# ON BALL COVERINGS FOR PRODUCTS OF MANIFOLDS

By

YASUYUKI TSUKUI

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**1. Introduction.** The concept of ball coverings was introduced in [2] to study compact connected $PL$ manifolds. Suppose $M^m$ is a compact connected $PL$ $m$-manifold and let $\mathscr{B}=\{B_i\}$ be a finite set a finite set of $m$-balls in $M$. Then $\mathscr{B}$ is called a ball covering of $M$, if (i) $\cup B_i=M$ and (ii) $B_i\cap B_j=\partial B_i\cap\partial B_j$ is an $(m-1)$-manifold (may not be connected) for $i\neq j$. Define that$b$ $(M)=\min.\{\#\mathscr{B};\ \mathscr{B}$ is a ball covering of $M\}$, where $\#\mathscr{B}$ means the number of elements of $\mathscr{B}$ [2]. We use other definitions and notations in [2], and throughout this paper all considerations are upon the piecewise linear point of view.

The purpose of this paper is to show a relation among $b(M)$, $b(N)$ and $b(M\times N)$, as follows. Also, by the same technique as the proof of Theorem 1, we obtain Theorem 2.

**Theorem 1.** *For any compact connected manifolds $M$ and $N$,*

$$b(M\times N)\leqq b(M)+b(N)-1\ .$$

**Theorem 2.** *For any connected compact manifold $V$ with nonempty boundary $\partial V$,*

$$b(D(V))\leqq b(V)+1\ ,$$

*where $D(V)$ is the double of $V$.*

Let $c(M)$ be the *Lusternik-Schnirelmann* category of a compact connected manifolold $M$ (that is, the minimum number of contractible sets in $M$ which cover $M$) [1]. Since a ball is contractible and $c(S^{p_1}\times S^{p_2}\times\cdots\times S^{p_k})\geqq k+1$, we have the following from Theorem 1.

**Corollary 1.** $b(S^{p_1}\times S^{p_2}\times\cdots\times S^{p_k})=k+1$, $(p_i\geqq 1)$.

Corollary 1 shows that [2, (2.7)] and [2, (2.11)] are not the best, since $S^1\times S^2\times\cdots\times S^k$ requires $(1/2)k(k+1)+1$ kind handles in its any handle decomposition and since $\{(S^1\times S^2\times\cdots\times S^k)-\text{int}\,B^{k(k+1)/2}\}$ does not have a spine of dimension less than $(1/2)k(k+1)-1$.

For integers $2\leqq m\leqq n+1$, $b(T^{m-2}\times S^{n-m+2})=b(S^1\times\cdots\times S^1\times S^{n-m+2})=(m-1)+1=m$, from Corollary 1. Hence,

**Corollary 2.** *For any integers $m$ and $n$, $2 \leqq m \leqq n+1$, there exists a closed connected $n$-manifold $M$ such that $b(M)=m$.*

**Remark 1.** Mr. Okawa also proved Corollary 1 by a non-standard handle decomposition of $S^{p_1} \times S^{p_2} \times \cdots \times S^{p_k}$.

**2.** Since $b(T^n)=b(S^1 \times S^1 \times \cdots \times S^1)=n+1$, and from following Lemma 3, there exists a connected compact bounded $n$-manifold $N$ with $b(N)=m$ for any $m$, $1 \leqq m \leqq n$.

**3.** Let $V$ be the compact contractible Mazur's 4-manifold, with boundary [3], then $b(V)=3$,

$$b(V \times I)=b(I^5)=1<3=b(V)+b(I)-1 \text{ , and}$$
$$b(D(V))=b(S^4)=2<4=b(V)+1 \text{ .}$$

But the author do not know if there exist closed connected manifolds $M$ and $N$ with $b(M \times N)<b(M)+b(N)-1$. If $b(M^m \times N^{5-m})=b(M)+b(N)-1$ for all closed $m$- and $(5-m)$-manifolds $M$ and$N$, respectively, $(m=3, 4)$, then 3- and 4-Poincare conjectures are true.

**2. Preliminaries.** In this section we will provide some lemmata to prove the theorems. It is sufficient to understand that a regular neighborhood $N(X; Y)$ of $X$ in $Y$ is a second derived neighborhood of $X$ in $Y$ for suitable triangulation of $Y$.

