ALMOST SURE INVARIANCE PRINCIPLES FOR ABSOLUTELY REGULAR PROCESSES

By

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(Received December 16, 1976)

1. Introduction. Let $\{\xi_j\}$ be a strictly stationary sequence of random variables which are defined on a probability space $(\Omega, \mathscr{A}, P)$. For $a \leqq b$, *let* $\mathscr{M}_a^b$ denote the σ- algebra of events generated by $\xi_a, \cdots, \xi_b$. As in [7]-[10], we shall say that the sequence is absolutely regular if

$$\beta(n) = E\{\sup_{A \in \mathscr{M}_n^\infty} |P(A|\mathscr{M}_{-\infty}^0) - P(A)|\} \downarrow 0 \qquad (n \to \infty) \tag{1.1}$$

For any T $(0 < T \leqq \infty)$, let $C_T = [0, T]$ be the space of all continuous functions on $[0, T]$. We give the uniform topology by defining the distance between two points x and y in C_T as

$$\rho_T(x, y) = \sup_{0 \leqq t \leqq T} |x(t) - y(t)| . \tag{1.2}$$

Let $S_k = \sum_{j=1}^k \xi_j$ and $S_0 = 0$, and define a random element $\bar{S}_T = \{S_T(t) : 0 \leqq t \leqq T\}$ in C_T by

$$S_T(t) = \begin{cases} S_k & \text{for } t = k,\ k = 0, 1, \cdots, [T] \\ S_{[T]} & \text{for } [T] \leqq t \leqq T , \\ \text{linearly interpolated } t \in [k-1, k),\ k = 1, \cdots, [T] \end{cases} \tag{1.3}$$

where $[s]$ denotes that largest integer m such that $m \leqq s$. Put $S(t) = S_\infty(t)$ for all $t \geqq 0$.

In [6], Strassen proved the almost sure invariance principle for martingales and in [3], Jain, Jogdeo and Stout extended the Strassen's results. On the other hand, in [5], Philipp and Stout showed the almost sure invariance principle for mixing sequences.

In this paper, we shall prove the almost sure invariance principle for absolutely regular processes. In particular, we shall prove the following

Theorem 1. *Assume that* $\{\xi_i\}$ *is a strictly stationary, absolutely regular process which satisfies the following conditions;*

$$\text{(i)} \quad E\xi_i = 0,\ E|\xi_i|^{4+\delta} < \infty \quad \textit{for some } \delta > 0, \textit{ and} \tag{1.4}$$

$$\text{(ii)} \quad \beta(n) = O(e^{-\gamma n}) \quad \textit{for some } \gamma > 0 , \tag{1.5}$$

Then, redefining $\{S(t), t \geqq 0\}$, if necessary, on a new probability space, there exists a standard Wiener process $w = \{w(t): 0 \leqq t < \infty\}$ such that

$$|S(t) - \sigma w(t)| = o(t^{1/2}(\log\log t)^{(1+\alpha)/2}) \quad a.s. \tag{1.6}$$

as $t \to \infty$; where α is an arbitrary positive number and

$$\sigma^2 = E\xi_0^2 + 2\sum_{j=1}^{\infty} E\xi_0\xi_j > 0 . \tag{1.7}$$

Remark. It is known that under (i) and (ii) the series in (1.7) converges absolutely, (cf. [9]).

2. **Auxiliary results.** In what follows, we shall agree to denote by the letter K, with or without indices, various positive constants. The followings were proved in [8], [9] and [10] under the conditions of Theorem 1; for all $n \geqq 1$

(2.1) (I) $\quad |Var\,(S_n) - n\sigma^2| \leqq K_1$;

(2.2) (II) $\quad ES_n^4 \leqq K_2 n^2$;

(2.3) (III) $\quad P(|S_n| \geqq 2a\sigma_0 n^{1/2} \log n) \leqq K_3 a^{-4} n^{-1}$

where $\sigma_0^2 = Var\,(\xi_1) > 0$ and a is an arbitrary positive constant;

(2.4) (IV)
$$P(\max_{1 \leqq j \leqq n} |S_j| \geqq \lambda) \leqq 2P\left(|S_n| > \frac{\lambda}{2}\right) + 2nr^{-1}\beta(r)$$
$$+ 2([nr^{-1}]+1)P\left(|\xi_1| + \cdots + |\xi_{2r}| \geqq \frac{\lambda}{4}\right)$$

where r is an arbitrary positive integer such that $r < n$;

(V) If $\eta_1, \eta_2, \cdots, \eta_m$ are absolutely regular with $\beta_1(j) = \beta(jq)$, then

$$P(\max_{1 \leqq j \leqq m} |\sum_{i=1}^{j} \eta_i| \geqq z) \leqq P(\max_{1 \leqq j \leqq m} |\sum_{i=1}^{j} X_i| \geqq z) + m\beta(q) \tag{2.5}$$

where, for each $i (1 \leqq i \leqq m)$, X_i has the same df as that of η_i (cf. Lemma 2.4 in [8]).

