CONVERGENCE RATES FOR INTEGRAL TYPE FUNCTIONALS OF ABSOLUTELY REGULAR PROCESSES

By

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1. Introduction. Let $\{\xi_j\}$ be a strictly stationary sequence of random variables which are defined on a probability space $(\Omega, \mathscr{A}, P)$. For $a \leqq b$, let $\mathscr{M}_a^b$ denote the σ-algebra of events generated by $\xi_a, \cdots, \xi_b$. As in [7]-[9], we shall say that the sequence is absolutely regular if

$$\beta(n) = E\{\sup_{A \in \mathscr{M}_n^\infty} |P(A|\mathscr{M}_{-\infty}^0) - P(A)|\} \downarrow 0 \quad (n \to \infty) \tag{1.1}$$

For any $T(0 < T \leqq \infty)$, let $C_T = C[0, T]$ be the space of all continuous functions on $[0, T]$. We give the uniform topology by defining the distance between two points x and y in C_T as

$$\rho_T(x, y) = \sup_{0 \leqq t \leqq T} |x(t) - y(t)| . \tag{1.2}$$

Set $S_k = \sum_{j=1}^k \xi_j$ and $S_0 = 0$. Let $w = \{w(t):\ 0 \leqq t < \infty\}$ be a standard Wiener process.

In [5], Sawyer proved that if $\{\xi_i\}$ is a sequence of independent and identically distributed random variables, then under some additional conditions

$$\sup_z \left| P\left(\frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}, \frac{S_k}{\sqrt{n}\,\sigma}\right) \leqq z\right) - P\left(\int_0^1 f(t, w(t))dt \leqq z\right)\right| = O(n^{-1/2} (\log n)^\alpha)$$

where α is a positive constant (cf. [4]), and, recently, Borisov [2] extended Sawyer's result.

In this paper, we shall consider the analogous problem for absolutely regular processes.

2. Conditions and some auxiliary results. In this and following sections we shall denote by the letter K, with or without indices, various positive constants.

Condition I. $f(s, x)$ is a function such that f and its partial derivatives of order one are of slow growth in x; i.e., f satisfies inequalities of the form

(2.1) $$|Df(s, x)| \leqq K(1+|x|^{\alpha})$$

where D denotes either the identity operator or a first derivative and α is some positive constant.

Condition II. The distribution function

(2.2) $$F(z)=P\left(\int_0^1 f(t, w(t))dt \leqq z\right)$$

satisfies a first order Lipschitz condition.

Condition III. $\{\xi_i\}$ is a strictly stationary, absolutely regular process such that

(2.3) (i) $E\xi_i=0,\ E|\xi_i|^{4+\delta}<\infty$ for some $\delta>0$,

(2.4) (ii) $\beta(n)=O(e^{-\gamma n})$ for some $\gamma>0$,

and

(2.5) (iii) $\sigma^2=E\xi_0^2+2\sum_{j=1}^{\infty} E\xi_0\xi_j>0$

(It is known that the series in (2.5) converges absolutely if (2.3) and (2.4) hold.)

The following Lemmas are proved in [8] and [9].

Lemma 2.1. *Under Condition* III

(2.6) $$|\mathrm{Var}\,(S_n)-n\sigma^2| \leqq K$$

(cf. Lemma 3.1 in [8])

Lemma 2.2. *Under Condition* III

(2.7) $$ES_n^4 \leqq Kn^2$$

(cf. Lemma 3.2 in [8])

Lemma 2.3. *Under Condition* III

(2.8) $$P(|n^{-1/2}S_n| \geqq 2\sigma_0 a \log n) \leqq Kn^{-1}a^{-4} \log n$$

where $a>0$ *and* $\sigma_0^2=\mathrm{Var}\,(\xi_1)>0$. (cf. Theorem 5.2 in [8])

3. The main result. For each $n(\geqq 1)$, define a random element $X_n=\{X_n(t): 0\leqq t\leqq 1\}$ in C_1 by

(3.1) $$X_n(t)=\begin{cases} \dfrac{1}{\sqrt{n}\sigma}S_k & \text{for } t=\dfrac{k}{n}, \quad k=0, 1, \cdots, n \\ \text{linearly interpolated for } t\in\left[\dfrac{k-1}{n}, \dfrac{k}{n}\right), & k=1, \cdots, n \end{cases}$$

where σ is the constant defined by (2.5). We shall prove the following theorem using results in [9].

