes its minimum value at an interior point of Ω_1. Let $y^{(k)}$ be an interior point of Ω_1 at which $w^{(k)}$ takes its minimum value. By taking a subsequence if necessary, we can assume that $\{y^{(k)}\}$, $k=1, 2, \cdots$, converges to a point $\bar{y} \in \Omega_1$.

By (2.3), (2.4) and (2.10), we have at each interior point x of Ω_1

$$B(x, u(x), p(x)) \leqq c_1 F(\|p(x)\|) . \tag{2.17}$$

Since u is a supersolution of the equation (2.1) and $u=w^{(k)}+\varepsilon_k\phi$ for each $\varepsilon_k (k=1, 2, \cdots)$, we have at each interior point x of Ω_1

$$\sum_{i,j=1}^{n} A_{ij}(x, u(x), p(x))(w_{ij}^{(k)}(x)+\varepsilon_k\phi_{ij}(x)) \leqq c_1 F(\|p(x)\|) . \tag{2.18}$$

We shall estimate the inequality (2.18) at $y^{(k)} (k=1, 2, \cdots)$. From (2.12) we have

$$\begin{aligned} &\phi_i(y^{(k)}) = -2\alpha z_i^{(k)} \xi(y^{(k)}) , \qquad 1 \leqq i \leqq n , \\ &\phi_{ij}(y^{(k)}) = -2\alpha(\delta_{ij} - 2\alpha z_i^{(k)} z_j^{(k)}) \xi(y^{(k)}) , \qquad 1 \leqq i, j \leqq n , \end{aligned} \tag{2.19}$$

where we put $z^{(k)}=(z_1^{(k)}, \cdots, z_n^{(k)})=y^{(k)}-x_0$ and $\xi(x)=\exp(-\alpha \|x-x_0\|^2)$. Since $w^{(k)}$ takes its minimum value on Ω_1 at $y^{(k)}$, we have

$$u_i(y^{(k)}) = \varepsilon_k \phi_i(y^{(k)}) , \qquad 1 \leqq i \leqq n , \tag{2.20}$$

and

$$\sum_{i,j=1}^{n} A_{ij}(y^{(k)}, u(y^{(k)}), p(y^{(k)})) w_{ij}^{(k)}(y^{(k)}) \geqq 0 . \tag{2.21}$$

From (2.9), (2.19), (2.20) and (2.21), at $y^{(k)}$ we have

$$\begin{aligned} \text{the left-hand side of (2.18)} \ &\geqq \mu_k (2\alpha\lambda_1 \|z^{(k)}\|^2 - n\lambda_2) \\ &\geqq \mu_k (2\alpha\lambda_1 r_2^2 - n\lambda_2) \end{aligned}$$

where we put $\mu_k = 2\alpha\varepsilon_k \xi(y^{(k)}) (>0)$ and we note that the last inequality follows from (2.8). Thus we have

$$2\alpha\lambda_1 r_2^2 - n\lambda_2 \leqq c_1 F(\|p(y^{(k)})\|)/\mu_k . \tag{2.22}$$

By (2.19), (2.20) we have $\|p(y^{(k)})\| = 2\alpha\varepsilon_k \xi(y^{(k)}) \|z^{(k)}\| = \mu_k \|z^{(k)}\|$. Since $\lim_{k\to\infty} y^{(k)} = \bar{y} \in \Omega_1$, $0 < r_2 \leqq \|z^{(k)}\| \leqq r_3 (k=1, 2, \cdots)$ and $\|\bar{y}-x_0\| \leqq r_3$, by (2.5) we have

$$\lim_{k\to\infty} \frac{F(\|p(y^{(k)})\|)}{\mu_k} = c \|\bar{y}-x_0\| \leqq c r_3 .$$

Thus from (2.22) we have

$$2\alpha\lambda_1 r_2^2 - n\lambda_2 \leqq c c_1 r_3 .$$

This contradicts (2.10). Hence we complete the proof.