(2.6) (VI) $\quad \sup_z |P(S_n < \sigma z \sqrt{n}) - \Phi(z)| \leqq K n^{-1/4} (\log n)^{3/2}$

where

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt .$$

Now, we shall prove the following theorems concerning non-uniform estimates.

Theorem 2. *If the conditions of Theorem 1 hold, then for all n sufficiently large and for all x*

$$|P(S_n<n^{1/2}\sigma z)-\Phi(z)|\leqq\frac{Kn^{-1/4}(\log n)^3}{1+z^2}\,. \tag{2.7}$$

Proof. Since from (I)

$$E\,|S_n/(\sqrt{n}\,\sigma)|^2=1+O(n^{-1})\,,$$

so

$$\lambda_0=\left|E\,|S_n/(\sqrt{n}\,\sigma)|^2-\int_{-\infty}^{\infty}x^2d\Phi(x)\right|=O(n^{-1})\,.$$

Hence, from Theorem 9, Chap. 5 in [4] and (VI) it follows that

$$\begin{aligned}&|P(S_n<n^{1/2}\sigma z)-\Phi(z)|\\&\quad\leqq\frac{Kn^{-1/4}(\log n)^3+\lambda_0}{1+z^2}\leqq\frac{Kn^{-1/4}(\log n)^3}{1+z^2}\,,\end{aligned}$$

and the proof is completed.

Let $s_n^2=E\,|S_n|^2$. Let $F_n(u)$ be the df of $S_n/(\sqrt{n}\,\sigma)$ and $\Phi_1(u)$ be the normal df with the same first two moments as $F_n(u)$. Set

$$X=F_n^{-1}(Z)\,,\qquad Y=\Phi_1^{-1}(Z) \tag{2.8}$$

where F^{-1} is the function inverse to F and Z is a random variable uniformly distributed in $[0,1]$. It is well known that X is distibuted like $S_n/\sqrt{n}\,\sigma$ and Y is normally distributed random variables with the same first two moments as X, i.e.,

$$EY=0\,,\qquad EY^2=s_n^2/(n\sigma^2)\,.$$

Theorem 3. *If the conditions of Theorem* 1 *hold, then*

$$E\,|X-Y|\leqq\int_{-\infty}^{\infty}|F_n(u)-\Phi_1(u)|\,du \tag{2.9}$$

for all n sufficiently large, where X and Y are random variables defined by (2.8).

Proof. Let a be the median of $S_n/\sqrt{n}\,\sigma$. Then, the median of $X-a$ is zero and $Y-a$ is normally distributed with parameters $(-a, s_n^2/n\sigma^2)$. Thus, applying Lemma 1 in [2] to the random variable $(X-a)-(Y-a)$

$$\begin{aligned}E\,|X-Y|&=E\,|(X-a)-(Y-a)|\\&\leqq\int_{-\infty}^{\infty}|F_n^{(a)}(u)-\Phi_1^{(a)}(u)|\,du\end{aligned} \tag{2.10}$$

where

$$F_n^{(a)}(u)=P(X-a\leqq u)=F(u+a)$$

and

$$\Phi_1^{(a)}(u)=P(Y-a\leqq u)=\Phi_1(u+a)\ .$$

Hence, (2.9) follows from (2.10) and the proof is completed.

3. Proof of Theorem 1. To prove Theorem 1, we need some lemmas.

Lemma 1. *Let the conditions of Theorem 1 be satisfied. For any fixed n, put $r=[n^{2/3}(\log n)^6]$ and $m=[nr^{-1}]$. Let $\tilde{S}_n=\{\tilde{S}_n(t): 0\leqq t\leqq n\}$ be the random element in C_n defined by*

$$\tilde{S}_n(t)=\begin{cases} S_{jr} & \text{for } t=jr,\ j=0,1,\cdots,m,\\ S_{mr} & \text{for } mr\leqq t\leqq n,\\ \text{linearly interpolated for } t\in[(j-1)r, jr),\ j=1,\cdots,m\ . \end{cases} \tag{3.1}$$

Then, for any $\varepsilon>0$

$$P(\rho_n(\bar{S}_n, \tilde{S}_n)\geqq\varepsilon z_n)=o(n^{-2/3}) \tag{3.2}$$

where $z_n=n^{1/2}(\log\log n)^{(1+\alpha)/2}$.

