LIMIT DISTRIBUTIONS OF TWO-DIMENSIONAL POINT PROCESSES GENERATED BY STRONG-MIXING SEQUENCES

By

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1. Introduction.

Let $m_n^{(k)}$ denote the k-th maximum of the first n terms from a sequence $\{\xi_n, n\geq 1\}$ of real random variables and write m_n for $m_n^{(1)}$. When ξ_n's are i.i.d. the class of all possible nondegenerate limit laws of $(m_n-b_n)/a_n$, where $a_n>0$ and b_n are constants, is well-known. In this case Lamperti [4] showed that if $(m_n-b_n)/a_n$ has a nondegenerate limit distribution then the sequence $\{M_n, n\geq 1\}$ of random functions M_n defined by $M_n(t)=(m_{[nt]}-b_n)/a_n$, each M_n being considered as an element of the space $D[a, b]$, $0<a<b<\infty$, of nondecreasing right-continuous functions endowed with the Skorohod J_1 topology, converges in distribution to an extremal process.

Recently Resnik [9], Mori and Oodaira [8] considered point process Φ_n on R^2 consisting of random points $(k/n, (\xi_k-b_n)/a_n)$, $k\geq 1$. They showed that if ξ_n's are i.i.d. and if $(m_n-b_n)/a_n$ has a nondegenerate limit distribution then Φ_n converges to a Poisson point process. Weissman [10, 11] used this approach to nonidentically distributed ξ_n's.

When $\{\xi_n\}$ is a stationary strong-mixing sequence Loynes [5] proved that the only possible limit laws for $(m_n-b_n)/a_n$ are the same types that occur in the independent case. Welsch [12] showed that the class of possible limit laws of $((m_n^{(1)}-b_n)/a_n, (m_n^{(2)}-b_n)/a_n)$ is larger than that occur in the independent case.

The purpose of this paper is to determine the class of all possible limit distributions of point processes Φ_n when underlying sequences $\{\xi_n\}$ are stationary strong-mixing. Contrary to the i.i.d. case weak convergence of $(m_n-b_n)/a_n$ does not imply weak convergence of Φ_n, and the class of all limit laws for Φ_n is fairly larger than that occur in the independent case. In fact it contains infinitely divisible point processes invariant under certain transformations.

In § 2 we describe a class of infinitely divisible point processes which may appear as limits of point processes Φ_n. The main theorems are stated in § 3 and proved in § 4 and § 5. Our method relies heavily on the theory of KLM

(=Kerstan-Lee-Matthes) measures (or queue measures) for infinitely divisible point processes which was developed extensively in the recent book of Kerstan, Matthes and Mecke [3].

2. A class of infinitely divisible point processes.

Throughout this paper α is a real parameter. Let R_α denote $R_+=(0,\infty)$, $R=(-\infty,\infty)$ or $R_-=(-\infty,0)$ according as $\alpha>0$, $\alpha=0$ or $\alpha<0$. Let $X_\alpha=R_+\times R_\alpha$. A point of X_α is denoted by $x=(t,u)$. The topology of X_α is the usual Euclidean topology. The topological σ-algebra of X_α is denoted by $\mathscr{X}_\alpha$.

Let us define mappings $T_{\alpha,\tau}$, $\tau\in R_+$, and $S_{\alpha,\sigma}$, $\sigma\in R_+$, from X_α to itself by

$$T_{\alpha,\tau}(t,u)=(t+\tau,u)$$

and

$$S_{\alpha,\sigma}(t,u)=\begin{cases}(\sigma t,\sigma^{1/\alpha}u) & \text{if } \alpha>0\\ (\sigma t,u+\log\sigma) & \text{if } \alpha=0\\ (\sigma t,\sigma^{1/\alpha}u) & \text{if } \alpha<0\,.\end{cases}$$

respectively.

Let M_α denote the set of all integer-valued nonnegative locally finite measures φ on $\mathscr{X}_\alpha$. For $a\in X_\alpha$ denote by $\delta_a=\delta(\cdot;a)$ the probability measure concentrated on $\{a\}$. Every $\varphi\in M_\alpha$ is represented as $\varphi=\sum_i m_i\delta(\cdot;x_i)$, where $\{x_i, i\geq 1\}$ is an enumeration of all atoms of φ and $m_i=\varphi(\{x_i\})$. The mappings $T_{\alpha,\tau}$ and $S_{\alpha,\sigma}$ induce mappings from M_α to M_α which are also denoted by $T_{\alpha,\tau}$ and $S_{\alpha,\tau}$ resp. In this paper the topology of M_α is the vague topology. The topological σ-algebra of M_α is denoted by $\mathscr{M}_\alpha$. A point process P on X_α is a probability measure on $\mathscr{M}_\alpha$.

For a finite number of point processes $P_1,\cdots,P_n$ on X_α their convolution is denoted by $P_1*\cdots*P_n$ or $\prod_{i=1}^n P_i$. When $P_1=\cdots=P_n=P$ write P^n for $\prod_{i=1}^n P_i$. If the convolution of a sequence $\{P_n, n\geq 1\}$ of point processes on X_α exists then it is denoted by $\prod_{n=1}^\infty P_n$. A point process P on X_α is infinitely divisible if for every $n\geq 1$ there exists a point process P_n on X_α such that $P_n^n=P$. Denote by $\tilde{P}$ the KLM measure of an infinitely divisible point process P. This is a σ-finite measure on $\mathscr{M}_\alpha$. For a definition of KLM measure see [2, 3, 6].

