RELATIVE K-THEORY AND SURGERY ON n-ADS

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1. Introduction.

Wall [6] has algebraically defined functors L_m of rings with involution measuring the surgery obstruction of a map, relative to the boundary. The groups L_{2*} are K_0 of a category of Hermitian forms and the L_{2*+1} are the corresponding K_1.

This has been extended to maps between rings in two seemingly different ways. *Wall* [6] defines the odd dimensional functors to be the relative K-theory associated to L_{2*+1}, and *Sharpe* [3] considers the K_2 analogue in the even dimensional case. In either situation, the algebraic object is a (and hence the only) surgery obstruction group for a map relative to a codimension 0 submanifold of the boundary.

In general, there is a geometrically defined surgery obstruction group for n-ads. The purpose of this paper is to give an algebraic construction of these groups.

The problems encountered are best illustrated by a well-known theorem of *Wall* [5]. Suppose $M^{4k} \cap N^{4k} = V$ is a condimension 0 submanifold of ∂M and ∂N. Then Sign (M)+Sign (N)−Sign $(M \cup N)$ is the signature of a symmetric bilinear form associated to the intersection pairing of ∂V (which is skew-symmetric). It is precisely this type of difficulty that is encountered in the even dimensional relative case: how to pass from a trivialization of an automorphism of an η-kernel to a quadratic form, which now must be $(-\eta)$-Hermition.

We define a generalized notion of relative K-theory which is the analogue of the geometric situation—at each level, except the top, we have a specific trivialization (i.e. null homotopy to BG)—and coincides with the usual relative K-theory if there is only one stage.

Our result is that the algebraic L-groups of a ring of type 2^n are given, as in [6], by K_0 or K_1 of an appropriate relativized category.

2. Relative K-Theory.

We define here a "period 2" K-theory.

Define an *n-system* X to be given by the following data:

(i) a space X_α for each $\alpha\in 2^n$,

(ii) an element $a\in F_2$, called the *parity* of X

(iii) maps $f^X_{\alpha,\beta}$: $X_\alpha\to X_\beta$ if $\alpha\subset\beta$,
$|\alpha|+1=|\beta|\equiv a \bmod (2)$

and (iv) maps $g^X_{\alpha,\beta}$: $X_\beta\to \Omega X_\alpha$ if $\alpha\subset\beta$,
$|\beta|-1=|\alpha|\equiv a \bmod (2)$.

Here $|\alpha|$ denotes the cardinality of α. Any pair of spaces defines a 1-system of parity 1. A *basepoint* for X is a collection of $x_\alpha\in X_\alpha$ so that $f^X_{\alpha,\beta}(x_\alpha)=x_\beta$ and $g^X_{\alpha,\beta}(x_\beta)(t)=x_\alpha$ for all $t\in I$.

If X is an n-system and A is a space, then $X\times A$ denotes the n-system $\{X_\alpha\times A\}$, $f^{X\times A}_{\alpha,\beta}=f^X_{\alpha,\beta}\times 1_A$, $g^{X\times A}_{\alpha,\beta}=g^X_{\alpha,\beta}\circ i_A$, where $\circ$: $\Omega X_\alpha\times\Omega A\to\Omega(X_\alpha\times A)$ and i_A:$A\to \Omega A$ are the obvious maps.

If X and Y are n-systems of the same parity, a *morphism* between X and Y is a collection ϕ_α: $X_\alpha\to Y_\alpha$ so that

$$f^Y_{\alpha,\beta}\circ\phi_\alpha=\phi_\beta\circ f^X_{\alpha,\beta}$$

and

$$g^Y_{\alpha,\beta}\circ\phi_\alpha=\Omega\phi_\beta\circ g^X_{\alpha,\beta}\ .$$

This defines a category of n-systems of a given parity.

Let $S^k_{a,n}$ denote the n-system with parity a defined by

$$(S^k_{a,n})_\alpha=\begin{cases} S^k & |\alpha|\not\equiv a \bmod (2)\\ S^{k+1} & |\alpha|\equiv a \bmod (2)\ ,\end{cases}$$

where $f_{\alpha,\beta}$ is the inclusion and $g_{\alpha,\beta}$ is adjoint to Λ $S^kS^1\cong S^{k+1}\xrightarrow{1}S^{k+1}$.

Let X be an n-system of parity a with basepoint. Define $\pi_k(X)$ to be the set of homotopy classes of basepoint-preserving maps $S^k_{a,n}\to X$. By compatibility of the basepoints of X_α, the usual construction inductively defines a group structure on $\pi_k(X)$ for $k>0$.