We suppose that the inhomogeneous term B of the equation (2.1) is of class C^1 for the variables $p_i (1 \leqq i \leqq n)$. Then we have

$$|B(x, u, p)-B(x, u, 0)| \leqq \left[\sum_{k=1}^{n} \int_0^1 \left| \frac{\partial B}{\partial p_k}(x, u, tp) \right| dt\right] \|p\| .$$

Therefore we see that the conditions (2.4) and (2.5) of Theorem 2.1 are satisfied. Thus we have

Corollary 2.1. *Suppose that for the equation* (2.1) *the inhomogeneous term* B *is of class* C^1 *for the variables* $p_i (1 \leqq i \leqq n)$ *and that the inequality* $B(x, u, 0) \leqq 0$ *holds on* $\Omega \times R$. If $u \in C^2(\Omega)$ *is a supersolution of the equation* (2.1), *then* u *can not take its minimum value in* Ω *unless* u *is constant.*

Theorem 2.2. *Suppose that for the equation* (2.1) *the conditions of Theorem* 2.1 *hold except the condition* (2.3) *and that the inequality* $B(x, u, 0) \geqq 0$ *holds on* $\Omega \times R$. *If* $u \in C^2(\Omega)$ *is a subsolution of the equation* (2.1), *then* u *can not take its maximum value in* Ω *unless* u *is constant.*

The proof can be proved by a similar argument as in the proof of Theorem 2.1. In this case we adopt $w^{(k)} = u + \varepsilon_k \phi$, $k = 1, 2, \cdots$, as auxiliary functions corresponding to (2.11).

Corollary 2.2. *Suppose that for the equation* (2.1) *the inhomogeneous term* B *is of class* C^1 *for the variables* $p_i (1 \leqq i \leqq n)$ *and that the inequality* $B(x, u, 0) \geqq 0$ *holds on* $\Omega \times R$. *If* $u \in C^2(\Omega)$ *is a subsolution of the equation* (2.1), *then* u *can not take its maximum value in* Ω *unless* u *is constant.*

Remark. For the conditions (2.3) and (2.4) in Theorem 2.1 we obtained an idea from *R. Redheffer's* paper [7]. It will be indicated in Sections 3 and 5 that for the equations (1.8) and (5.13) the condition (2.3) closely relates to a geometrical condition which is connected with the mean curvature.

3. Proof of Theorem 1.2.

The notations which will be used in this section are all same as in Section 1 unless otherwise stated. In the following we shall put $\nabla u = p = (p_1, \cdots, p_n)$ for a $u \in C^2(\Omega)$.

We put $\bar{B}(x, p) = B(x, p, H_0)$. Then from (1.6) and (1.9) we have

Lemma 3.1. *For the equation* (1.8), $A_{ij} (1 \leqq i, j \leqq n)$ *and* $\bar{B}$ *are continuous* *on* $\Omega\times R^{n}$ *and* $\bar{B}$ *is of class* C^{1} *for the variables* $p_i (1\leqq i\leqq n)$.

Lemma 3.2.

$$\bar{B}(x, 0)=n(H_0-\mathscr{H})(g-g^{ij}g_i g_j)^{1/2}\det(g_{ij})\ , \qquad x\in\Omega\ .$$

Proof. From (1.6), (1.7) and (1.9) we have

$$B(x, 0, H_0)=nH_0(g-g^{ij}g_i g_j)^{1/2}\det(g_{ij}) \\ -\det(g_{ij})g^{ij}\Gamma^{\alpha}_{ij}(g_{\alpha n+1}-g_{\alpha k}g^{kl}g_l)\ .$$

Then the lemma follows from Lemma 1.1.

Proof of Theorem 1.2. For a real-valued continuous function H' on Ω, we set

$$L_{H'}(u)=\sum_{i,j=1} A_{ij}(x, p)u_{ij}-B(x, p, H')$$

where $u\in C^{2}_{*}(\Omega,\tau)$, $A_{ij}(1\leqq i, j\leqq n)$ and B are given by (1.9). Let $u\in C^{2}_{*}(\Omega,\tau)$ be a solution of the equation (1.8). Since $|H|\leqq H_0$, we have

$$L_{H_0}(u)=L_{H_0}(u)-L_{H}(u)=n(H-H_0)\sqrt{G}\det(\bar{g}_{ij})\leqq 0\ .$$

Hence u is a supersolution of the equation $L_{H_0}(v)\equiv 0$ on Ω. Since $\mathscr{H}\geqq H_0$, by Lemma 3.2 we have $\bar{B}(x,0)\leqq 0$. Therefore by Lemma 3.1 we can apply Corollary 2.1 to the equation $L_{H_0}(v)\equiv 0$. Then the theorem follows from Corollary 2.1.