Let P_α denote the Poisson point process on X_α with intensity measure π_α, where

$$\pi_\alpha(dtdu)=\begin{cases}u^{-\alpha}dtdu & \text{if } \alpha>0\\ e^{-u}dtdu & \text{if } \alpha=0\\ |u|^{-\alpha}dtdu & \text{if } \alpha<0\end{cases}$$

It is easy to prove the following:

Lemma 1. *A σ-finite measure ν on $\mathscr{X}_\alpha$ is invariant under $T_{\alpha,\tau}$ and $S_{\alpha,\sigma}$, $\tau>0$, $\sigma>0$, iff ν is a scalar multiple of π_α.*

By the mapping $x\to\delta_x$ π_α induces a measure on $\mathscr{M}_\alpha$ which is denoted by Q_α. The Poisson point process P_α is infinitely divisible and $\tilde{P}_\alpha=Q_\alpha$. Both P_α and Q_α are invariant under $T_{\alpha,\tau}$ and $S_{\alpha,\sigma}$.

Let N be the set of all integer-valued (nonnegative, locally finite) measures ψ on $\bar{R}_+=[0,\infty)$ such that $\psi(\{0\})\geq 1$. Endow N with the vague topology and let $\mathscr{N}$ denote the topological σ-algebra of N. Denote by $\mathscr{Q}$ the set of all probability measures q on $\mathscr{N}$.

For each $x=(t,u)\in X$ define a mapping $f_{\alpha,x}$: $R_+\to X_\alpha$ by

$$f_{\alpha,x}(v)=\begin{cases}(t, ue^{-v/\alpha}) & \text{if } \alpha>0\\ (t, u-v) & \text{if } \alpha=0\\ (t, ue^{v/\alpha}) & \text{if } \alpha<0\end{cases}$$

for $v\in\bar{R}_+$. Every $f_{\alpha,x}$ induces a mapping from N to M_α which is also denoted by $f_{\alpha,x}$. It is easy to verify that

$$T_{\alpha,\tau}f_{\alpha,x}=f_{\alpha,x'}\ ,\qquad x'=T_{\alpha,\tau}x\ , \tag{2.1}$$

and

$$S_{\alpha,\sigma}f_{\alpha,x}=f_{\alpha,x''}\ ,\qquad x''=S_{\alpha,\sigma}x\ . \tag{2.2}$$

By a mapping $f_{\alpha,x}$ from N to M_α every $q\in\mathscr{Q}$ induces a probability measure $qf_{\alpha,x}^{-1}$ on $\mathscr{M}_\alpha$ which will be denoted by $\kappa_{\alpha,q}(\cdot|x)$. For each $q\in\mathscr{Q}$ and $\alpha\in R$ the class $\{\kappa_{\alpha,q}(\cdot|x),\ x\in X\}$ of point processes on X_α determines a shower field on X_α with the state space X_α (see [3]).

Given $q\in\mathscr{Q}$ and $\varphi=\sum_i m_i\delta(\cdot;\ x_i)\in M_\alpha$ define $\kappa_{\alpha,q}(\cdot|\varphi)$ by

$$\kappa_{\alpha,q}(\cdot|\varphi)=\prod_i\{\kappa_{\alpha,q}(\cdot|x_i)\}^{m_i}\ ,$$

if the convolution on the right exists. In particular

$$\kappa_{\alpha,q}(\cdot|\delta_x)=\kappa_{\alpha,q}(\cdot|x)\ ,\qquad x\in X_\alpha\ .$$

The set of $\varphi\in M_\alpha$ for which $\kappa_{\alpha,q}(\cdot|\varphi)$ exists is equal to $M'_\alpha=\{\varphi;\ \varphi\in M_\alpha,\ \varphi((0,t)\times(u,u_\alpha))<\infty$ for every $t>0$ and $u\in R_\alpha\}$, where $u_\alpha=\infty$ or 0 according as $\alpha\geq 0$ or $\alpha<0$. Since P_α is concentrated on M'_α we can define a point process $P_{\alpha,q}$ on X_α by

$$P_{\alpha,q}=\int\kappa_{\alpha,q}(\cdot|\varphi)P_\alpha(d\varphi) \tag{2.3}$$

(see 5.1.2 of [3]). It is known that $P_{\alpha,q}$ is infinitely divisible and

$$\tilde{P}_{\alpha,q}=\int \kappa_{\alpha,q}(\cdot|\varphi)\tilde{P}_\alpha(d\varphi)=\int \kappa_{\alpha,q}(\cdot|\varphi)Q_\alpha(d\varphi)$$

(see 5.2.3 of [3]). Hence

$$\tilde{P}_{\alpha,q}=\int \kappa_{\alpha,q}(\cdot|x)\pi_\alpha(dx)\ . \tag{2.4}$$

In particular if $q=\delta(\cdot;\ \delta_0)$ then $\kappa_{\alpha,q}(\cdot|\varphi)=\delta_\varphi$ and therefore $P_{\alpha,q}=P_\alpha$, $\tilde{P}_{\alpha,q}=Q_\alpha$. Let $\Lambda_\alpha=\{\varphi;\ \varphi\in M_\alpha,\ \varphi=f_{\alpha,x}\psi$ for some $x\in X_\alpha$ and $\psi\in N\}$. Then $\tilde{P}_{\alpha,q}$ is concentrated on $\Lambda_\alpha\in\mathscr{M}_\alpha$. Define a mapping g_α: $X_\alpha\times N\to M$ by

$$g_\alpha(x,\psi)=f_{\alpha,x}(\psi)\ ,\quad x\in X_\alpha\ ,\quad \psi\in N\ ,$$

which is a homeomorphism from $X_\alpha\times N$ onto Λ_α. Denote by h_α the inverse of g_α. Then we have

Lemma 2. $\tilde{P}_{\alpha,q}h_\alpha^{-1}=\pi_\alpha\times q$ *where* $\pi_\alpha\times q$ *is the product measure on* $\mathscr{X}_\alpha\times\mathscr{N}$.