Theorem. *Let Conditions* I-III *be satisfied. Then*

$$\sup_z \left| P\left(\int_0^1 f(t, X_n(t))dt \leqq z\right) - P\left(\int_0^1 f(t, w(t))dt \leqq z\right)\right| = O(n^{-1/5}(\log n)^{1+a'}) \tag{3.2}$$

where $a' > a > 0$.

To prove Theorem, we need some lemmas. Let $[s]$ be the largest integer p such that $p \leqq s$. For any integer N, define random elements

$$\bar{S}_N = \{S_N(t):\ 0 \leqq t \leqq N\} \text{ and } \tilde{S}_N = \{\tilde{S}_N(t):\ 0 \leqq t \leqq N\} \text{ in } C_N ,$$

respectively, by

$$S_N(t) = \begin{cases} S_k \text{ for } t = k, k = 0, 1, \cdots, N \\ \text{linearly interpolated for } t \in [k-1, k),\ k = 1, \cdots, N \end{cases}$$

and

$$\tilde{S}_N(t) = \begin{cases} S_{jr} \text{ for } t = jr,\ j = 0, 1, \cdots, m \\ S_{mr} \text{ for } mr \leqq t \leqq N, \\ \text{linearly interpolated for } t \in [(j-1)r, jr),\ j = 1, \cdots, m \end{cases} \tag{3.4}$$

where m and r are some integers defined below.

Lemma 3.1. *Let* $m = [n^{2/5}]$ *and* $r = [nm^{-1}]$. *Then, under Conditions* I-III

$$P\left(\sup_{0 \leqq t \leqq 1} \left| f(t, X_n(t)) - f\left(t, \frac{\tilde{S}_n(nt)}{\sqrt{n}\sigma}\right)\right| \geqq \varepsilon_n\right) = o(n^{-1/5} (\log n)^{1+a'}) \tag{3.5}$$

where $\varepsilon_n = n^{-1/5}(\log n)^{1+a}$ *and* $a' > a > 0$.

Proof. By Condition I

$$|f(t, x) - f(t, x')| \leqq K|x - x'|(1 + \max(|x|^a, |x'|^a))$$

for all t, x and x'. So, putting $b = a'/a > 1$,

$$\begin{aligned} &P\left(\sup_{0 \leqq t \leqq 1} \left| f(t, X_n(t)) - f\left(t, \frac{\tilde{S}_n(nt)}{\sqrt{n}\sigma}\right)\right| \geqq \varepsilon_n\right) \\ &\leqq P\left(\sup_{0 \leqq t \leqq 1} K \left| X_n(t) - \frac{\tilde{S}_n(nt)}{\sqrt{n}\sigma}\right| \left(1 + \max(|X_n(t)|^a, \left|\frac{\tilde{S}_n(nt)}{\sqrt{n}\sigma}\right|^a\right) \geqq \varepsilon_n\right)\right) \\ &\leqq P(\rho_n(\bar{S}_n, \tilde{S}_n) \geqq 4\varepsilon_n n^{1/2}\sigma(\log n)^{-a'}) + 2P\left(\max_{1 \leqq j \leqq n} |S_j| \geqq \frac{1}{2} n^{1/2}\sigma(\log n)^b\right). \end{aligned} \tag{3.6}$$

From the method of the proof of Lemma 4.1 in [9]

$$P(\rho_n(\bar{S}_n, \tilde{S}_n) \geqq \varepsilon_n n^{1/2}\sigma(\log n)^{-a}) = o(n^{-1/5}(\log n)^{1+a'}) \tag{3.7}$$

On the other hand, since from Lemma 2.2

$$E|r^{-1/2}\sum_{i=1}^{r}\xi_i|^4 \leqq K\,,$$

so from the method of the proof of Lemma 3.3 in [9] and Lemma 2.3 that

$$\begin{aligned} P\Big(\max_{1\leqq j\leqq n}|S_j| \geqq \frac{1}{2}n^{1/2}\sigma(\log n)^b\Big) &\leqq 2P\Big(|S_n| \geqq \frac{1}{2}n^{1/2}\sigma(\log n)^b\Big) + 2nr^{-1}\beta(r) \\ &\leqq K(mr)^{-1} + o(n^{-1}) = o(n^{-1})\,. \end{aligned} \tag{3.8}$$

Combining (3.5)-(3.7), we have (3.5) and the proof is completed.