Proof. As in the proof of Lemma 2 in [1], we have that

$$\begin{aligned}\rho_n(\bar{S}_n, \tilde{S}_n)&=\sup_{0\leq t\leq n}|S_n(t)-\tilde{S}_n(t)|\\ &\leqq 2\max_{1\leq i\leq m}\ \max_{(i-1)r\leq j\leq ir}|S_j-S_{(i-1)r}|+\max_{mr\leq j\leq n}|S_j-S_{mr}|\ .\end{aligned}$$

So, from (IV)

$$\begin{aligned}&P(\rho_n(\bar{S}_n, \tilde{S}_n)\geqq\varepsilon z_n) \\ &\leqq\sum_{i=1}^{m}P\Big(\max_{(i-1)r\leq j\leq ir}|S_j-S_{(i-1)r}|\geqq\frac{\varepsilon}{4}z_n\Big)+P\Big(\max_{mr\leq j\leq n}|S_j-S_{mr}|\geqq\frac{\varepsilon}{2}z_n\Big)\\ &\leqq(m+1)P\Big(\max_{0\leq j\leq r}|S_j|\geqq\frac{\varepsilon}{4}z_n\Big)\\ &\leqq(m+1)\Big\{2P\Big(|S_r|\geqq\frac{\varepsilon}{8}z_n\Big)+2rq^{-1}\beta(q)+2(rq^{-1}+1)P\Big(|\xi_1|+\cdots+|\xi_{2q}|\geqq\frac{\varepsilon}{16}z_n\Big)\Big\}\ .\end{aligned} \tag{3.3}$$

where $q=[c\log n]$ $(c\gamma>2)$.
Firstly, we note that by (1.5)

$$(m+1)rq^{-1}\beta(q)=o(n^{-1}) \tag{3.4}$$

and from (III)

$$mP\Big(|S_r|\geqq\frac{\varepsilon}{8}z_n\Big)=o(n^{-1})\ . \tag{3.5}$$

Secondly, let

$$\bar{\xi}_1=\begin{cases}\xi_1 & \text{if } |\xi_1|\leqq m\ ,\\ 0 & \text{otherwise,}\end{cases}$$

and $\bar{\bar{\xi}}_1=\xi_1-\bar{\xi}_1$. Then, we have

$$P\Big(|\bar{\xi}_1|+\cdots+|\bar{\xi}_{2q}|\geqq\frac{\varepsilon}{16}z_n\Big)=0$$

for all n sufficiently large and so

$$\begin{aligned}(3.6)\qquad mrq^{-1}P\Big(|\xi_1|+\cdots+|\xi_{2q}|\geqq\frac{\varepsilon}{8}z_n\Big)&\leqq mrq^{-1}P\Big(|\bar{\bar{\xi}}_1|+\cdots+|\bar{\bar{\xi}}_{2q}|\geqq\frac{\varepsilon}{16}z_n\Big)\\ &\leqq Kmrq^{-1}z_n^{-2}E(\sum_{j=1}^{2q}|\bar{\bar{\xi}}_j|)^2\leqq Emrq\,z_n^{-2}E\,|\bar{\bar{\xi}}_1|^2\\ &\leqq Kmrq\,z_n^{-2}m^{-(2+\delta)}E\,|\bar{\bar{\xi}}_1|^{4+\delta}=o(n^{-2/3})\ .\end{aligned}$$

Hence from (3.3)-(3.6) we have (3.2) and the proof is completed.

Lemma 2. *Let the conditions of Lemma* 1 *be satisfied. Then, if necessary, on a new probability space, there exists a standard Wiener process* $w=\{w(t): 0\leqq t<\infty\}$ *such that for all* n *sufficiently large*

$$P(\rho_n(\tilde{S}_n, \sigma\tilde{w}_n)\geqq\varepsilon z_n)=O((\log n)^{-3}) \tag{3.7}$$

where $\tilde{w}_n=\{\tilde{w}_n(t): 0\leqq t\leqq n\}$ *is the radom element in* C_n *defined by*

$$\tilde{w}_n(t)=\begin{cases}w(jr) & \textit{for}\quad t=jr,\ j=0,1,\cdots,m\ ,\\ w(mr) & \textit{for}\quad mr\leqq t\leqq n\ ,\\ \textit{linearly interpolated for } t\in[(j-1)r, jr),\ j=1,\cdots,m\ ,\end{cases} \tag{3.8}$$

and, z_n*,* m *and* r *are integers defined in Lemma* 1.