Proof. Since $h_\alpha f_{\alpha,x}(\psi)=(x,\psi)$ we have for $A\in\mathscr{X}_\alpha$ and $E\in\mathscr{N}$

$$f_{\alpha,x}^{-1}h_\alpha^{-1}(A\times E)=\{\psi;\ (x,\psi)\in A\times E\}\ ,$$

and therefore

$$\kappa_{\alpha,q}(h_\alpha^{-1}(A\times E)|x)=qf_{\alpha,x}^{-1}h_\alpha^{-1}(A\times E)=\chi_A(x)q(E)\ .$$

where χ_A is the indicated of A. Hence

$$\tilde{P}_{\alpha,q}h_\alpha^{-1}(A\times E)=\int \kappa_{\alpha,q}(h_\alpha^{-1}(A\times E)|x)\pi_\alpha(dx)=\pi_\alpha(A)q(E)\ .$$

Lemma 3. $P_{\alpha,q}$ *and* $\tilde{P}_{\alpha,q}$ *are invariant under* $T_{\alpha,\tau}$ *and* $S_{\alpha,\sigma}$.

Proof. By (2.1) we have

$$\kappa_{\alpha,q}(T_{\alpha,\tau}^{-1}\cdot|x)=qf_{\alpha,x}^{-1}T_{\alpha,\tau}^{-1}(\cdot)=qf_{\alpha,x'}^{-1}(\cdot)=\kappa_{\alpha,q}(\cdot|x')$$

where $x'=T_{\alpha,\tau}x$. Hence by (2.4) and Lemma 1 we have

$$\tilde{P}_{\alpha,q}T_{\alpha,\tau}^{-1}=\int \kappa_{\alpha,q}(T_{\alpha,\tau}^{-1}\cdot|x)\pi_\alpha(dx)$$

$$=\int \kappa_{\alpha,q}(\cdot|T_{\alpha,\tau}x)\pi_\alpha(dx)=\int \kappa_{\alpha,q}(\cdot|x)\pi_\alpha(dx)=\tilde{P}_{\alpha,q}.$$

By the same argument as in [3] p. 148 $P'_{\alpha q}=P_{\alpha,q}T_{\alpha,\tau}^{-1}$ is infinitely divisible and $\tilde{P}'_{\alpha,q}=\tilde{P}_{\alpha,q}T_{\alpha,\tau}^{-1}=\tilde{P}_{\alpha,q}$. Hence $P_{\alpha,q}T_{\alpha,\tau}^{-1}=P_{\alpha,q}$. Invariance under $S_{\alpha,\sigma}$ is similarly proved.

3. Strong-mixing sequences and main results.

Let $\{\xi_n, n\geq 1\}$ be a stationary strong-mixing sequence of real random variables defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. Let $\{\beta(m), m\geq 1\}$ be mixing coefficients for $\{\xi_n\}$, i.e. $\beta(m)$ satisfies $\beta(m)\to 0$ and

$$|\mathbf{P}\{A\cap B\}-\mathbf{P}\{A\}\mathbf{P}\{B\}|\leq\beta(m) \tag{3.1}$$

whenever $n\geq 1$, $A\in\mathscr{F}(\xi_1,\cdots,\xi_n)$ and $B\in\mathscr{F}(\xi_{n+m},\cdots)$, where $\mathscr{F}(\xi_1,\cdots)$ stands for the σ-algebra generated by $\xi_1,\cdots$.

Let $m_n=\max(\xi_1,\cdots,\xi_n)$. Loynes [5] proved that the nondegenerate limit distribution function of $(m_n-b_n)/a_n$ is of the same type as one of the following G_α:

$$G_\alpha(x)=\begin{cases}0 & x\leq 0\\ \exp(-x^{-\alpha}) & x>0\end{cases} \quad \text{if } \alpha>0\ ,$$

$$G_0(x)=\exp(-e^{-x})\ , \tag{3.2}$$

$$G_\alpha(x)=\begin{cases}\exp(-(-x)^{-\alpha}) & x<0\\ 1 & x\geq 0\end{cases} \quad \text{if } \alpha<0\ .$$

Throughout the rest we assume that there exist two sequences $\{a_n\}$, $a_n>0$, and $\{b_n\}$ such that

$$(m_n-b_n)/a_n\xrightarrow{d}G_\alpha\ . \tag{3.3}$$

For each $\omega\in\Omega$ let $p_{nj}(\omega)$ denote the point $(j/n,\ (\xi_j(\omega)-b_n)/a_n)$, $j\geq 1$, $n\geq 1$, and let $\Phi_n(\omega)=\sum_j^*\delta(\cdot;\ p_{nj}(\omega))$, where $\sum_j^*$ denote the sum over all j satisfying $p_{nj}(\omega)\in X_\alpha$. Then Φ_n is an M_α-valued random element. Let $P_n=\mathbf{P}\Phi_n^{-1}$ denote the distribution of Φ_n.

When ξ_n's are independent identically distributed the following theorem was obtained by Resnik [9], Mori and Oodaira [8].

Theorem 0. *Let $\{\xi_n\}$ be a sequence of i.i.d. random variables. If* (3.3) *holds for some α then $P_n\xrightarrow{w}P_\alpha$.*

It is well known that for every $\alpha\in R$ there exists a sequence $\{\xi_n\}$ of i.i.d. random variables satisfying (3.3) with suitable $\{a_n\}$ and $\{b_n\}$. Hence the class of all limit laws of Φ_n generated by i.i.d. random variables ξ_n coincides with $\{P_\alpha, \alpha\in R\}$ (except for scale and location parameters).

When ξ_n's are not independent (3.3) does not imply the weak convergence of P_n. Example 2 of [7] serves as a counterexample. In this case the class of all possible limit laws of Φ_n is given by the following two theorems.