For any n, let $m=[n^{2/5}]$, $r=[m^{-1}n]$ and $q=[c\log n]$ $(c>1/\gamma)$. Let $p=r-q$ and put

$$\begin{aligned} \eta_j &= \sum_{i=1}^{p}\xi_{(j-1)r+i} \qquad (j=1,\cdots,m)\,, \\ \zeta_j &= \sum_{i=1}^{q}\xi_{(j-1)r+p+i} \quad (j=1,\cdots,m); \quad \zeta_{m+1} = \sum_{i=1}^{p-mr}\xi_{mr+i}\,. \end{aligned} \tag{3.9}$$

Further, let $\hat{S}_n=\{\hat{S}_n(t)\colon 0\leqq t\leqq n\}$ be the random element in C_n defined by

$$\hat{S}_n(t) = \begin{cases} \sum_{i=1}^{j}\eta_i & \text{for } t=jr,\ j=0,1,\cdots,m\,, \\ \sum_{i=1}^{m}\eta_i & \text{for } mr\leqq t\leqq n\,, \\ \text{linearly interpolated for } t\in[(j-1)r, jr),\ j=1,\cdots,m\,. \end{cases} \tag{3.10}$$

Lemma 3.2. *Let $\hat{S}_n$ be defined by* (3.10). *Then, under Conditions* I-III

$$P\Big(\sup_{0\leqq t\leqq 1}\Big|f\Big(t, \frac{\tilde{S}_n(nt)}{\sqrt{n}\sigma}\Big) - f\Big(t, \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\Big)\Big| \geqq \varepsilon_n\Big) = O(n^{-1/5}\log n)^{1+a'}) \tag{3.11}$$

where $\varepsilon_n = n^{-1/5}(\log n)^{1+a'}$.

Proof. As in the proof of Lemma 3.1 we have

$$\begin{aligned} &P\Big(\sup_{0\leqq t\leqq 1}\Big|f\Big(t, \frac{\tilde{S}_n(nt)}{\sqrt{n}\sigma}\Big) - f\Big(t, \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\Big)\Big| \geqq \varepsilon_n\Big) \\ &\leqq P\Big(\sup_{0\leqq t\leqq 1} K\Big|\frac{\tilde{S}_n(nt)}{\sqrt{n}\sigma} - \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\Big|\Big(1+\max\Big(\Big|\frac{\tilde{S}_n(nt)}{\sqrt{n}\,\sigma}\Big|^a, \Big|\frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\Big|^a\Big)\Big) \geqq \varepsilon_n\Big) \\ &\quad + P\Big(\sup_{0\leqq t\leqq 1}|\hat{S}_n(nt)| \geqq \frac{1}{2}n^{1/2}\sigma(\log n)^b\Big) \end{aligned} \tag{3.12}$$

where $b=a'/a>1$. From the proof of Lemma 4.2 in [9] and Lemma 2.3

$$(3.13)\qquad \begin{aligned} &P(\rho_n(\tilde{S}_n, \hat{S}_n)\geqq \varepsilon_n n^{1/2}\sigma(\log n)^{-a}) \\ &\leqq P\Big(\max_{1\leqq j\leqq m} |\sum_{i=1}^{j} \zeta_i|\geqq \frac{1}{2}\varepsilon_n n^{1/2}\sigma(\log n)^{-a}\Big) \\ &\quad + P\Big(|\zeta_{m+1}|\geqq \frac{1}{2}\varepsilon_n n^{1/2}\sigma(\log n)^{-a}\Big) \\ &\leqq 2P\Big(|\sum_{i=1}^{m} \zeta_i|\geqq \frac{1}{4}\varepsilon_n n^{1/2}\sigma(\log n)^{-a}\Big)+m\beta(r) \\ &\quad + P\Big(|\sum_{i=1}^{n-mr} \xi_i|\geqq \frac{1}{2}n^{3/10}(\log n)^{1+(a-a')}\Big) \\ &\leqq Kn^{-1}\varepsilon_n^{-2}m(\log n)^{2a}E|\zeta_1|^2+Kn^{-3/5}=o(n^{-1/5})\ . \end{aligned}$$

Similarly, we have

$$(3.14)\qquad \begin{aligned} &P\Big(\sup_{0\leqq t\leqq 1}|\hat{S}_n(nt)|\geqq \frac{1}{2}n^{1/2}\sigma(\log n)^{b}\Big) \\ &\quad = P\Big(\max_{1\leqq j\leqq m} |\sum_{i=1}^{j} \eta_i|\geqq \frac{1}{2}n^{1/2}\sigma(\log n)^{b}\Big) \\ &\quad \leqq 2P\Big(|\sum_{i=1}^{m} \eta_i|\geqq \frac{1}{4}n^{1/2}\sigma(\log n)^{b}\Big)+m\beta(q) \\ &\quad = o(m^{-1})+mo(n^{-1})=o(n^{-2/5})\ . \end{aligned}$$

Thus, from (3.8) and (3.12)-(3.14), we have the desired result.