Proof. Put

$$\eta_j=S_{jr}-S_{(j-1)r+q}\ (j=1,\cdots,m)\ ,\qquad \eta_{m+1}=S_n-S_{mr}\ ; \tag{3.9}$$

and

$$\eta_j^*=\frac{\sqrt{r}\,\sigma}{s_{r-q}}\eta_j\qquad (j=1,\cdots,m)\ . \tag{3.10}$$

It is clear that $\{\eta_j^*, 1\leqq j\leqq m\}$ is an absolutely regular sequence, each η_j^* $(j=2,\cdots,m)$ having the same df as that of η_1^*.

$$\tilde{S}_n^*(t)=\begin{cases}\sum_{i=1}^{j}\eta_i^* & \text{for}\quad t=jr,\ j=0,1,\cdots,m\ ,\\ \sum_{i=1}^{m}\eta_i^* & \text{for}\quad mr\leqq t\leqq n\ ,\\ \text{linearly interpolated for } t\in[(j-1)r, jr),\ j=1,\cdots,m\ .\end{cases} \tag{3.11}$$

Since

$$\rho_n(\tilde{S}_n, \tilde{S}_n^*) \leqq \max_{1\leqq j\leqq m} |\sum_{i=1}^{j} (S_{(i-1)r+q} - S_{(i-1)r})| + \max_{1\leqq j\leqq m} |\sum_{i=1}^{j} (\eta_i - \eta_i^*)| + |\gamma_{m+1}| ,$$

and from (I)

$$\begin{aligned} |\sum_{i=1}^{j} (\eta_i - \eta_i^*)| &= \left|1 - \frac{s_{r-q}}{\sqrt{r}\,\sigma}\right| |\sum_{i=1}^{j} \eta_i^*| \leqq \frac{|s_{r-q}^2 - r\sigma^2|}{r\sigma^2} |\sum_{i=1}^{j} \eta_i^*| \\ &\leqq K r^{-1} q |\sum_{i=1}^{j} \eta_i^*| , \end{aligned}$$

so for any $\varepsilon>0$

$$\begin{aligned} (3.12) \qquad & P(\rho_n(\tilde{S}_n, \tilde{S}_n^*) \geqq \varepsilon z_n) \\ & \leqq P\left(\max_{1\leqq j\leqq m} K |\sum_{i=1}^{j} \eta_i^*| \geqq \frac{\varepsilon}{2} z_n r q^{-1}\right) + P\left(|\eta_{m+1}| \geqq \frac{\varepsilon}{4} z_n\right) \\ & \quad + P\left(\max_{2\leqq j\leqq m+1} |\sum_{i=1}^{j-1} (S_{ir+q} - S_{ir})| \geqq \frac{\varepsilon}{4} z_n\right) \end{aligned}$$

We note here that the followings hold:

$$P\left(|\eta_{m+1}| \geqq \frac{\varepsilon}{4} z_n\right) = o(n^{-1}) \qquad \text{(by (III))} ,$$

$$\begin{aligned} (3.13) \qquad & P\left(\max_{2\leqq j\leqq m+1} |\sum_{i=0}^{j-1} (S_{ir+q} - S_{ir})| \geqq \frac{\varepsilon}{4} z_n\right) \leqq P\left(\sum_{i=1}^{m} |S_{ir+q} - S_{ir}| \geqq \frac{\varepsilon}{4} z_n\right) \\ & \leqq K m z_n^{-1} E|S_q| \leqq K m z_n^{-1} q^{1/2} = o(n^{-1/6}) , \\ & P\left(\max_{1\leqq j\leqq m} K |\sum_{i=1}^{j} \eta_i^*| \geqq \frac{\varepsilon}{2} z_n r q^{-1}\right) \leqq P\left(K \sum_{j=1}^{m} |\eta_j^*| \geqq \frac{\varepsilon}{2} z_n r q^{-1}\right) \\ & \leqq K (z_n r)^{-1} q \sum_{j=1}^{m} E|\eta_j^*| \leqq K (z_n r)^{-1} q \sum_{j=1}^{m} \{E|\eta_j^*|^2\}^{1/2} = o(n^{-1/8}) . \end{aligned}$$

Hence, by (3.12) and (3.13) we have

$$(3.14) \qquad P(\rho_n(\tilde{S}_n, \tilde{S}_n^*) \geqq \varepsilon z_n) = o(n^{-1/6})$$

Now, let F_* be the df of $\eta_1^*/\sqrt{r}\,\sigma$ and $\Phi(u)$ the normal df with the same first two moments as $F_*(u)$, i.e. the normal df N(0, 1). Consider

$$\sqrt{r}\,\sigma\{F_*^{-1}(Z) - \Phi^{-1}(Z)\}$$

where Z is a random variable uniformly distributed in [0, 1]. We choose normally and identically distributed random variables $\{\zeta_1, \cdots, \zeta_m\}$ having the following properties;

(a) each $\zeta_j/(\sqrt{r})(j=1, \cdots, m)$ has the normal df $\Phi^{-1}(Z)$,

(b) $\eta_j^* - \sigma\zeta_j (j=1, \cdots, m)$ satisfy the absolute regularity condition with $\beta_1(j) = \beta(jq)$,