Theorem 1. *Suppose that a stationary strong-mixing sequence $\{\xi_n\}$ satisfies* (3.3). *If P_n converges weakly to a point process P on X_α then there exists $q\in\mathcal{Q}$ such that $P=P_{\alpha,q}$.*

Theorem 2. *For every $\alpha\in R$ and $q\in\mathcal{Q}$ there exists a strong-mixing sequence $\{\xi_n\}$ such that for suitably chosen $\{a_n\}$ and $\{b_n\}$* (3.3) *holds and $P_n\xrightarrow{w}P_{\alpha,q}$.*

Let D_α denote the space of R_α-valued nondecreasing right-continuous functions on R_+. Endow D_α with the weak topology which is derived from the Lévy metric. We may choose this metric as the one defined in [13] p. 204 when $\alpha=0$. The modification needed for the case $\alpha\neq0$ will be obvious.

For each integer $k\geq1$ let

$$m^{(k)}(\varphi, t)=\sup\{u;\ \varphi((0,t]\times(u,u_\alpha))\geq k\}\,,\quad \varphi\in M_\alpha\,,\quad t>0\,.$$

Note that

$$m^{(k)}(\Phi_n, t)=(m^{(k)}_{[nt]}-b_n)/a_n\,,\quad n\geq1\,,\quad t\geq0\,.$$

Since $\varphi\to m^{(k)}(\varphi,\cdot)$ is a continuous mapping from M_α to D_α (see [8]) we have immediately from Theorem 1 the following:

Corollary 1. *If $\{\xi_n\}$ satisfies* (3.3) *and $P_n\xrightarrow{w}P_{\alpha,q}$ then $m^{(k)}(\Phi_n,\cdot)$ converges in distribution to the distribution of $m^{(k)}(\varphi,\cdot)$ with respect to $P_{\alpha,q}$.*

Remark 1. Note that the limit distribution of $m^{(1)}(\Phi_n,\cdot)$ is the distribution of an extremal process [4] and does not depend on q.

Remark 2. Let $D_{\alpha,S}$ denote the space D_α with Skorohod J_1 topology. If $\{\xi_n\}$ satisfies (3.3) and if $P_n\xrightarrow{w}P_\alpha$ as in the i.i.d. case then $m^{(1)}(\Phi_n,\cdot)$(and also every $m^{(k)}(\Phi_n,\cdot)$) converges in distribution in $D_{\alpha,S}$ (see [4, 9]). This follows from the fact that the set E of discontinuity points of $\varphi\to m^{(1)}(\varphi,\cdot)$ considered as a mapping from M_α to $D_{\alpha,S}$ has P_α-measure zero. However $P_{\alpha,q}(E)>0$ unless $P_{\alpha,q}=P_\alpha$. Hence in the space $D_{\alpha,S}$ the sequence $m^{(1)}(\Phi_n,\cdot)$ does not converge in general.

Corollary 2. *Let $m^{(k)}_n$ denote the k-th maximum of $\xi_1,\cdots,\xi_n$. If $\{\xi_n\}$ satisfies* (3.3) *and $P_n\xrightarrow{w}P_{\alpha,q}$ then the sequence $((m^{(1)}_n-b_n)/a_n,\cdots,(m^{(k)}_n-b_n)/a_n)$ converges in distribution to the distribution of $(m^{(1)}(\varphi,1),\cdots,m^{(k)}(\varphi,1))$ with respect to $P_{\alpha,q}$.*

Remark 3. As a particular case of this corollary we can see that the assumptions of Corollary 2 imply

$$\lim \mathbf{P}\{(m_n^{(1)}-b_n)/a_n \le u_1,\ (m_n^{(2)}-b_n)/a_n \le u_2\} = \begin{cases} G_\alpha(u_1) & \text{if } u_1 \le u_2 \\ G_\alpha(u_2)[1-\rho(\log G_\alpha(u_1)/\log G_\alpha(u_2)) \log G_\alpha(u_2)] & \text{if } u_1 > u_2 \end{cases} \tag{3.4}$$

where

$$\rho(s)=\int_s^1 (1-F(-\log u))du\ , \quad 0\le s\le 1\ ,$$

and

$$F(t)=q\{\psi:\ \psi[0,t]\ge 2\}\ , \quad t\ge 0\ .$$

Welsch [12] proved that if $\{\xi_n\}$ satisfies (3.3) and if the limit on the left of (3.4) exists then the limit must have the form on the right of (3.4). Although this corollary does not include Welsch's result it seems to give an insight into his result.

4. Proof of Theorem 1.

We begin with some lemmas. Throughout this section suppose that $\{\xi_n\}$ satisfies (3.1), (3.3) and $\{\Phi_n\}$ converges in distribution to a point process P on X_α:

$$P_n \xrightarrow{w} P\ . \tag{4.1}$$

For every interval $I\subset R_+$ denote by $\mathscr{M}_\alpha(I)$ the smallest σ-algebra of subsets of M_α with respect to which every mapping $\varphi\to\varphi(A)$ is measurable, where A is a Borel subset of $I\times R_\alpha$. In particular $\mathscr{M}_\alpha(R_+)=\mathscr{M}_\alpha$.