We need to define a slightly more relative homotopy group, as the above construction does not give null-homotopies for the maps g. This can be done in the abstract way, as above, but we shall write down the explicit formula.

Consider the set of all maps $\phi_{\alpha,\beta}$: $(D^{k+1}, S^k)\to(M_{f_{\alpha\ \beta}}, X_\alpha)$, $\phi_{\alpha,\beta}$: $(D^{k+2}, S^{k+1})\to (M_{g_{\alpha\ \beta}}, X_\beta)$ so that the maps $S^k\to X_\alpha$, $S^{k+1}\to X_\beta$ define a map $S^k_{a,n}\to X$ and

$$\begin{array}{ccc} D^{k+1} & \longrightarrow & M_{f_{\alpha,\beta}} \\ \downarrow & & \downarrow \\ S^{k+1} & \longrightarrow & X_\beta \end{array}$$

commutes. Let $\pi_k^{rel}(X)$ be the set of homotopy classes of such maps. This has a natural group structure for $k>0$, and we have a zero sequence

$$\pi_k^{rel}(X) \longrightarrow \pi_k(X) \longrightarrow \bigoplus \pi_{k+1}(g^X_{\alpha,\beta}) .$$

We have the following categorical analogue. An *n-category* $\mathscr{C}$ with parity a is given by

(i) a category $\mathscr{C}_\alpha$ for each $\alpha \in 2^n$,

(ii) a functor $T^{\mathscr{C}}_{\alpha,\beta}$: $\mathscr{C}_\alpha \to \mathscr{C}_\beta$ if
$|\alpha|+1=|\beta| \equiv a \bmod (2)$, and

(iii) a functor $U^{\mathscr{C}}_{\alpha,\beta}$: $\mathscr{C}_\beta \to \mathrm{Aut}\ (\mathscr{C}_\alpha)$ if
$|\beta|-1=|\alpha| \equiv a \bmod (2)$.

An n-category $\mathscr{C}$ is *exact* if each $\mathscr{C}_\alpha$ is exact, and the $T^{\mathscr{C}}_{\alpha,\beta}$, $U^{\mathscr{C}}_{\alpha,\beta}$ are exact functors (with the induced exact sequences in Aut ($\mathscr{C}_\alpha$)).

Let $\mathscr{C}$ be exact and $Q\mathscr{C}_\alpha$ the "subobject-quotient" category of [2] associated to C_α.

Lemma 1. *$\mathscr{C}$ defines an n-system $BQ\mathscr{C}$.*

Proof: Since an exact functor $\mathscr{C}_\alpha \to \mathscr{C}_\beta$ induces a map $BQ\mathscr{C}_\alpha \to BQ\mathscr{C}_\beta$, we need only produce a map BQ (Aut ($\mathscr{D}$))$\to \Omega BQ\mathscr{D}$ for an exact category $\mathscr{D}$.

Define a functor Q(Aut ($\mathscr{D}$))$\to$Aut ($Q\mathscr{D}$) by sending an object f:$A\to A$ of Q(Aut ($\mathscr{D}$)) to the automorphism $A \overset{1}{\leftarrow} A \overset{f}{\rightarrow} A$ of $Q\mathscr{D}$, and the morphism

$$\begin{array}{ccccc} A & \overset{\phi}{\longleftarrow} & C & \overset{\psi}{\longrightarrow} & B \\ f\downarrow & & \downarrow h & & \downarrow g \\ A & \underset{\phi'}{\longleftarrow} & C & \underset{\psi'}{\longrightarrow} & B \end{array}$$

to

$$\begin{array}{ccccc} A & \overset{\phi}{\longleftarrow} & C & \overset{\psi}{\longrightarrow} & B \\ 1\downarrow & & \downarrow 1 & & \downarrow 1 \\ A & \overset{\phi}{\longleftarrow} & C & \overset{\psi}{\longrightarrow} & B \\ f\downarrow & & \downarrow h & & \downarrow g \\ A & \overset{\phi'}{\longleftarrow} & C & \overset{\psi'}{\longrightarrow} & B \end{array}$$