**Lemma 1.** *Suppose $\{B'_1, B'_2, \cdots, B'_n\}$ is a set of $m$-balls in a connected compact $m$-manifold $M$ such that (i) $\cup B'_i=M$ and (ii) $B'_i \cap B'_j=\partial B'_i \cap \partial B'_j$ for $i \neq j$. Then there exists a ball covering $\{B_1, B_2 \cdots, B_n\}$ of $M$ such that $B_i \cap B_j=\emptyset$ if $B'_i \cap B'_j=\emptyset$.*

**Proof.** Let $B_1=N(B'_1; M)$, $B_i=N(\mathrm{Cl}\,(B'_i-\bigcup_{k<i} B_k); \mathrm{Cl}\,(M-\bigcup_{k<i} B_k))$, $i=2, 3, \cdots, n-1$, and let $B_n=\mathrm{Cl}\,(M-\bigcup_{k<n} B_k)$. Hence $\{B_1, B_2, \cdots, B_n\}$ is obviously a ball covering of $M$. If $B'_i \cap B'_j=\emptyset$, $N(B'_i; M) \cap N(B'_j; M)=\emptyset$. Therfore, $B_i \cap B_j=\emptyset$.

Next lemma is a special case of the above.

**Lemma 2.** *Suppose $\mathscr{B}=\{B_i; i=1, 2, \cdots, k\}$ is a ball covering of a connected compact manifold $M$ and let $B^*_i=N(B_i; M)$ be a regular neighborhood of $B_i$ in $M$. Then $\mathscr{B}^*=\{B^*_j; j=1, 2, \cdots, k\}$ is a ball covering of $M$, where $B^*_j=\mathrm{Cl}\,(B_j-B^*_i)$ for $j \neq i$, $1 \leqq j \leqq k$.*

**Lemma 3.** *If $\{B_1, B_2, \cdots, B_k\}$ is a ball covering of a connected compact manifold $M$ of $b(M)=k$, then $M'=\mathrm{Cl}\,(M-B_1)=\bigcup_{1<i} B_i$ is a connected compact manifold with $b(M')=k-1$.*

**Proof.** Since $B_i \cap B_j = \partial B_i \cap \partial B_j (i \neq j)$ is a manifold of dimension=dim. $(M)-1$, $M'$ is a compact manifold. If $M'$ is not connected, there are two balls $B_i$ and $B_j$ so that $B_i \cap B_j = \emptyset$. This contradicts that $b(M)=k$ [2, (2.3)]. Suppose $b(M')< k-1$, then $b(M)=b(M' \cup B_1) \leqq (k-2)+1=k-1$. It is also a contradiction to $b(M)= k$. Hence, $M'$ is a connected compact manifold with $b(M')=k-1$.

**Lemma 4.** *Let $M$ be a connected compact manifold with $b(M)=k$. For any integer $n$, there exist $n$ ball coverings $\mathscr{B}_i=\{B_{i,j};\ j=1, 2, \cdots, k\}$ of $M$, $i= 1, 2, \cdots, n$, such that*

(*) $$B_{i,j} \cap B_{p,q} = \emptyset, \text{ if } i+j=p+q \text{ and } (i, j) \neq (p, q) .$$

**Proof.** We will prove the lemma by the induction on $b(M)$. If $b(M)=1$, $M$ is homeomorphic to a ball. Let $B_{i,1}=M$ and let $\mathscr{B}_i=\{B_{i,1}\}$, $i=1, 2, \cdots, n$. Hence $\mathscr{B}_i$'s satisfy the condition (*).

Suppose now that the lemma is true for any connected compact manifold $N$ with $b(N) \leqq k-1$, $k \geqq 2$, and assume that $M$ is a connected compact manifold with $b(M)=k$. Then there is a ball covering $\{C_1, C_2, \cdots, C_k\}$ of $M$. From lemma 3, $M'=\mathrm{Cl}\,(M-C_1)$ satisfies the inductive hypothesis. Hence there exist $n$ ball coverings $\mathscr{B}'_i=\{B'_{i,j};\ j=2, 3, \cdots, k\}$, $i=1, 2, \cdots, n$, of $M'$ with property (*). Let $B_{n,1}=C_1$ and $B_{n,j}=B'_{n,j}$, $j=2, 3, \cdots, k$. Then $\{B_{n,j};\ j=1, 2, \cdots, k\}$ is a ball covering of $M$.