Lemma 4. *If $I_j=(c_j, d_j]$, $1\le j\le r$, are nonempty disjoint bounded intervals then $\mathscr{M}_\alpha(I_1), \cdots, \mathscr{M}_\alpha(I_r)$ are independent with respect to P.*

Proof. Since $\mathscr{M}_\alpha(I)\subset\mathscr{M}_\alpha(J)$ if $I\subset J$, it suffices to show that $\mathscr{M}_\alpha(I_1)$ and $\mathscr{M}_\alpha(I_2)$ are independent where $I_1=(c_1, c_2]$ and $I_2=(c_2, c_3]$, $0\le c_1<c_2<c_3<\infty$. Let $I_2^{(m)}=(c_2+m^{-1}, c_3]$. For $A\in\mathscr{M}_\alpha(I_1)$ and $B\in\mathscr{M}_\alpha(I_2^{(m)})$ it follows from (3.1) that

$$|P_n(A\cap B)-P_n(A)P_n(B)|<\beta([n/m])\ . \tag{4.2}$$

Hence if $P(\partial A)=P(\partial B)=0$ then $P(A\cap B)=P(A)P(B)$. The class of sets $A\in\mathscr{M}_\alpha(I)$ such that $P(\partial A)=0$ forms an algebra and generates $\mathscr{M}_\alpha(I)$. This shows that $\mathscr{M}_\alpha(I_1)$ and $\mathscr{M}_\alpha(I_2^{(m)})$ are independent for $m\ge 1$. Since $\cup_{m=1}^\infty \mathscr{M}_\alpha(I_2^{(m)})$ is an algebra generating $\mathscr{M}_\alpha(I_2)$, $\mathscr{M}_\alpha(I_1)$ and $\mathscr{M}_\alpha(I_2)$ must be independent.

Lemma 5. *P is invariant under $T_{\alpha,\tau}$.*

Proof. Let $\tau>0$ and $\tau_n=[n\tau]/n$, $n\ge 1$. By the stationarity of $\{\xi_n\}$ we have

$$P_n T^{-1}_{\alpha,\tau_n} = P_n \ .$$

If $\{\varphi_n\}$, $\varphi_n \in M_\alpha$, is a sequence tending to $\varphi \in M_\alpha$ then $T_{\alpha,\tau_n}\varphi_n \to T_{\alpha,\tau}\varphi$. Thus by the continuous mapping theorem ([1] Theorem 5.5) we have $PT^{-1}_{\alpha,\tau} = P$.

Lemma 6. *P is invariant under $S_{\alpha,\sigma}$.*

Proof. For each $n \geq 1$ define a mapping S^*_n from X_0 onto X_0 by

$$S^*_n(t, u) = (t/n, (u - b_n)/n) \ .$$

By the convergence of types theorem it can be shown that for each integer $k \geq 1$ $S^*_n S^{*-1}_{kn}$ converges as $n \to \infty$ to $S^*_{\alpha,k}$ defined by

$$S^*_{\alpha,k}(t, u) = \begin{cases} (kt, k^{1/\alpha}u) & \text{if } \alpha > 0 \\ (kt, u + \log k) & \text{if } \alpha = 0 \\ (kt, k^{1/\alpha}u) & \text{if } \alpha < 0 \end{cases}$$

for $(t, u) \in X_0$ (see Welsch [10]). The restriction of $S^*_{\alpha,k}$ to X_α coincides with $S_{\alpha,k}$.

The mapping from M_0 to M_0 induced by S^*_n is also denoted by S^*_n. Let P^*_n denote the distribution of $\Phi^*_n(\omega) = \sum_i \delta(\cdot\,; p_{ni}(\omega))$ on X_0. It is easy to see that $P^*_n = P^*_1 S^{*-1}_n$. Hence for $k \geq 1$

$$P^*_{kn} = P^*_n S^*_n S^{*-1}_{kn} \ .$$

Let $A_1, \cdots, A_r$ be disjoint intervals of X_α such that $\bar{A}_j$ is compact in X_α. Suppose $P\{\varphi;\ \varphi(\partial A_j) \geq 1\} = 0$, $1 \leq j \leq r$. Then noticing that by the mapping from M_0 to M_0 sending φ to $\varphi(\cdot \cap X_\alpha)$ P^*_n induces P_n we have

$$\begin{aligned} &P_n\{\varphi;\ \varphi \in M_\alpha,\ \varphi(A_j) = k_j,\ 1 \leq j \leq r\} \\ &= P_n\{\varphi;\ \varphi \in M_\alpha,\ \varphi(S^*_n S^{*-1}_{kn} A_j) = k_j,\ 1 \leq j \leq r\} \ . \end{aligned}$$

By letting $n \to \infty$ we obtain

$$\begin{aligned} &P\{\varphi;\ \varphi \in M_\alpha,\ \varphi(A_j) = k_j,\ 1 \leq j \leq r\} \\ &= P\{\varphi;\ \varphi \in M_\alpha,\ \varphi(S_{\alpha,k} A_j) = k_j,\ 1 \leq j \leq r\} \ . \end{aligned}$$

This shows $P = PS^{-1}_{\alpha,k}$ for $k \geq 1$. Thus for every pair of integers $k \geq 1$ and $l \geq 1$ $PS^{-1}_{\alpha,k} = PS^{-1}_{\alpha,l}$ and therefore

$$P = PS^{-1}_{\alpha,\sigma} \tag{4.3}$$

for rational $\sigma > 0$. Suppose $\sigma_n \to \sigma > 0$. If $\varphi_n \to \varphi$ then $S_{\alpha,\sigma_n}\varphi_n \to S_{\alpha,\sigma}\varphi$ and therefore by the continuous mapping theorem $PS^{-1}_{\alpha,\sigma_n} \xrightarrow{w} PS^{-1}_{\alpha,\sigma}$. This shows that (4.3) holds for every $\sigma > 0$.

Lemma 7. *P is an infinitely divisible point process.*

Proof. By Satz 1.4.12 of [3] it suffices to show that the distribution of random vector $(\varphi(A_1), \cdots, \varphi(A_r))$ on the probability space $(M_\alpha, \mathcal{M}_\alpha, P)$ is infinitely divisible, where $A_1, \cdots, A_r$ are disjoint semi-open intervals of X_α. For this purpose suppose at first that $A_1=I\times J_1, \cdots, A_r=I\times J_r$, where $I=(c, c+h]\subset R_+$ and $J_1, \cdots, J_r$ are disjoint intervals of R_α. For each $n\geq 1$ let $A_{jk}=(c+(k-1)h/n, c+kh/n]\times J_j$, $1\leq j\leq r$, $1\leq k\leq n$. By Lemma 4 and Lemma 5 the random vectors $(\varphi(A_{1k}), \cdots, \varphi(A_{rk}))$, $1\leq k\leq n$, are i.i.d. and

$$(\varphi(A_1), \cdots, \varphi(A_r))=\sum_{k=1}^{n} (\varphi(A_{1k}), \cdots, \varphi(A_{rk})) .$$

Hence in this particular case our assertion is proved. In view of Lemma 4 the general case follows from this particular case.