(c) for each $j (j=1, \cdots, m)$, $\eta_j^* - \sigma\zeta_j$ has the same df as that of

$$\sqrt{r}\,\sigma\{F_*^{-1}(Z) - \Phi^{-1}(Z)\}\ , \tag{3.15}$$

Define $\tilde{X}_n = \{\tilde{X}_n(t) : 0 \leqq t \leqq n\}$ in C_n

$$\tilde{X}_n(t) = \begin{cases} \sum_{i=1}^{j} \sigma\zeta_i & \text{for } t = jr,\ j = 0, 1, \cdots, m\ , \\ \sum_{i=1}^{m} \sigma\zeta_i & \text{for } mr \leqq t \leqq n\ , \\ \text{linearly interpolated for } t \in [(j-1)r, jr),\ j = 1, \cdots, m\ . \end{cases} \tag{3.16}$$

Using the method of the proof of (IV)

$$P\Big(\max_{1 \leqq j \leqq m} |\sum_{i=1}^{j} (\eta_i^* - \sigma\zeta_i)| \geqq \frac{1}{2}\varepsilon z_n\Big) \leqq 2P\Big(|\sum_{i=1}^{m} (\eta_i^* - \sigma\zeta_i)| \geqq \frac{1}{4}\varepsilon z_n\Big) + m\beta(q)\ . \tag{3.17}$$

Since by construction random variables $\eta_i^* - \sigma\zeta_i (i=1, \cdots, m)$ satisfy the absolute regularity condition coefficient $\beta_1(j) = \beta(jq)$, so it follows from Theorem 6.2 in [8] that

$$\begin{aligned} &P\Big(|\sum_{i=1}^{m} (\eta_i^* - \sigma\zeta_i)| \geqq \frac{1}{8}\varepsilon z_n\Big) \\ &\qquad \leqq \sum_{i=1}^{m} P(|\eta_i^* - \sigma\zeta_i| \geqq n^{1/2}(\log n)^{-1}) + Kq\Big(\frac{qB_1}{z_n}\Big)^2 + 4m\beta(q) \end{aligned} \tag{3.18}$$

where

$$B_1 = \sum_{i=1}^{m} E\,|\eta_i^* - \sigma\zeta_i| \tag{3.19}$$

and $q = [c \log n]$ $(c\gamma > 1)$.

It follows from (III) that

$$\begin{aligned} &P(|\eta_i^* - \sigma\zeta_i| \geqq n^{1/2}(\log n)^{-1}) \\ &\qquad \leqq P\Big(|\eta_i^*| \geqq \frac{1}{2} n^{1/2}(\log n)^{-1}\Big) + P\Big(\sigma|\zeta_i| \geqq \frac{1}{2} n^{1/2}(\log n)^{-1}\Big) = o(n^{-2/8})\ . \end{aligned} \tag{3.20}$$

On the other hand, from Theorems 2 and 3 it follows that for all n sufficiently large and for each $j (j=1, \cdots, m)$

$$\begin{aligned} E\,|\eta_i^* - \sigma\zeta_i| &\leqq r^{1/2}\sigma \int_{-\infty}^{\infty} |F_*(u) - \Phi(u)|\, du \\ &\leqq r^{1/2}\sigma \int_{-\infty}^{\infty} \frac{Kr^{-1/4}(\log r)^3}{1+u^2} du = O(r^{1/4}(\log r)^3) = O(n^{1/6}(\log n)^3) \end{aligned}$$

and so

$$qB_1 = q\sum_{i=1}^{m} E\,|\eta_i^* - \sigma\zeta_i| = qmO(n^{1/6}(\log n)^3) = O(n^{1/2}(\log n)^{-2})\,. \tag{3.21}$$

Hence, from (3.18)-(3.21)

$$\begin{aligned} &P(\rho_n(\tilde{S}_n^*, \tilde{X}_n) \geqq \varepsilon z_n) \\ &\qquad \leqq P\Big(\max_{1\leqq j\leqq m} |\sum_{i=1}^{j} (\eta_i^* - \sigma\zeta_i)| \geqq \frac{1}{2}\varepsilon z_n\Big) + m\beta(p) = O((\log n)^{-3})\,. \end{aligned} \tag{3.22}$$

Next, let $\{X_1, \cdots, X_m\}$ and $\{Y_1, \cdots, Y_m\}$ be two independent collections of random variables which are independently and identically distributed with df $N(0, r)$. Define a random element $\tilde{Y}_n = \{\tilde{Y}_n(t) : 0 \leqq t \leqq n\}$ in C_n by

$$\tilde{Y}_n(t) = \begin{cases} \sigma \sum_{i=1}^{j} Y_i & \text{for } t = jr,\ j = 0, 1, 2, \cdots, m\,, \\ \sigma \sum_{i=1}^{m} Y_i & \text{for } mr \leqq t \leqq n\,, \\ \text{linearly interpolated for } t \in [(j-1)r, jr),\ j = 1, \cdots, m\,, \end{cases} \tag{3.23}$$

