HEEGAARD SPLITTINGS OF $F\times S^1$

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A closed 3-manifold M with a Heegaard splitting of genus one such that $\pi_1(M)$ is Z, an infinite cyclic group, is homeomorphic to $S^2\times S^1$ where S^2 is a 2-sphere and S^1 is a 1-sphere. We extend this result to $F\times S^1$ where F is a closed orientable 2-manifold. We work in the piecewise linear category throughout the paper.

Definition. Let $(H_1, H_2; V)$ be a triad such that H_1, H_2 are solid tori with genus n and V is a closed 2-manifold. Then the triad $(H_1, H_2; V)$ is said to be a Heegaard splitting with genus n for a closed orientable 3-manifold M if followings hold;

$$M=H_1\cup H_2 \tag{1}$$

and

$$\partial H_1=\partial H_2=H_1\cap H_2=V\,. \tag{2}$$

Definition. Let H be a compact 3-manifold. We say that H is irreducible when any 2-spheres embedded in H bound 3-cells in H.

Proposition. *Let F be a closed orientable 2-manifold with genus n and S^1 a 1-sphere. Then $F\times S^1$ has a Heegaard splitting of genus $2n+1$.*

Proof. Let $(m_1, \cdots, m_n; l_1, \cdots l_n)$ be longitudemeridian system of F, i.e.,

(1) $m_i(i=1, \cdots, n)$ and $l_k(k=1, \cdots, n)$ are simple closed curves in F,

(2) $m_i\cap m_k=l_i\cap l_k=q$ a point in F $(i\neq k)$,

(3) $m_i\cap l_k=q$ for all i, k,

(4) $F-\bigcup_{i=1}^{n}\mathring{N}(m_i, F)-\bigcup_{k=1}^{n}\mathring{N}(l_k, F)$ is a 2-disk where $N(m_i, F)\{N(l_k, F)\}$ is a regular neighborhood of $m_i(l_k)$ in F. Then

$$N((\bigcup_{i=1}^{n} m_i)\cup(\bigcup_{k=1}^{n} l_k)\cup(q\times S^1), F\times S^1)\,,$$

a regular neighborhood of $(\bigcup_{i=1}^{n} m_i)\cup(\bigcup_{k=1}^{n} l_k)\cup(q\times S^1)$ in $F\times S^1$, is a solid torus with genus $2n+1$ and $F\times S^1-\mathring{N}((\bigcup_{i=1}^{n} m_i)\cup(\bigcup_{k=1}^{n} l_k)\cup(q\times S^1), F\times S^1)$ is also a solid torus with genus $2n+1$ since the condition (4) holds. Q.E.D.

Definition. The connected sum $M \# M'$ of two closed orientable 3-manifolds M, M' is obtained by removing the interior of a 3-cell from each, and then matching the resulting boundaries, using an orientation reversing homeomorphism. (See *Waldhausen* [7].)

Hereafter let M be a closed orientable 3-manifold and F a closed orientable 2-manifold with genus n. Then we have;

Lemma 1. *If $\pi_1(M)$ is isomorphic to $\pi_1(F)\times Z$, then M has no Heegaard splittings whose genus is less than $2n+1$.*

Proof. At first, $H_1(F)$, the first homology group of F, has rank $2n$. Here if M has a Heegaard splitting whose genus is less than $2n+1$, it follows from the definition of the Heegaard splitting that $\pi_1(M)$ has generators of less than $2n+1$. Consequently $H_1(M)$ has rank of less than $2n+1$, it is impossible since $\pi_1(M)$ is isomorphic to $\pi_1(F)\times Z$. Q.E.D.

Next in the theory of groups we have *Bear-Levi* theorem [3], that is, if any groups are non-trivial direct products then they are not non-trivial free products and conversely. Then we have;

Lemma 2. *If $\pi_1(M)$ is isomorphic to $\pi_1(F)\times Z$ then $\pi_1(M)$ is not a non-trivial free product.*

Note that Lemma 2 is true geometrically; Let F be not a 2-sphere. Then by *Stallings* [6] $\pi_2(M)$ is trivial and so by *Epstein* [1] $\pi_1(M)$ is not a non-trivial free product.