4. Proof of Theorem 1.1.

We first give a proof of Theorem 1.1 and in the next place we study minimal hypersurfaces in a homogeneous Riemannian manifold.

Proof of Theorem 1.1. Suppose that $|H(m)|\leqq H_0$ holds at each point m of M. Let m and m_1 be points of M and ∂D, respectively. Since N is homogeneous, there exists an element φ_1 of $I_0(N)$ such that $\varphi_1(f(m))=m_1$ where $I_0(N)$ denotes the identity component of the isometry group of N. Let φ: $[0,1]\to I_0(N)$ be a continuous curve in $I_0(N)$ such that $\varphi(0)$ is the identity transformation and $\varphi(1)=\varphi_1$. We set $t_0=\sup\{t\in[0,1];\ \varphi_{t'}(f(M))\subset D$ for any $t'\in[0,t]\}$ where we put $\varphi_t=\varphi(t)$ for each $t\in[0,1]$. Since M is compact, ∂D is closed in N and φ is continuous, we see that $\varphi_{t_0}(f(M))\subset\bar{D}$ and $\varphi_{t_0}(f(M))\cap\partial D$ is non-empty. We put $\bar{f}=\varphi_{t_0}\circ f$. Then we want to show $\bar{f}(M)\subset\partial D$. We set $M'=\{m\in M;\bar{f}(m)\in\partial D\}$. Let m be a point of M' and we put $m_0=\bar{f}(m)$. Now we note that the following notations have all same meaning as in Section 1 unless otherwise stated. As we have shown in Section 1 (see Section 1) there exist a local coordinate neighborhood U of m_0 in ∂D and a positive τ such that $U\times(-\tau,\tau)$ is a local coordinate neighborhood of m_0

in N. Since $\bar{f}(M)$ is tangent to ∂D at m_0, by the theorem of implicit function, there exist a domain $\Omega(\subset U)$ in ∂D which contains m_0 and a $u \in C^2_*(\Omega, \tau)$ such that $u \geqq 0$ holds on Ω, $u(m_0)=0$ and $\bar{f}(M)$ is locally expressed by the form (1.3). Then, by the argument in Section 1, u is a solution of the equation (1.8) on Ω. Since u takes its minimum value at $m_0 \in \Omega$, by Theorem 1.2 we can conclude that there exists an open neighborhood V of m in M such that $\bar{f}(V) \subset \partial D$. Thus we have proved that M' is open in M. Since ∂D is closed in N, M' is closed in M. Hence, by connectedness of M, $\bar{f}(M)$ must be contained in ∂D. Thus we complete the proof.

Let M and N be Riemannian manifolds of dimension $n(n \geqq 1)$ and of dimension $n+1$, respectively. Let $f: M \to N$ be an isometric immersion. If the mean curvature of M for f vanishes at each point of M, M is called a minimal hypersurface in N. In particular, for $n=1$ we say that $f: M \to N$ is a goedesic in N.

From Theorem 1.1 we have

Theorem 4.1. *Let N be a homogeneous Riemannian manifold of dimension $n+1 (n \geqq 1)$. Let D be a domain in N with regular smooth boundary ∂D and $\mathscr{H}$ the mean curvature of ∂D with respect to the inward unit normal vector to ∂D. Suppose that $\mathscr{H} > 0$ holds at each point of ∂D. Then there are no compact minimal hypersurfaces in N which is contained in $\bar{D}$ where $\bar{D} = D \cup \partial D$.*

Theorem 4.2. *Let N be a homogeneous Riemannian manifold of dimension $n+1 (n \geqq 1)$ and M_0 a closed hypersurface in N. Let M be a compact Riemannian manifold of dimension n and $f: M \to N$ an isometric immersion. Suppose that M_0 and M are minimal hypersurfaces in N and that M is not a Riemannian covering manifold of M_0. Then for any isometry φ of N $\varphi(f(M))$ and M_0 must intersect* each other.*