Since $\zeta_i - Y_i (i=1, \cdots, m)$ are absolutely regular with $\beta_1(j) = \beta(jq)$, so from (V)

$$\begin{aligned} P(\rho_n(\tilde{X}_n, \tilde{Y}_n) \geqq \varepsilon z_n) &= P\Big(\max_{1\leqq j\leqq m} |\sum_{i=1}^{j} (\zeta_i - Y_i)| \geqq \frac{\varepsilon}{\sigma} z_n\Big) \\ &\leqq P\Big(\max_{1\leqq j\leqq m} |\sum_{i=1}^{j} (X_i - Y_i)| \geqq \frac{\varepsilon}{\sigma} z_n\Big) + m\beta(q) \end{aligned} \tag{3.24}$$

We note that, by construction, the random variables $X_i - Y_i (i=1, \cdots, m)$ are independently and identically distributed with the normal df $N(0, 2r)$ and so $\sum_{i=1}^{m} (X_i - Y_i)$ is normally distributed with df $N(0, 2mr)$. Hence

$$\begin{aligned} &P\Big(\max_{1\leqq j\leqq m} |\sum_{i=1}^{j} (X_i - Y_i)| \geqq \frac{\varepsilon}{\sigma} z_n\Big) \\ &\quad \leqq 2P\Big(|\sum_{i=1}^{m} (X_i - Y_i)| \geqq \frac{\varepsilon}{\sigma} z_n\Big) \\ &\quad = 2\int_{|u| \geqq (\varepsilon/\sigma) z_n} \frac{1}{2\sqrt{\pi m r}} \exp\Big(-\frac{u^2}{8mr}\Big) du \\ &\quad = o((\log n)^{-3}) \end{aligned} \tag{3.25}$$

for all n sufficiently large. As $m\beta(q) = o(n^{-2/3})$, so from (3.24) and (3.25)

$$P(\rho_n(\tilde{X}_n, \tilde{Y}_n) \geqq \varepsilon z_n) = o((\log n)^{-3})\,. \tag{3.26}$$

Since, $Y_1, \cdots, Y_m$ are independently and identically distributed with df $N(0, r)$, so, if necessary, on a new probability space, there exists a standard Wiener process $w=\{w(t): 0\leqq t<\infty\}$ such that

$$\sum_{i=1}^{j}\zeta_i=\sum_{i=1}^{j}(\tilde{w}(ri)-\tilde{w}((r-1)i))=\tilde{w}(rj) \qquad (j=1,\cdots,m) \tag{3.27}$$

with probability one where

$$\tilde{w}(rj)=w(rj) \qquad (j=1,\cdots,m) \tag{3.28}$$

Define the random element $\tilde{w}_n=\{\tilde{w}_n(t): 0\leqq t\leqq n\}$ in C_n by

$$\tilde{w}_n(t)=\begin{cases}\tilde{w}(jr) & \text{for} \quad t=jr,\ j=0,1,\cdots,m\ ,\\ \tilde{w}(mr) & \text{for} \quad mr\leqq t\leqq n\ ,\\ \text{linearly interpolated for } t\in[(j-1)r, jr) & (j=1,\cdots,m)\ .\end{cases} \tag{3.29}$$

Then, $\tilde{Y}_n=\sigma\tilde{w}_n$ with probability one, and so

$$P(\rho_n(\tilde{Y}_n, \sigma\tilde{w}_n)\varepsilon\geqq z_n)=o((\log n)^{-8})\ . \tag{3.30}$$

Hence, the conclusion of Lemma 2 follows from (3.14), (3.22), (3.26) and (3.30).

Lemma 3. *Let m and r be integers defined in Lemma 1. Then for any $\varepsilon>0$ and all n sufficiently large*

$$P(\rho_n(\tilde{w}_n, w)\geqq\varepsilon z_n)=o(n^{-2/8}) \tag{3.31}$$

where $z_n=n^{1/2}(\log\log n)^{(1+\alpha)/2}$.