This induces BQ (Aut $(\mathscr{D})) \to B$ Aut $(Q(\mathscr{D}))$. Given a category $\mathscr{D}'$, we define B Aut $(\mathscr{D}') \to \Omega B\mathscr{D}'$ by sending the k-simplex

$$\begin{array}{ccccccc} A_0 & \longrightarrow & A_1 & \longrightarrow \cdots \longrightarrow & A_k \\ \downarrow & & \downarrow & & \downarrow \\ A_0 & \longrightarrow & A_1 & \longrightarrow \cdots \longrightarrow & A_k \end{array}$$

to the (simplicial) loop $I \to B\mathscr{D}$ whose 1-simplex maps to the morphism defined by the k-simplex. *QED.*

We define the ***higher K-groups*** of the exact n-category $\mathscr{C}$ by

$$K_i(\mathscr{C}) = \pi_{i+1}^{rel} BQ\mathscr{C} .$$

Examples: (1) If $\mathscr{C}$ is an exact category, regarded as a 0-category of parity 1, then this is the usual definition.

(2) If $\mathscr{C}$ is an exact 1-category of parity 1 of projective modules, then $K_1(\mathscr{C})$ is the relative K_1 functor of [1].

(3) If $\mathscr{C}$ is a certain pull-back of a "Steinberg category" by the automorphisms of a Hermitian form category, regarded as a 1-category with parity 0, then $K_0(\mathscr{C})$ is the group constructed by *Sharpe* [3]. This will be discussed in more detail later.

3. A Review of Algebraic L-Theory.

Let Λ be a ring with involution *. Define $\mathscr{H}^{\pm}(\Lambda)$ to be the category with objects non-singular ($\pm$)-Hermitian forms on a free Λ-module and morphisms generated by matrices

$$\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \begin{pmatrix} a \\ b \end{pmatrix}\right)$$

where $a \mp a^* = \alpha^*\gamma$, $b \mp b^* = \delta^*\beta$ if $\alpha^*\gamma$, $\delta^*\beta$ are of this form and a and b zero otherwise. Composition is defined in [4], § 3.

Define the exact sequences of $\mathscr{H}^{\pm}(\Lambda)$ to be generated by

(i) split exact sequences,

(ii) $0 \to K \to 0$ if K is a kernel,

and (iii) $0 \to K \xrightarrow{1} K \xrightarrow{A} K \to 0$ where A is of the form

$$\left(B, \begin{pmatrix} 0 \\ b \end{pmatrix}\right), \quad B \in TU(\Lambda), \quad \left(\begin{pmatrix} C & 0 \\ 0 & C^{*-1} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right), \quad C \in SL(\Lambda),$$

or

$$\left(\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).$$

Proposition 2. $L^h_{2k}(\Lambda)\cong K_0[\mathscr{H}^{(-1)^k}(\Lambda)]$
$L^h_{2k+1}(\Lambda)\cong K_1[\mathscr{H}^{(-1)^k}(\Lambda)]$.

If A is a self-dual subgroup of $K_1(\Lambda)$, then we can similarly construct $L^A_*(\Lambda)$. The following is essentially Sharpe's unitary Steinberg construction [3]:

Let $\mathscr{R}^{\pm}(\Lambda)$ be the category with objects (u, P, Q), $u\in St(\Lambda)$, P, Q $(\pm)$-Hermitian forms over Λ, and morphisms defined as above on P and Q, and from Aut$(St(\Lambda))$ on u, regarding $St(\Lambda)$ as a category. We make $\mathscr{R}^{\pm}(\Lambda)$ into an exact category, using the relations (iii)-(vi) on pg. 456 of [4], as above. Inclusion to the second factor defines a functor $\mathscr{H}^{\pm}(\Lambda)\to\mathscr{R}^{\pm}(\Lambda)$, and the construction

$$(P, Q)\mapsto\begin{pmatrix} I & 0 \\ P & I \end{pmatrix}\begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix}\begin{pmatrix} I & 0 \\ Q & I \end{pmatrix}$$

defines a functor $\mathscr{R}^{\pm}(\Lambda)\to$Aut $\mathscr{H}^{\mp}(\Lambda)$. It is this functor which provides an algebraic stepping stone between η- and $(-\eta)$-Hermitian forms.

4. Surgery Groups for m-ads.

In this section we construct an m-category whose relative K-groups define obstruction groups for relative surgery.

Let Λ be a ring of type 2^m with involution. Thus, for $\alpha\in 2^m$, we have a ring $\Lambda(\alpha)$ with involution, together with morphisms $i_{\alpha,\beta}$: $\Lambda(\alpha)\to\Lambda(\beta)$ for $\alpha\subset\beta$.