Proof. It is sufficient to show that $f(M)$ and M_0 intersect each other. Suppose for contradiction that $f(M)$ and M_0 do not intersect. Let D be the connected component of $N - M_0$ whose closure $\bar{D}$ contains $f(M)$. We note $\partial D = \bar{D} - D \subset M_0$. Then, using a similar argument as in the proof of Theorem 1.1, there exists an isometry φ of N such that $\varphi(f(M)) \subset \bar{D}$ and $\varphi(f(M)) \cap M_0$ is non-empty. We put $\bar{f} = \varphi \circ f$, and set $M' = \{m \in M; \bar{f}(m) \in M_0\}$. Since M_0 is a closed hypersurface in N, M' is closed in M. Let m be a point of M', and we put $m_0 = \bar{f}(m)$. Now we

* Here, for example, by $f(M)$ and M_0 intersect each other we mean that there exists a point m of M such that $f(m) \in M_0$ and for an open neighborhood U of $f(m)$ in N which is divided by M_0 just two connected components, say U_1, U_2, we can choose an open neighborhood V of m in M so that $f(V) \subset U$ and $f(V)$ has certainly common points with U_1 and U_2, respectively.

note that the following notations have all same meaning as in Section 1 unless otherwise stated. As we have shown in Section 1 (see Section 1) there exist a local coordinate neighborhood U of m_0 in M_0 and a positive τ such that $U\times(-\tau,\tau)$ is a local coordinate neighborhood of m_0 in N. Since $\bar{f}(M)$ is tangent to M_0 at m_0, by the theorem of implicit function, we see that there exist a domain $\Omega(\subset U)$ in M_0 which contains m_0 and a $u\in C^2_*(\Omega,\tau)$ such that $u\geqq 0$ holds on Ω, $u(m_0)=0$ and $f(M)$ is locally expressed by the form (1.3). Since M is a minimal hyersurface in N, by the argument in Section 1 we see that u is a solution of the following equation on Ω:

$$\sum_{i,j=1}^{n} A_{ij}(x,\nabla u)u_{ij}=B(x,\nabla u,0)$$

where $A_{ij}(1\leqq i,j\leqq n)$ and B are given by (1.9). Since M_0 and M are minimal hypersurfaces in N, we can apply Theorem 1.2 to the equation above. Since u takes its minimum value at $m_0\in\Omega$, it follows from Theorem 1.2 that $u\equiv 0$ on Ω. Therefore there exists an open neighborhood V of m in M such that $\bar{f}(V)\subset M_0$. Thus we have proved that M' is open in M. By connectedness of M, $\bar{f}(M)$ must be contained in M_0. Then we can conclude that the isometric immersion $\bar{f}\colon M\to M_0$ is a covering mapping. This contradicts the hypothesis.

Remark. In the case N is a complete Riemannian manifold with positive Ricci curvature, *T. Frankel* obtained a similar result as Theorem 2.2 in [4].

5. Non-parametric hypersurfaces in the Euclidean sphere.

The purpose of this section is to generalize the results obtained in [6]. Let r be a positive constant, and let Ω be a domain in the n-dimensional $(n\geqq 2)$ Euclidean space R^n which is contained in the open ball of radius r centred at the origin. We denote by $C^2(\Omega)$ the set of real-valued functions of class C^2 on Ω. Let $C^2(\Omega,r)$ be the subset of $C^2(\Omega)$ whose each element u satisfies the condition

$$r^2>\|x\|^2+(u(x))^2 \quad \text{for any} \quad x\in\Omega \tag{5.1}$$

where $\|\ \|$ stands for the Euclidean norm of R^n. We denote by $S^{n+1}(r)$ the $(n+1)$-dimensional Euclidean sphere of radius r centred at the origin. For a $u\in C^2(\Omega,r)$, let us codsider a non-parametric hypersurface M in $S^{n+1}(r)$ defined by

$$\bar{u}(x)=(x_1,\cdots,x_n,\sqrt{r^2-\|x\|^2-(u(x))^2}\,,u(x))\,,\qquad x\in\Omega\,, \tag{5.2}$$

where $x_i(1\leqq i\leqq n)$ denotes the i-th coordinate of x. We put

$$X_i=(0,\cdots,1,\cdots,0,U_i,p_i)\ (1\leqq i\leqq n)\,,\qquad g_{ij}=X_i\cdot X_j\quad(1\leqq i,j\leqq n) \tag{5.3}$$

where $U=(r^2-\|x\|^2-(u(x))^2)^{1/2}$, $U_i=\partial U/\partial x_i$, $p_i=\partial u/\partial x_i(1\leqq i\leqq n)$ and where the dod stands for the inner product of R^n. Then we have