Proof. We use the method of the proof of Lemma 2 in [1]. For each $j(1\leqq j\leqq m)$, let

$$\theta_j=\max_{(j-1)r\leqq t\leqq jr}|w(t)-\tilde{w}_n(t)|\leqq 2\max_{(j-1)r\leqq t\leqq jr}|w(t)-w((j-1)r)|\ .$$

It follows that for all n sufficiently and $j(1\leqq j\leqq m)$

$$\begin{aligned}P(\theta_j\geqq\varepsilon z_n)&\leqq P\Big(\max_{0\leqq t\leqq r}|w(t)|\geqq\frac{\varepsilon}{2}z_n\Big)\\ &\leqq\sqrt{\frac{2}{\pi}}(1-\Phi(n^{1/7}))=o(n^{-4/8})\ .\end{aligned}$$

Since $\rho_n(\tilde{w}_n, w)=\max_{1\leqq j\leqq m}\theta_j$,

$$P(\rho_n(\tilde{w}_n, w)\geqq\varepsilon z_n)\leqq mo(n^{-4/8})=o(n^{-1})\ .$$

Thus, the proof is completed.

From Lemma 1, 2 and 3, we have the following

Lemma 4. *Let the conditions of Theorem 1 be satisfied. Then redefining $\{S(t), t\geqq 0\}$, if necessary, on a new probability space, there exists a standard*

Wiener process $w=\{w(t): 0\leqq t<\infty\}$ *such that for each* n *sufficiently large and any* $\varepsilon>0$

$$P(\rho_n(\bar{S}, \sigma w)\geqq \varepsilon z_n)=O((\log n)^{-3}) \tag{3.32}$$

where $z_n=n^{1/2}(\log\log n)^{(1+\alpha)/2}$.

Now, we proceed to prove Theorem 1. Let $n_k=2^k$, $k=1, 2, \cdots$. To prove Theorem 1, it is enough to show that for the sequeuce $\{n_k\}$

$$P(\rho_{n_k}(\bar{S}, \sigma w)\geqq \varepsilon z_{n_k} \quad \text{i.o.})=0 \tag{3.33}$$

where $\bar{S}$, w and z_n are the ones in Lemma 4. But, from Lemma 4, we have

$$\sum_{k=1}^{\infty} P(\rho_{n_k}(\bar{S}, \sigma w)\geqq \varepsilon z_{n_k})\leqq K\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty .$$

Thus, from Borel-Cantelli's lemma we have (3.33), which completes the proof of Theorem 1.

4. Integral tests. As in [3] using Theorem 1, we have the following theorems concerning integral tests.

Theorem 4. *Let the conditions of Theorem* 1 *be satisfied. Let* $\varphi>0$ *be a non-decreasing function. Then, we have the followings*: (a)

$$P(S_n>n^{1/2}\sigma\varphi(n) \quad \text{i.o.})=0 \quad or \quad 1 \tag{4.1}$$

according as

$$I(\varphi)=\int_1^{\infty}\frac{\varphi(t)}{t}\exp(-\varphi^2(t)/2)dt<\infty \quad or \quad =\infty . \tag{4.2}$$

(b)

$$P(\max_{1\leqq i\leqq n}|S_i|<n^{1/2}\sigma\{\varphi(n)\}^{-1} \quad \text{i.o.})=1 \quad or \quad 0 \tag{4.3}$$

according as

$$I_1(\varphi)=\int_1^{\infty}\frac{\varphi^2(u)}{u}\exp\{-8\varphi^2(u)/\pi^2\}du=\infty \quad or \quad <\infty . \tag{4.4}$$

The proofs of these theorems are analogous to Theorems 5.1 and 6.1 in [3] and so are omitted.

5. Almost sure convergence for U-statistics. In [8], the author proved the almost sure invariance principle for U-statistics generated by ϕ-mixing sequences. But, the results are easily extended by Theorem 1 and the methods of the proofs of Theorems 4 and 5 in [7] as follows: Let $F(x)$ be the df of ξ_i and consider a functional

$$\theta(F)=\int\cdots\int g(x_1,\cdots,x_m)dF(x_1)\cdots dF(x_m) \tag{5.1}$$

defined over $\mathscr{F}=\{F: |\theta(F)|<\infty\}$ where $g(x_1,\cdots,x_m)$ is symmetric in its $m(\geqq 1)$ arguments. As an estimator of $\theta(F)$, we define a U-statistic

$$U_n=\binom{n}{m}^{-1}\sum_{(i)}^{(n)} g(\xi_{i_1},\cdots,\xi_{i_m}) , \qquad n\geqq m \tag{5.2}$$

where the summation $\sum_{(i)}^{(n)}$ extends over all possible $1\leqq i_1<\cdots<i_m\leqq n$. As another estimator of $\theta(F)$, we shall consider von Mises' differentiable functional $\theta(F_n)$ defined by

$$\theta(F_n)=n^{-m}\sum_{i_1=1}^{n}\cdots\sum_{i_m=1}^{n} g(\xi_{i_1},\cdots,\xi_{i_m}) . \tag{5.3}$$