Lemma 3.3. *Under the conditions of Lemma 3.2, there exists a sequence of nonnegative, independent and identically distributed random variables $\tau_1, \cdots, \tau_m$ with the following properties:*

$$(3.15)\qquad \text{(i)}\quad E\tau_1=\operatorname{Var}\Big(\frac{1}{\sqrt{p}\sigma}\eta_1\Big),\quad E\tau_1^j\leqq K_j\Big(\frac{1}{\sqrt{p}\sigma}\Big)^{j}E|\eta_1|^{2j}\quad (j=1, 2, \cdots)$$

$$(3.16)\qquad \begin{aligned} \text{(ii)}\quad &P\Big(\frac{1}{m}\sum_{j=1}^{m} f\Big(\frac{j}{m}, w\Big(\sum_{i=1}^{j}\tau_i\Big)\Big)\leqq z-\varepsilon_n\Big)+o(n^{-1/5}) \\ &\leqq P\Big(\int_0^1\Big(f\Big(t, \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\Big)dt\leqq z\Big) \\ &\leqq P\Big(\frac{1}{m}\sum_{j=1}^{m} f\Big(\frac{j}{m}, w\Big(\sum_{i=1}^{j}\tau_i\Big)\Big)\leqq z+\varepsilon_n\Big)+o(n^{-1/5}) \end{aligned}$$

where $\varepsilon_n=n^{-1/5}(\log n)^{1+a'}$.

Proof. Since from Condition I

$$|f(t, x)-f(t', x')|\leqq K(|t-t'|+|x-x'|)(1+\max(|x|^a, |x'|^a))\ ,$$

so choosing $a''(a'>a''>a)$ we have that for all n sufficiently large

$$\begin{aligned}(3.17)\quad & P\left(\sup_{0\leqq t\leqq 1}\left|f\left(t, \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\right)-f\left(\frac{[mt]}{m}, \sum_{i=1}^{[mt]}\frac{\eta_i}{\sqrt{n}\sigma}\right)\right|\geqq\varepsilon_n\right)\\
&\leqq P\left(\sup_{0\leqq t\leqq 1} K\left(m^{-1}+\left|\frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}-\sum_{i=1}^{[mt]}\frac{\eta_i}{\sqrt{n}\sigma}\right|\right)\right.\\
&\qquad\left.\times\left(1+\max\left(\left|\frac{\hat{S}_n(nt)}{\sqrt{n}\,\sigma}\right|, \left|\sum_{i=1}^{[mt]}\frac{\eta_i}{\sqrt{n}\sigma}\right|^a\right)\right)\geqq\varepsilon_n\right)\\
&\leqq P\left(\sup_{0\leqq t\leqq 1}\left|\frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}-\sum_{i=1}^{[mt]}\frac{\eta_i}{\sqrt{n}\sigma}\right|\geqq\varepsilon_n(\log n)^{-a''}\right)\\
&\qquad+P\left(\sup_{0\leqq t\leqq 1}\max\left(\hat{S}_n(nt), \left|\sum_{i=1}^{[mt]}\eta_i\right|\right)\geqq\frac{1}{2}n^{1/2}\sigma(\log n)^{a''/a}\right)\\
&\qquad+P\left(\sup_{0\leqq t\leqq 1}\left(1+\max\left(\left|\frac{\hat{S}_n(nt)}{\sqrt{n}\,\sigma}\right|, \left|\sum_{i=1}^{[mt]}\frac{\eta_i}{\sqrt{n}\sigma}\right|^a\right)\right)\geqq\frac{1}{2}\varepsilon_n m\right)\\
&\leqq P\left(\max_{1\leqq j\leqq m}|\eta_j|\geqq n^{1/2}\sigma\varepsilon_n(\log n)^{-a''}\right)\\
&\qquad+P\left(\max_{1\leqq j\leqq m}\left|\sum_{i=1}^{j}\eta_i\right|\geqq\frac{1}{2}n^{1/2}\sigma(\log n)^{a''/a}\right)\\
&\qquad+P\left(\max_{1\leqq j\leqq m}\left|\sum_{i=1}^{j}\frac{\eta_i}{\sqrt{n}\sigma}\right|\geqq\left(\frac{1}{2}\varepsilon_n m\right)^{a''/a}\right)\\
&\qquad+2P\left(\sup_{0\leqq t\leqq 1}|\hat{S}_n(nt)|\geqq\frac{1}{4}n^{1/2}\sigma(\log n)^{a''/a}\right)\\
&\leqq mP(|\eta_1|\geqq n^{1/2}\sigma\varepsilon_n(\log n)^{-a''})+2P\left(\max_{1\leqq j\leqq m}\left|\sum_{i=1}^{j}\eta_i\right|\geqq\frac{1}{2}n^{1/2}\sigma(\log n)^{a''/a}\right)\\
&\qquad+2P\left(\sup_{0\leqq t\leqq 1}|\hat{S}_n(nt)|\geqq\frac{1}{4}n^{1/2}\sigma(\log n)^{a''/a}\right).\end{aligned}$$