$$U^2 \det(g_{ij})=r^2(1+\|p\|^2)-(u-p\cdot x)^2 \tag{5.4}$$

where $p=(p_1, \cdots, p_n)$. We put

$$\mathscr{G}=r^2(1+\|p\|^2)-(u-p\cdot x)^2>0 . \tag{5.5}$$

Now we take the unit normal vector field $\eta=(\eta_1, \cdots, \eta_{n+2})$ on M as follows:

$$\begin{aligned} &\eta_i=-\{r^2p_i+(u-p\cdot x)x_i\}/r\sqrt{\mathscr{G}} , \quad 1\leqq i\leqq n , \\ &\eta_{n+1}=-(u-p\cdot x)U/r\sqrt{\mathscr{G}} , \quad \eta_{n+2}=\{r^2-(u-p\cdot x)u\}/r\sqrt{\mathscr{G}} . \end{aligned} \tag{5.6}$$

Let H be the mean curvature of M with respect to η. We denote by D the Riemannian connection on $S^{n+1}(r)$ defined by the standard Riemannian metric on $S^{n+1}(r)$. Then we have

$$\frac{\partial\eta}{\partial x_i}=D_{X_i}\eta , \quad 1\leqq i\leqq n . \tag{5.7}$$

By the Weingarten's formula $D_{X_i}\eta(1\leqq i\leqq n)$ are expressed as

$$D_{X_i}\xi=-\sum_{j=1}^{n} a_{ij}X_j , \quad 1\leqq i\leqq n , \tag{5.8}$$

where $a_{ij}(1\leqq i, j\leqq n)$ are continuous functions on Ω. Then the mean curvature of M at $\bar{u}(x)$ with respect to η is defined by

$$H(x)=\frac{1}{n}\sum_{i=1}^{n} a_{ii}(x) , \quad x\in\Omega , \tag{5.9}$$

By (5.6)$\sim$(5.9) we have at each point $\bar{u}(x)$ of M

$$H(x)=\frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left\{\frac{r^2p_i+(u-p\cdot x)x_i}{r\sqrt{\mathscr{G}}}\right\} . \tag{5.10}$$

Theorem 5.1. *Assume that Ω is an open ball in R^n of radius R which is contained in the open ball in R^n of radius r centred at the origin. For a $u\in C^2(\Omega, r)$ let M be a non-parametric hypersurface in $S^{n+1}(r)$ defined by* (5.2) *and H the mean curvature of M with respect to η which is given by* (5.6). *Suppose that for a positive constant H_0 the condition $|H|\geqq H_0$ holds at each point of M. Then we have*

$$H_0R\leqq 1 .$$

Proof. Let $\Omega_{r'}$ be the closed ball of radius r' in R^n whose centre is the same as Ω where $0<r'<R$. We may assume without loss of generality that $H\geqq H_0$

holds at each point of M. By the divergence formula, (5.6) and (5.10), we have

$$\int_{\Omega_{r'}} nHdx=\int_{\partial\Omega_{r'}}\left[\sum_{i=1}^{n}(-\eta_i)\nu_i\right]dS\,, \tag{5.11}$$

where $dx=dx_1\wedge\cdots\wedge dx_n$, $\nu=(\nu_1,\cdots,\nu_n)$ is the outward unit normal vector field on the boundary $\partial\Omega_{r'}$ of $\Omega_{r'}$ and dS denotes the volume element of $\partial\Omega_{r'}$. Since $H\geqq H_0$ and $\sum_{i=1}^{n}(\eta_i)^2<1$, we have

$$\begin{aligned}&\int_{\Omega_{r'}} nHdx\geqq nH_0\times(\text{volume of } \Omega_{r'})\\&\int_{\partial\Omega_{r'}}\Big[\sum_{i=1}^{n}(-\eta_i)\nu_i\Big]dS<\text{volume of } \partial\Omega_{r'}\,.\end{aligned} \tag{5.12}$$

By (5.11) and (5.12) we have $H_0r'<1$. Thus we have $H_0R\leqq 1$ as $r'\to R$.

Remark. Theorem 5.1 implies the following: Let Ω be a domain in R^n which is contained in the open ball of radius r centred at the origin. For a $u\in C^2(\Omega, r)$, let M be a non-parametric hypersurface in $S^{n+1}(r)$ defined by (5.2). Suppose that for a positive constant H_0 the mean curvature H (defined up to a sign) of M satisfies the inequality $|H|\geqq H_0$ at each point of M. Then Ω can not contain a closed ball of radius $1/H_0$.