Finally, we define random processes $V=\{V(t); 0\leqq t<\infty\}$ and $V^*=\{V^*(t): 0\leqq t<\infty\}$, respectively, by

$$V(t)=\begin{cases}0 & \text{for } t=k,\ 0\leqq k\leqq m-1 ,\\ k[U_k-\theta(F)] & \text{for } t=k,\ k\geqq m ,\\ \text{linearly interpolated for } t\in[k,k+1],\ k\geqq 0 ,\end{cases} \tag{5.4}$$

and

$$V^*(t)=\begin{cases}k[\theta(F_k)-\theta(F)] & \text{for } t=k,\ k\geqq 0 ,\\ \text{linearly interpolated for } t\in[k,k+1],\ k\geqq 0 .\end{cases} \tag{5.5}$$

Theorem 5. *Let $\{\xi_n\}$ be a strictly stationary, absolutely regular process. Suppose that*

$$\int\cdots\int |g(x_1,\cdots,x_m)|^{4+\delta}dF(x_1)\cdots dF(x_m)\leqq M_0<\infty \tag{5.6}$$

and for all integers $i_1, i_2,\cdots, i_m (i_1<i_2<\cdots<i_m)$

$$E\,|g(\xi_{i_1},\xi_{i2},\cdots,\xi_{i_m})|^{4+\delta}\leqq M_0<\infty . \tag{5.7}$$

If the conditions of Theorem 1 are satisfied, then, upon redefining $\{V(t): 0\leqq t<\infty\}$ and $\{V^*(t): 0\leqq t<\infty\}$ respectively on a new probability space, if necessary, there exists a standard Wiener process $w=\{w(t): 0\leqq t<\infty\}$ such that as $t\to\infty$

$$|V(t)-m\sigma_1 w(t)|=o(t^{1/2}(\log\log t)^{(1+\alpha)/2}) \quad a.s., \tag{5.8}$$

$$|V^*(t)-m\sigma_1 w(t)|=o(t^{1/2}(\log\log t)^{(1\alpha+)/2}) \quad \text{a.s.} \tag{5.9}$$

and

(5.10) $$|V(t)-V^*(t)|=o(t^{1/2}(\log\log t)^{(1+\alpha)/2}) \quad a.s.$$

where $\alpha>0$ *is arbitrary,*

$$\sigma_1^2=\{Eg_1^2(\xi_1)-\theta^2(F)\}+2\sum_{k=1}^{\infty}\{Eg_1(\xi_1)g_1(\xi_{k+1})-\theta^2(F)\},$$

and

$$g_1(x_1)=\int\cdots\int g(x_1, x_2, \cdots, x_m)dF(x_2)\cdots dF(x_m).$$

Theorem 6. *Let the conditions of Theorem* 5 *be satisfied. Let* φ *be any nondecreasing function such that* $0<\varphi\uparrow$. *Then the followings hold:* (a)

(5.11) $$P(V(n)>n^{1/2}\sigma_1\varphi(n) \text{ i.o.})=0(or\ 1)$$

and

(5.12) $$P(V^*(n)>n^{1/2}\sigma_1\varphi(n) \text{ i.o.})=0(or\ 1)$$

according as $I(\varphi)<\infty$ $(or=\infty)$.

(b)

(5.13) $$P(\max_{1\leq i\leq n}|V(i)|\leqq n^{1/2}\sigma_1\{\varphi(n)\}^{-1} \text{ i.o.})=0(or\ 1)$$

and

(5.14) $$P(\max_{1\leq i\leq n}|V^*(i)|\leqq n^{1/2}\sigma_1\{\varphi(n)\}^{-1} \text{ i.o.})=0(or\ 1)$$

according as $I_1(\varphi)<\infty(or=\infty)$.

Remark. If $\{\xi_n\}$ is a strictly stationary and ϕ-mixing sequence with $\sum_{n=1}^{\infty}\{\phi(n)\}^{1/2}<\infty$, and (5.6) and (5.7) hold, then the stronger conclusions than those of Theorem 5 are obtained. More specifically, instead of (5.8)-(5.10), the following relations hold:

(5.15) $$|V(t)-m\sigma_1 w(t)|=o(t^{1/2-\mu}) \quad \text{a.s.},$$

(5.16) $$|V^*(t)-m\sigma_1 w(t)|=o(t^{1/2-\mu}) \quad \text{a.s.},$$

and

(5.17) $$|V(t)-V^*(t)|=o(t^{1/2-\mu}) \quad \text{a.s.}$$

where μ is any number such that $\mu<(2+\delta)/(48+12\delta)$.

The proof of this fact is easily obtained by Theorem 4.1 in [5] and Lemmas 7 and 8 in [8].

Acnowledgement. The author is very grateful to the referee for his useful comments and suggestions.

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