Lemma 8. $\tilde{P}(\Lambda_\alpha^c)=0$.

Proof. Let I and J be disjoint subintervals of R_+ and let $K\subset R_\alpha$ be compact. By Lemma 4 $\varphi(I\times K)$ and $\varphi(J\times K)$ are independent with respect to P. This implies by Hilfssatz 2.3.6 of [3] that

$$\tilde{P}(\{\varphi;\ \varphi(I\times K)>0,\ \varphi(J\times K)>0\})=0 . \tag{4.4}$$

Let for $n\geq 1$

$$\Lambda_{\alpha,n}=\{\varphi;\ \varphi\in M_\alpha,\ \varphi\neq 0,\ \varphi(((k-1)2^{-n}, k2^{-n}]\times R_\alpha)=0 \text{ except for one } k\geq 1\} ,$$

and for a sequence $\{c_n\}$ such that $c_n\uparrow u_\alpha$ let

$$N_{\alpha,m,n}=\{\varphi;\ \varphi\in M_\alpha,\ \varphi((0, m]\times(c_n, u_\alpha))=0\} , \quad m\geq 1 , \quad n\geq 1 .$$

Then

$$\Lambda_\alpha=(\cap_{n=1}^{\infty}\Lambda_{\alpha,n})\cap(\cap_{m=1}^{\infty}\cup_{n=1}^{\infty}N_{\alpha,m,n}) .$$

It follows from (4.4) that $\tilde{P}(\Lambda_{\alpha,n}^c)=0$, $n\geq 1$. It is known that (see [3] p. 93)

$$\begin{aligned}\exp[-\tilde{P}\{\varphi;\ \varphi((0,1]\times(c_n, u_\alpha))>0\}]\\ =P\{\varphi;\ \varphi((0,1]\times(c_n, u_\alpha))=0\} = 1-G_\alpha(c_n) .\end{aligned}$$

Hence for every $m\geq 1$, $n\geq 1$ $\tilde{P}(N_{\alpha,m,n}^c)=0$ and therefore $\tilde{P}(\Lambda_\alpha^c)=0$.

Lemma 9. *P is invariant under $T_{\alpha,\tau}$ and $S_{\alpha,\sigma}$.*

Proof. Apply the same argument as Hilfssatz 3.3.1 of [3].

Proof of Theorem 1. The following two relations are easily verified:

$$T_{\alpha,\tau}^{-1}h_\alpha^{-1}(A\times E)=h_\alpha^{-1}((T_{\alpha,\tau}^{-1}A)\times E)$$

and

$$S_{\alpha,\sigma}^{-1}h_\alpha^{-1}(A\times E)=h_\alpha^{-1}((S_{\alpha,\sigma}^{-1}A)\times E)$$

for every $A\in\mathscr{X}_\alpha$ and $E\in\mathscr{N}$.

Fix $E\in\mathscr{N}$ and write $\tilde{P}h_\alpha^{-1}(A\times E)=\nu_E(A)$. ν_E is a σ-finite measure on $\mathscr{X}_\alpha$ and by Lemma 9 ν_E satisfies

$$\begin{aligned}\nu_E(T_{\alpha,\tau}^{-1}A)&=\tilde{P}h_\alpha^{-1}(T_{\alpha,\tau}^{-1}A\times E)=\tilde{P}T_{\alpha,\tau}^{-1}h_\alpha^{-1}(A\times E)\\&=\tilde{P}h_\alpha^{-1}(A\times E)=\nu_E(A)\ ,\quad A\in\mathscr{X}_\alpha\ .\end{aligned}$$

Similarly

$$\nu_E(S_{\alpha,\sigma}^{-1}A)=\nu_E(A)\ ,\quad A\in\mathscr{X}_\alpha\ .$$

It follows from Lemma 1 that there exists a constant $q(E)\geq 0$ such that

$$\tilde{P}h_\alpha^{-1}(A\times E)=\nu_E(A)=q(E)\pi_\alpha(A)\ .$$

It is easy to see that q determines a measure on $\mathscr{N}$. If $A\in\mathscr{X}_\alpha$ is compact then since

$$h_\alpha^{-1}(A\times N)=\{\varphi;\ \varphi\in\Lambda_\alpha,\ \varphi(A)>0\}\ ,$$

we have by Hilfssatz 2.2.1 of [3] that

$$q(N)\pi_\alpha(A)\leq\tilde{P}\{\varphi;\ \varphi(A)>0\}<\infty\ .$$

Hence q is finite. By (3.3) and (4.1) we have

$$\begin{aligned}P[\varphi;\ \varphi(A_u)=0\}&=\lim_n P_n\{\varphi;\ \varphi(A_u)=0\}\\&=\lim_n \mathbf{P}\{(m_n-b_n)/a_n\leq u\}=G_\alpha(u)\ ,\quad A_u=(0,1)\times(u,u_\alpha)\ ,\quad u\in R_\alpha\ .\end{aligned}$$

Hence by an argument in p. 93 of [3]

$$\begin{aligned}q(N)\pi_\alpha(A_u)&=\tilde{P}h_\alpha^{-1}(A_u\times N)=\tilde{P}\{\varphi;\ \varphi(A_u)>0\}\\&=-\log P\{\varphi;\ \varphi(A_u)=0\}=-\log G_\alpha(u)\ .\end{aligned}$$

By the definition of π_α and G_α we have $q(N)=1$, i.e. $q\in\mathscr{Q}$. Thus in view of Lemma 2 we have

$$\tilde{P}h_\alpha^{-1}=\tilde{P}_{\alpha,q}h_\alpha^{-1}\ .$$

Since $\tilde{P}(\Lambda_\alpha^c)=\tilde{P}_{\alpha,q}(\Lambda_\alpha^c)=0$ we have $\tilde{P}=\tilde{P}_{\alpha,q}$. This completes the proof.