For $\alpha\subset\beta$, let $P^{\pm}_{\alpha,\beta}$ be the pull-back of

$$\begin{array}{ccc} & & \mathscr{R}^{\pm}(\Lambda)(\beta)) \\ & & \downarrow \\ \text{Aut}\ (\mathscr{H}^{\mp}(\Lambda)(\alpha) & \longrightarrow & \text{Aut}\ (\mathscr{H}^{\mp}(\Lambda(\beta))) . \end{array}$$

Define, for $\eta=\pm 1$ and $a\in F_2$, an m-category $\mathscr{H}^{\eta}_a(\Lambda)$ by

(i) $\mathscr{H}^{\pm}(\Lambda(\alpha))\to\mathscr{H}^{\pm}(\Lambda(\beta))$ if $m-|\beta|\not\equiv a \bmod (2)$

(ii) $\mathscr{H}^{\pm}(\Lambda(\beta))\to P^{\pm}_{\alpha,\beta}\to$Aut$(\mathscr{H}^{\mp}(\Lambda(\alpha)))$ if
$m-|\beta|\equiv a \bmod (2)$, $|\beta|\neq m$

(iii) $P^{\pm}_{\alpha,\beta}\to$Aut $(\mathscr{H}^{\mp}(\Lambda(\alpha)))$ if $|\beta|=m$ and $a=0$, starting at $\mathscr{H}^{\eta}(\Lambda(\phi))$. We define

$$L_n(\Lambda)=\begin{cases} K_0(\mathscr{H}^{(-1)^k}_0(\Lambda)) & n \text{ even} \\ K_1(\mathscr{H}^{(-1)^k}_1(\Lambda)) & n \text{ odd} \end{cases}$$

where $k=[(n-m)/2]$. These are the *algebraic relative L-groups.*

In the geometric situation, these groups contain too much information. Suppose π is a group of type 2^m and $Z[\pi]$ its integral group ring. If n is odd, let $L_n(\pi)=L_n(Z[\pi])$. If n is even and $|\alpha|=m-1$, then $\pi(\alpha)$ acts on $L_n(Z[\pi])$. Define inductively

$$L_n^0(\pi)=L_n(Z[\pi])$$
$$L_n^k(\pi)=(\pm\pi(\alpha)\times_\alpha L_n^{k-1}(\pi))/[\pi(\alpha),\ \pi(\alpha)]$$
$$\text{as in [3], where } \alpha=\{1,\cdots,\hat{k},\cdots,m\}\ .$$

Finally, for $\beta=\{1,\cdots,m\}$, let

$$L_n(\pi)=\text{coker}\ (Wh_2(\pi(\beta))\to L_n^m(\pi)/(\pm\pi(\beta))[L_n^m(\pi),\ L_n^m(\pi)]).$$

Proposition 3. *There is an exact sequence*

$$\cdots\to L_n(\partial_i\pi)\to L_n(\delta_i\pi)\to L_n(\pi)\to L_{n-1}(\partial_i\pi)\to\cdots$$

Proof: This is defined inductively by the sequences of [6], [3], and exactness is an easy consequence. *QED.*

Theorem 4. *Let K be a CW m-ad. Then, for $n-m\geq 4$, there is an isomorphism*

$$L_n(K)\to L_n(\pi_1 K)\ .$$

Proof: Let ϕ: $M^n\to X^n$ be a normal map between manifold $(m+1)$-abs, with a map $\delta_m X\to K$ inducing an isomorphism on π_1. Then ϕ represents its geometric surgery obstruction in $L_n(K)$.

Define its algebraic obstruction in $L_n(\pi_1 K)$ as follows. Let $\alpha\subset\beta$, $|\beta|=|\alpha|+1$, $\alpha,\beta\in 2^m$. Then we have a normal map of pairs

$$\phi(\alpha):\ ((\delta_m M)(\alpha),\ \partial(\delta_m M)(\alpha))\to((\delta_m X)(\alpha),\ \partial(\delta_m X)(\alpha))$$

defined by ϕ which is a homotopy equivalence when restricted to the boundary. This map is bounded by the corresponding restriction for β, and thus we have a trivialization of the classifying map for the surgery obstruction of $\phi(\alpha)$. If $|\beta|=m$, we simply get the classifying map for $(|M|,\ \partial M)\to(|X|,\partial X)$. Inductively, this defines, by [6] and [3], an element of $L_n(\pi_1 K)$. By Propositions 2, 3 and the five-lemma, we have an isomorphism.

References

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