Theorem. *Let $\pi_1(M)$ be isomorphic to $\pi_1(F)\times Z$. Then if M has a Heegaard splitting of genus $2n+2$, M is homeomorphic to $F\times S^1$.*

Proof. Two cases happen since n is a non-negative integer.

Case (1). Suppose that n is positive. If M is irreducible, then M is homeomorphic to $F\times S^1$ by *Neuwirth* [5]. Hence we may assume that M is not irreducible. Let $(H_1, H_2; V)$ be a Heegaard splitting of genus $2n+2$ of M. Then by *Haken* [4] it follows from the non-irreducibility of M that there is a simple closed curve c in V which is not homotopic to zero in V and bounds a 2-disk in H_1, H_2 respectively. There are two cases in which c separates V into two components and otherwise.

At first, let $V-c$ be connected, then M has a connected sum decomposition $M_1 \# S^1\times S^2$ where M_1 has a Heegaard splitting of genus $2n+1$, induced from that of M. By *van Kampen* [2], $\pi_1(M)=\pi_1(M_1)^*Z=\pi_1(F)\times Z$ and so $\pi_1(M_1)$ is trivial by Lemma 2. But it is impossible that $Z=\pi_1(F)\times Z$ and $\pi_1(F)$ is non-trivial because of F being not a 2-sphere. Hence we have the only one case that c separates V into two components. Then M has a connected sum decomposition $M_1 \# M_2$ such that $M_i(i=1,2)$ has a Heegaard splitting whose genus is less than $2n+2$, induced from that of M, because c is not homotopic to zero in V and so D_1, D_2 separate H_1, H_2 respectively where $D_i(i=1,2)$ is a 2-disk in $H_i(i=1,2)$ such that $D_i\cap\partial H_i=\partial D_i=c$. By *van Kampen* [2], $\pi_1(M)=\pi_1(M_1)*\pi_1(M_2)=\pi_1(F)\times Z$, consequently which of $\pi_1(M_i)(i=1,2)$ is trivial by Lemma 2. We may assume that $\pi_1(M_1)$ is trivial. Then the Heegaard splitting of M_1 induced from that of M has genus one, otherwise M_2 has a Heegaard splitting whose genus is less than $2n+1$, since the genus of the Heegaard splitting of $M=$(the genus of the one of M_1)+ (the genus of the one of M_2). But it is impossible by Lemma 1, since $\pi_1(M_2)$ is isomorphic to $\pi_1(F)\times Z$. Here M_1 is homeomorphic to a 3-sphere since M_1 has a Heegaard splitting of genus one and $\pi_1(M_1)$ is trivial. And so M_2 is homeomorphic to M. Hence M_2 is also not irreducible and $\pi_1(M_2)$ is isomorphic to $\pi_1(F)\times Z$ and M_2 has a Heegaard splitting of genus $2n+1$.

To repeat the preceding argument of the connected sum decomposition, M_2 has a Heegaard splitting whose genus is less than $2n+1$. But it is impossible by Lemma 1. Hence M is irreducible.

(2). Suppose that n equals zero. Then F is a 2-sphere and $\pi_1(M)$ is isomorphic to Z. By *Stallings* [6] M has a 2-sphere embedded in M which does not bound a 3-cell. Hence M is not irreducible.

To repeat the preceding argument of the connected sum decomposition, M is hcmeomorphic to $M_1\# M_2$ such that $M_i(i=1,2)$ has a Heegaard splitting of genus one. By *van Kampen* [2] and Nieisen-Schreier theorem [3], we may assume that $\pi_1(M_1)$ is trivial and $\pi_1(M_2)=Z$. Then M_1 is homeomorphic to a 3-sphere and M_2 is homeomorphic to $S^1\times S^2$.

Consequently M is homeomorphic to $S^1\times S^2$. Q.E.D.

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