5. Proof of Theorem 2.

Let $\{Z_n\}_{n\geq 1}$ be i.i.d. random variables having common exponential distribution with mean one. Let $q\in\mathscr{Q}$ and let $\zeta=(\zeta^{(1)},\zeta^{(2)},\cdots)$ be an infinite dimensional random vector such that $0\leq\zeta^{(1)}\leq\zeta^{(2)}\leq\cdots\leq\infty$, $\lim_k \zeta^{(k)}=\infty$ w.p.l, and the distribution of the point process $\sum_k \delta(\cdot;\ \zeta^{(k)})$ coincides with q. By convention $\delta(\cdot;\ \infty)$ is the zero measure. Let $\{\zeta_n\}_{n\geq 1}$ be i.i.d. random vectors each having the same distribution as ζ. Suppose $\{Z_n\}$ and $\{\zeta_n\}$ are independent.

Let $\varphi(u)$, $u\geq 0$, be a continuous increasing function satisfying

$$\lim_{u\to\infty} \varphi(u)=\infty\ , \qquad \lim_{u\to\infty} \varphi(u)/u=0\ ,$$

and

$$\sum_{k=1}^{\infty}\int_{\varphi^{-1}(k)}^{\infty} \mathbf{P}\{\zeta^{(k)}<u\}e^{-u}du<\infty\ . \tag{5.1}$$

The existence of such a function is obvious. Let

$$\zeta'_{nk}=\begin{cases}\zeta_{nk} & \text{if} \quad k\leq\varphi(Z_n)\\ \infty & \text{if} \quad k>\varphi(Z_n)\end{cases}$$

for $n\geq 1$, $k\geq 0$, and define

$$\xi_n=\sup_{k\geq 0}(Z_{n+k}-\zeta'_{n+k,k})\ , \quad n\geq 1\ .$$

We can show that $\xi_n<\infty$ w.p.l. In fact

$$\mathbf{P}\{\xi_n\leq x\}=\Pi_{k=0}^{\infty}\mathbf{P}\{Z_1-\zeta'_{1k}\leq x\}=\Pi_{k=0}^{\infty}[1-\mathbf{P}\{\zeta_{1k}<Z_1-x,\ \varphi(Z_1)\geq k\}]\ . \tag{5.2}$$

It follows from (5.1) and

$$\mathbf{P}\{\zeta_{1k}<Z_1-x,\ \varphi(Z_1)\geq k\}=\int_{\varphi^{-1}(k)}^{\infty}\mathbf{P}\{\zeta_{1k}<u-x\}e^{-u}du \tag{5.3}$$

that the infinite product in (5.2) converges. Since the integral on the right of (5.3) converges to zero monotonically as $x\to\infty$, we have

$$\lim_{x\to\infty}\mathbf{P}\{\xi_n\leq x\}=1\ .$$

It is easy to see that the sequence $\{\xi_n\}_{n\geq 1}$ is stationary. Furthermore we have

Lemma 10. *$\{\xi_n\}$ is strong-mixing.*

Proof. Let $n\geq 1$, $m\geq 1$. Define ξ'_k, $k\geq 1$, by

$$\xi'_k=\begin{cases}\xi_k & \text{if} \quad k\geq n+m\\ \sup_{0\leq j<n+m-k}(Z_{k+j}-\zeta'_{k+j,j}) & \text{if} \quad k<n+m\end{cases}$$

Let us define an event C_m, $m\geq 1$, by

$$C_m=\{\varphi(Z_{n+m+k})<m+k \text{ for every } k\geq 0\}\ .$$

Then

$$\mathbf{P}\{C_m\}=\Pi_{k=m}^{\infty}\{1-e^{-\varphi^{-1}(k)}\}\ .$$

Since $\varphi(k)<k$ for large k, the infinite product on the right converges. Hence for every $\varepsilon>0$ there exists m (independent of n) satisfying $P\{C_m\}>1-\varepsilon$. By definition we have

(5.4) $$\xi'_k=\xi_k \quad \text{on} \quad C_m \quad \text{for} \quad 1\leq k\leq n\ .$$

Let $A\in\mathscr{F}(\xi_1,\cdots,\xi_n)$ and $B\in\mathscr{F}(\xi_{n+m+1},\cdots)$. It follows from (5.4) that there exists $A''\in\mathscr{F}(\xi_1\cdots,\xi_n)$ such that

(5.5) $$A\cap C_m=A'\cap C_m\ .$$

By (5.4) and (5.5) we have

$$|\mathbf{P}\{A\}-\mathbf{P}\{A'\}|<2\varepsilon \text{ and } |\mathbf{P}\{A\cap B\}-\mathbf{P}\{A'\cap B\}|<2\varepsilon\ .$$

Since A' and B are independent this implies

$$|\mathbf{P}\{A\cap B\}-\mathbf{P}\{A\}\mathbf{P}\{B\}|<4\varepsilon\ .$$

This proves that $\{\xi_n\}$ is strong-mixing.

Finally we show that $\{\xi_n\}$ satisfies the assertion in Theorem 2. Let Φ_n be a point process consisting of random points $(k/n, \xi_k-\log n)$, $k\geq 1$. Let Φ'_n be a point process consisting of random points $((k-j)/n,\ Z_k-\zeta'_{k,j}-\log n)$, $j\geq 0$, $k\geq 1$, and let Ψ_n denote a point process consisting of random points $(k/n,\ Z_k-\log n)$, $k\geq 1$. We show that for every interval $A=(t_1,t_2)\times(u_1,u_2)$ $\mathbf{P}\{\Phi_n(A)\neq\Phi'_n(A)\}\to 0$.