Now, we will construct, inductively, ball coverings $\mathscr{B}_i$, $i=n-1, n-2, \cdots, 2, 1$, as follows. Let $B_{n-s,1}=N(B_{n-s+1,1};\ M)$ be a small regular neighborhood of $B_{n-s+1,1}$ in $M$, $1 \leqq s \leqq n-1$, and let $B_{n-s,j}=\mathrm{Cl}\,(B'_{n-s,j}-B_{n-s,1})$, $j=2, 3, \cdots, k$. Hence $\mathscr{B}_i= \{B_{i,j};\ j=1, 2, \cdots, k\}$ is a ball covering of $M$ by Lemma 2, $i=n, n-1, \cdots, 2, 1$. Note that $B_{i,j} \cap B_{p,q}=\emptyset$ if $j, p \neq 1$, $i+j=p+q$, and $(i, j) \neq (p, q)$, since $B_{i,j} \subset B'_{i,j} (j \neq 1)$. From the construction of $B_{i,1}$'s, $\mathscr{B}_i=\{B_{i,j};\ j=1, 2, \cdots, k\}$, $i=1, 2, \cdots, n$, have the property (*). This completes the proof.

**3. Proof of Theorem 1.** Suppose $M^m$ and $N^n$ are connected compact $m$- and $n$-manifolds with $b(M)=p$ and $b(N)=q$, respectively. From Lemma 4, there exist $p$ ball coverings $\mathscr{B}_i=\{B^n_{i,j};\ j=1, 2, \cdots, q\}$, $i=1, 2, \cdots, p$, of $N$ with property (*). Let $\{B^m_1, B^m_2, \cdots, B^m_p\}$ be a ball covering of $M$. Then,

$$\begin{aligned} M \times N &= (B_1 \cup B_2 \cup \cdots \cup B_p) \times N = \bigcup_i (B_i \times N) = \bigcup_i (B_i \times \bigcup_j B_{i,j}) \\ &= \bigcup_{\substack{1 \leqq i \leqq p \\ 1 \leqq j \leqq q}} (B^m_i \times B^n_{i,j}) . \end{aligned}$$

Note that $B^m_i \times B^n_{i,j}$ is an $(m+n)$-ball in $M \times N$, $1 \leqq i \leqq p$, $1 \leqq j \leqq q$, such that $B_i \times B_{i,j} \cap B_k \times B_{k,h} = \partial(B_i \times B_{i,j}) \cap \partial(B_k \times B_{k,h})$ for $(i, j) \neq (k, h)$.

Since $\{B_{i,j};\ 1\leqq i\leqq p,\ 1\leqq j\leqq q\}$ has property (*), $B_i\times B_{i,j}\cap B_k\times B_{k,h}=\emptyset$ if $(i, j)\neq(k, h)$ and $i+j=k+h$. By Lemma 1, there exists a ball covering $\mathscr{C}=\{C_{i,j};\ 1\leqq i\leqq p,\ 1\leqq j\leqq q\}$ with property (*). It is noted that $C_{1,2}\cap C_{2,1}=\emptyset$ and $\cup\{\partial C_{i,j};\ 1\leqq i\leqq p,\ 1\leqq j\leqq q,\ i+j\neq 3\}$ is connected. Choose a simple arc $J$ in $\cup\{\partial C_{i,j};\ 1\leqq i\leqq p,\ 1\leqq j\leqq q,\ i+j\neq 3\}$ so that $J$ intersects with $\partial C_{1,2}$ and $\partial C_{2,1}$, respectively, at its endpoint. Let $C_3'=N(C_{1,2}\cup C_{2,1}\cup J;\ M\times N)$ be a small regular neighborhood of $C_{1,2}\cup C_{2,1}\cup J$ in $M\times N$, which is an $(m+n)$-ball. Let $C_2'=\mathrm{Cl}\,(C_{1,1}-C_3')$ and $C_{i,j}'=\mathrm{Cl}\,(C_{i,j}-C_3')$ for $i+j\geqq 4$. Hence, $\mathscr{C}'=\{C_2',\ C_3',\ C_{i,j}';\ 1\leqq i\leqq p,\ 1\leqq j\leqq q,\ i+j\geqq 4\}$ is a ball covering of $M\times N$ such that $\#\mathscr{C}'=pq-1$ and $\{C_{i,j}';\ i+j\neq 4\}$ has property (*).