From Lemma 2.3

$$(3.18)\qquad mP(|\eta_1|\geqq n^{1/2}\sigma\varepsilon_n(\log n)^{-a''})=mo(r^{-1})=o(n^{-1/5})$$

On the other hand, using the method of the proof of Lemma 2.2 in [8]

$$\begin{aligned}(3.19)\quad & P\left(\max_{1\leqq j\leqq n}\left|\sum_{i=1}^{j}\eta_i\right|\geqq n^{1/2}\sigma(\log n)^{a''/a}\right)\\
&\leqq P\left(\max_{1\leqq j\leqq m}\left|\sum_{i=1}^{j}Y_i\right|\geqq n^{1/2}\sigma(\log n)^{a''/a}\right)+2m\beta(q)\\
&\leqq 2P\left(\left|\sum_{i=1}^{m}Y_i\right|\geqq\frac{1}{2}n^{1/2}\sigma(\log n)^{a''/a}\right)+2m\beta(q)\end{aligned}$$

where $Y_1, \cdots, Y_m$ are independent and identically distributed random variables, each Y_i having the same d.f. as that of $\eta_1/(\sqrt{p}\sigma)$. As $a''/a>1$ and

$$E|Y_i|^4=E\left|\frac{\eta_1}{\sqrt{p}\sigma}\right|^4\leqq K,$$

so from Theorem 17.11 in [1]

$$P\left(|\sum_{i=1}^{m} Y_i| \geqq \frac{1}{2} n^{1/2}\sigma(\log n)^{a''/a}\right) = o(n^{-2/5}) \tag{3.20}$$

Hence, from (3.17)-(3.20) we have

$$P\left(\sup_{0\leqq t\leqq 1}\left|f\left(t, \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\right) - f\left(\frac{[mt]}{m}, \sum_{i=1}^{[mt]}\frac{\eta_i}{\sqrt{n}\sigma}\right)\right| \geqq \varepsilon_n\right) = o(n^{-1/5}) \tag{3.21}$$

Thus, from (3.21)

$$\begin{aligned} &P\left(\int_0^1 f\left(t, \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\right)dt \leqq z\right) \\ &\leqq P\left(\int_0^1 f\left(\frac{[mt]}{m}, \sum_{i=1}^{[mt]}\frac{\eta_i}{\sqrt{n}\sigma}\right)dt \leqq z + \varepsilon_n\right) \\ &\quad + P\left(\sup_{0\leqq t\leqq 1}\left|f\left(t, \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\right) - f\left(\frac{[mt]}{m}, \sum_{i=1}^{[mt]}\frac{\eta_i}{\sqrt{n}\sigma}\right)\right| \geqq \varepsilon_n\right) \\ &= P\left(\frac{1}{m}\sum_{j=1}^{m} f\left(\frac{j}{m}, \sum_{i=1}^{j}\frac{\eta_i}{\sqrt{n}\sigma}\right) \leqq z + \varepsilon_n\right) + o(n^{-1/5}) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} &P\left(\int_0^1 f\left(t, \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\right)dt \leqq z\right) \\ &\geqq P\left(\int_0^1 f\left(\frac{[mt]}{m}, \sum_{i=1}^{[mt]}\frac{\eta_i}{\sqrt{n}\sigma}\right)dt \leqq z - \varepsilon_n\right) \\ &\quad - P\left(\sup_{0\leqq t\leqq 1}\left|f\left(t, \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\right) - f\left(\frac{[mt]}{m}, \sum_{i=1}^{[mt]}\frac{\eta_i}{\sqrt{n}\sigma}\right)\right| \geqq \varepsilon_n\right) \\ &= P\left(\frac{1}{m}\sum_{j=1}^{m} f\left(\frac{j}{m}, \sum_{i=1}^{j}\frac{\eta_i}{\sqrt{n}\sigma}\right) \leqq z - \varepsilon_n\right) - o(n^{-1/5}) \end{aligned} \tag{3.23}$$