Lemma 11. *For every $\varepsilon>0$ and $u_1\in R$ there exist $M>0$ and $h>0$ such that for large n*

(5.6) $$\mathbf{P}\{\Psi_n((0,t_2+1)\times(M,\infty))=0\}>1-\varepsilon$$

and

(5.7) $$\mathbf{P}\{\Psi_n(((t_1,t_1+h)\cup(t_2,t_2+h))\times(u_1,M))=0\}>1-\varepsilon\ .$$

Proof. It is well known that Ψ_n converges in distribution to the Poisson point process P_0 with intensity π_0. Hence the left sides of (5.6) and (5.7) converges as $n\to\infty$ to

$$\exp\{-(t_2+1)e^{-M}\} \quad \text{and} \quad \exp\{-2h(e^{-u_1}-e^{-M})\}$$

respectively. Choose M sufficiently large and then h sufficiently small so that both of the above quantities are less than $\varepsilon/2$. Then (5.6) and (5.7) holds for sufficiently large n.

Lemma 12. *For every* $\varepsilon>0$ *there exists* $h'>0$ *such that*

$$\mathbf{P}\{\Psi_n((jh',(j+2)h')\times(u_1,M))\leq 1 \text{ for every } j\leq k\}>1-\varepsilon, \tag{5.8}$$

for large n, *where* $k=[t_2/h']+1$.

Proof. The probability on the left of (5.8) is not smaller than

$$1-\sum_{j=0}^{k}\mathbf{P}\{\Psi_n((jh',(j+2)h')\times(u_1,M))\geq 2\}.$$

This converges as $n\to\infty$ to

$$\begin{aligned}1-(k+1)[1-\exp(-2h'(e^{-u_1}-e^{-M}))-2h'(e^{-u_1}-e^{-M})\exp(-2h'(e^{-u_1}-e^{-M}))]\\ \geq 1-(k+1)\{2h'(e^{-u_1}-e^{-M})\}^2\geq 1-5t_2(e^{-u_1}-e^{-M})^2h'.\end{aligned}$$

This implies the lemma.

Lemma 13. *Let* K_n *be the set* $\{k;(k/n,Z_k-\log n)\in(0,t_2)\times(u_1,M)\}$. *Then for every* $\varepsilon>0$ *and for large* n

$$\mathbf{P}\{\zeta_{j,\varphi(M+\log n)}>M-u_1 \text{ for every } j\in K_n\}>1-\varepsilon. \tag{5.9}$$

Proof. Choose K so large that $\mathbf{P}\{\text{card}(K_n)>K\}<\varepsilon/2$. Then for large n

$$\mathbf{P}\{\zeta_{1,\varphi(M+\log n)}\leq M-u_1\}<\varepsilon/(4K),$$

and the left side of (5.9) is greater than

$$\begin{aligned}\mathbf{P}\{\zeta_{j,\varphi(M+\log n)}>M-u_1 \text{ for } j\in K_n, \text{ and } \text{card}(K_n)\leq K\}\\ \geq(1-\varepsilon/2)(1-\varepsilon/(4K))^K>1-\varepsilon.\end{aligned}$$

Lemma 14. *For* $A=(t_1,t_2)\times(u_1,u_2)$

$$\lim_n \mathbf{P}\{\Phi_n(A)\neq\Phi'_n(A)\}=0.$$

Proof. Given $\varepsilon>0$ choose M, h, h' so as to satisfy Lemma 11 and Lemma 12. Let E_1,E_2,E_3,E_4 denote the events on the right of (5.6), (5.7), (5.8) and (5.9) resp. and $E=E_1\cap E_2\cap E_3\cap E_4$. Then it is easy to see that $\Phi_n(A)=\Phi'_n(A)$ on E. By Lemma 11, 12 and 13 $\mathbf{P}\{E\}>1-4\varepsilon$ for large n.

Proof of Theorem 2. Let $\kappa_n(\cdot|x)$ be the distribution of point process consisting of random points $(t-j/n,u-\zeta^{(j)})$, $0\leq j\leq[\varphi(u+\log n)]$. Then

$$\mathbf{P}\,\Phi_n'^{-1}=\int\kappa_n(\cdot|\varphi)P'_n(d\varphi).$$

Let E denote the set of the form $\{\varphi;\varphi(A_i)=k_i, 1\leq i\leq r\}$, where A_i's are bounded disjoint intervals of X_0 and k_i's are nonnegative integers. Choose a bounded interval A_0 such that $A_0 \supset \bigcup_{i=1}^{r} A_i$. Let $D=\{\varphi;\ \varphi(\partial A_i)>0$ for some $i,\ 0\leq i\leq r\}$. If a sequence $\{\varphi_n\}$ of elements of M converges to $\varphi\in M$ and if $\varphi\notin D$ then $\kappa_n(E|\varphi_n)\to\kappa_{0,q}(E|\varphi)$. Since $P'_n\to P_0$ and since $P_0(D)=0$ we have

$$\Phi'_n \xrightarrow{D} P_{0,q}=\int \kappa_{0,q}(\cdot|\varphi)P_0(d\varphi)\ .$$

Together with Lemma 14 this proves the theorem in the case $\alpha=0$. When $\alpha\neq 0$ we can replace ξ_n constructed above by $\exp(\alpha^{-1}\xi_n)$ or $-\exp(\alpha^{-1}\xi_n)$ according as $\alpha>0$ or $\alpha<0$ resp. to obtain examples of strong-mixing sequence satisfying the assertion of the theorem.

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