Theorem 5.2. *Let Ω be a domain in R^n whose closure $\bar{\Omega}$ is contained in the open ball of radius r centred at the origin. Let k and H_0 be constants such that $0<k<\sqrt{r^2-r_1^2}$ and $0\leqq H_0<k/\sqrt{r^2-k^2}$ where $r_1=\max_{x\in\bar{\Omega}}\|x\|$. For a $u\in C^2(\Omega)\cap C^0(\bar{\Omega})$ satisfying the condition $m_0:=r^2H_0/\sqrt{r^2H_0^2+1}\leqq u\leqq k$ on $\bar{\Omega}$, let M be a non-parametric hypersurface in $S^{n+1}(r)$ defined by* (5.2) *where* $C^0(\bar{\Omega})$ *denotes the set of real-valued continuous functions on $\bar{\Omega}$. Suppose that for the mean curvature H of M the inequality $|H|\leqq H_0$ holds at each point of M. If $\bar{u}(\partial\Omega)\subset Q_{m_1}^k$, then $\bar{u}(\bar{\Omega})\subset Q_{m_1}^k$ where m_1 is a constant such that $m_0<m_1<k$ and $Q_{m_1}^k=\{(x_1,\cdots,x_{n+2})\in S^{n+1}(r);\ x_{n+1}>0,\ m_1\leqq x_{n+2}\leqq k\}$.*

Proof. Let η be the unit normal vector field on M defined by (5.6). The mean curvature H of M with respect to η is expressed as the form (5.10). Then we can rewrite (5.10) as follows:

$$\sum_{i,j=1}^{n} A_{ij}(x,u,p)u_{ij}=B(x,u,p,H) \tag{5.13}$$

where

$$\begin{aligned}A_{ij}(x,u,p)&=r^2\{\mathscr{G}\delta_{ij}-r^2p_ip_j-(1+\|p\|^2)x_ix_j\\&\qquad-(u-p\cdot x)(x_ip_j+x_jp_i)\}\,,\qquad 1\leqq i,j\leqq n\,,\\B(x,u,p,H)&=n\mathscr{G}\{rH\sqrt{\mathscr{G}}-(u-p\cdot x)\}\,.\end{aligned} \tag{5.14}$$

By (5.5) and (5.14), $A_{ij}(1\leqq i, j\leqq n)$ and B are continuous on $\Omega\times R\times R^n$ and B is of class C^1 for the valiables $p_i(1\leqq i\leqq n)$. It is easy to see $A_{ij}=r^2\mathscr{G}g^{ij}(1\leqq i, j\leqq n)$ where $g^{ij}(1\leqq i\leqq n)$ is the (i, j)-component of the inverse matrix of the matrix (g_{ij}). If we regard H as a given continuous function on Ω, (5.13) is a quasilinear elliptic partial differential equation of second order on Ω. Now for a continuous function H' on Ω we put

$$L_{H'}(v)=\sum_{i,j=1}^{n} A_{ij}(x, v, p)v_{ij}-B(x, v, p, H')$$

where $v\in C^2(\Omega, r)$, $p=(\partial v/\partial x_1, \cdots, \partial v/\partial x_n)$ and $A_{ij}(1\leqq i, j\leqq n)$ and B are defined by (5.14). Since u is a solution of the equation (5.13) and $|H|\leqq H_0$, we have

$$L_{H_0}(u)=L_{H_0}(u)-L_H(u)=nr\mathscr{G}^{3/2}(H-H_0)\leqq 0\ .$$

Therefore u is a supersolution of the equation $L_{H_0}(v)\equiv 0$ on Ω. Since by the hypothesis $u\geqq r^2H_0/\sqrt{r^2H_0^2+1}$ holds on Ω, we have $B(x, u, 0, H_0)\leqq 0$. Then we can apply Corollary 2.1 to the equation $L_{H_0}(u)\equiv 0$. The present theorem follows from Corollary 2.1.

Remark. Theorem 5.2 is a generalization of Theorem 2.1 which was obtained in [6]. By a similar argument as in the proof of Theorem 5.2 and Corollary 2.2 we can also generalize Theorem 3.2 which was obtained in [6].

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