Since from Theorem 2.1 in [9], there exists a sequence of nonnegative, i.i.d. random variables $\tau_1, \cdots, \tau_m$ satisfying (3.14) and

$$\begin{aligned} &\left|P\left(\frac{1}{m}\sum_{j=1}^{m} f\left(\frac{j}{m}, \sum_{i=1}^{j}\frac{\eta_i}{\sqrt{n}\sigma}\right) \leqq z\right) - P\left(\frac{1}{m}\sum_{j=1}^{m} f\left(\frac{j}{m}, w\left(\sum_{i=1}^{j}\tau_i\right)\right) \leqq z\right)\right| \\ &\leqq 2m\beta(q) = o(n^{-3/5}), \end{aligned}$$

so from (3.22)-(3.24) we have the desired conclusion.

4. Proof of Theorem. Let n be fixed. We used the notations in the preceding section. Since

$$\begin{aligned} &P\left(\int_0^1 f\left(t, \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\right)dt \leqq z - 2\varepsilon_n\right) - P\left(\sup_{0\leqq t\leqq 1}\left|f(t, X_n(t)) - f\left(t, \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\right)\right| \geqq 2\varepsilon_n\right) \\ &\leqq P\left(\int_0^1 f(t, X_n(t))dt \leqq z\right) \\ &\leqq P\left(\int_0^1 f\left(t, \frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\right)dt \leqq z + 2\varepsilon_n\right) + P\left(\sup_{0\leqq t\leqq 1}\left|f(t, X_n(t)) - f\left(t, \left(\frac{\hat{S}_n(nt)}{\sqrt{n}\sigma}\right)\right| \geqq 2\varepsilon_n\right) \end{aligned} \tag{4.1}$$

so, from Lemmas 3. 1-3. 3, to prove Theorem, it is enough to show that

$$\sup_z \left| P\left(\frac{1}{m}\sum_{i=1}^{j} f\left(\frac{j}{m},\ w(\sum_{i=1}^{j}\tau_i)\right)\leqq z\right) - P\left(\int_0^1 f(t, w(t))dt \leqq z\right) \right. \tag{4.2}$$
$$= O(n^{-1/5}(\log n)^{1+\alpha'}) .$$

Thus, as in [5], it suffices to prove

$$P\left(\left|\frac{1}{m}\sum_{j=1}^{m} f\left(\frac{j}{m},\ w(\sum_{i=1}^{j}\tau_i)\right) - \int_0^1 f(t, w(t))\,dt\right| \geqq 6\varepsilon_n\right) \tag{4.3}$$
$$= O(n^{-1/5}(\log n)^{1+\alpha'})$$

But, $\tau_1, \cdots, \tau_m$ are independent and identically distributed,

$$E\tau_1 = \frac{1}{n\sigma^2}E\eta_1 = \frac{1}{n}p(1+O(p^{-1})) = m^{-1} + n^{-1}O(1) \tag{4.4}$$

and

$$\mathrm{Var}\,(\tau_1) \leqq E\tau_1^2 \leqq Kn^{-2}p^2 = O(n^{-4/5}) \tag{4.5}$$

(cf. the proof of Lemma 4.3 in [9]). So the Sawyer's method used in the proof of Equation (1.5) in [5] can be completely carried over to this case and we have (4.4). Hence, the proof is obtained.

5. Remark. As is [5], we have the following

Corollary. *Under the conditions of Theorem*

$$E\left[g\left(\int_0^1 f(t, X_n(t))dt\right)\right] = E\left[g\left(\int_0^1 f(t, w(t))dt\right)\right] + O(n^{-1/8}(\log n)^{1+\alpha'})$$

where $g(y)$ is an arbitrary continuous function on R with

$$\int |g'(y)|dy < \infty .$$

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