Suppose that $\mathscr{C}^{(r)}=\{C_2^{(r)},\ \cdots,\ C_{r+2}^{(r)},\ C_{i,j}^{(r)};\ 1\leqq i\leqq p,\ 1\leqq j\leqq q,\ i+j\geqq r+3\}$ is a ball covering of $M\times N$ such that $\{C_{i,j}^{(r)};\ i+j\geqq r+3\}$ has property (*). Let $s=\#\{C_{i,j}^{(r)};\ i+j=r+3,\ 1\leqq i\leqq p,\ 1\leqq j\leqq q\}$. We can choose a simple arc $J_{u,v}^{(t)}$ in $\cup\partial C_i^{(r)}\cup\cup\{\partial C_{i,j}^{(r)};\ i+j=r+3,\ i\neq u,\ v\}$, which connects a point of $\partial C_{u,r+3-u}^{(r)}$ and a point of $\partial C_{v,r+3-v}^{(r)}(u\neq v)$, so that $\cup\{J^{(t)};\ t=1, 2, \cdots, s-1\}\cup\cup\{C_{i,j}^{(r)};\ i+j=r+3\}$ is connected and $J^{(t)}\cap J^{(t')}=\emptyset$ for $t\neq t',\ t,t'=1, 2, \cdots, s-1$. Hence, a regular neighborhood $C_{r+3}^{(r+1)}=N(\cup J^{(t)}\cup\cup\{C_{i,j}^{(r)};\ i+j=r+3\};\ M\times N)$ is an $(m+n)$-ball. Let

$$C_i^{(r+1)}=\mathrm{Cl}\,(C_i^{(r)}-C_{r+3}^{(r+1)}),\ i=2,\ 3,\ \cdots,\ r+2, \text{ and let}$$
$$C_{i,j}^{(r+1)}=\mathrm{Cl}\,(C_{i,j}^{(r)}-C_{r+3}^{(r+1)}),\ 1\leqq i\leqq p,\ 1\leqq j\leqq q,\ i+j\geqq r+4\,.$$

Then $\mathscr{C}^{(r+1)}=\{C_k^{(r+1)},\ C_{i,j}^{(r+1)};\ 2\leqq k\leqq r+3,\ 1\leqq i\leqq p,\ 1\leqq j\leqq q,\ i+j\geqq r+4\}$ is a ball covering of $M\times N$ such that $\#\mathscr{C}^{(r+1)}=\#\mathscr{C}^{(r)}-(s-1)$ and $\{C_{i,j}^{(r+1)};\ i+j\geqq r+4\}$ has property (*). For, $C_{i,j}^{(r+1)}\cap C_{k,h}^{(r+1)}\subset C_{i,j}^{(r)}\cap C_{k,h}^{(r)}$.

At last, we have a ball covering $\mathscr{C}^*=\{C_2^*,\ C_3^*,\ \cdots,\ C_{p+q-1}^*,\ C_{p+q}^*\}$ of $M\times N$, where $*=p+q-3$ and $C_{p+q}^*=C_{p,q}^{(p+q-3)}$. Hence the theorem was proved.

**Remark 4.** We can define the weak ball coverings of a connected compact manifold $M$ by removing the condition (ii) of the definition of the ball coverings and can define $\beta(M)$ similar as $b(M)$ [2]. We expect that the Theorem 1 holds for $\beta$. But the proof for Theorem 1, in this paper, do not succeed for $\beta$. For, the proof of Lemma 3 fails.

**5.** Let $M$ be a sphere $S^p$ bundle over a sphere $S^q$, $(p, q\geqq 1)$. Then, it is trivial that $b(M)\leqq 4$. If $M$ has a cross-section, then $b(M)\leqq 3$ from lemma 2 and [4, §18].

**4. Proof of theorem 2.** Let $V^n$ be a connected compact $n$-manifold with non-empty boundary $\partial V$. Let $V'$ be a copy of $V$ and $h: V\to V'$ be the identity homeomorphism. Then $D(V)=V\cup V'/(h|\partial V)$ is considered the double of $V$.

Suppose $b(M)=k$, then there are ball coverings $\mathscr{B}_i=\{B_{i,1},\ \cdots,\ B_{i,k}\}$, $i=1, 2,$

of $V$ which satisfy the condition (*) of Lemma 4. It is noted that $\cup\{B_{1,i}, h(B_{2,j});$ $1\leqq i\leqq k\}=D(V)$ and $B_{1,i}\cap h(B_{2,j})=\partial B_{1,i}\cap\partial h(B_{2,j})$. For, $B_{1,i}\cap h(B_{2,j})\subset\partial V\cap\partial V'$. Hence from Lemma 1, there exists a ball covering $\mathscr{C}=\{C_{i,j}; i=1, 2, 1\leqq j\leqq k\}$ of $D(V)$ with property (*).

By the same way as the proof of Theorem 1, one can obtain a ball covering $\mathscr{C}^*=\{C_2^*, C_3^*, \cdots, C_{k+2}^*\}$ of $D(V)$. Hence, $b(D(V))\leqq k+1=b(V)+1$.

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Department of Mathematical Engineering
Sagami Institute of Technology
Fujisawa 251, Japan