TWO KNOTS WITH THE SAME 2-FOLD BRANCHED COVERING SPACE

By

MOTO-O TAKAHASHI

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It has been proved by Birman-Hilden [3] that every orientable closed 3-manifold M of Heegaard genus $\leqq 2$ is homeomorphic to the 2-fold branched covering space over S^3 with a link L as its branching line.

Whether L is determined by M uniquely would be an interesting problem.

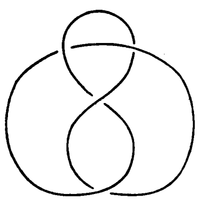
In this paper we show that this is not the case.*)

As M we shall take the 3-manifold obtained by removing a tubular neighbourhood of a figure-eight knot (1-1) and sewing it back differently. This was given by Bing [1].

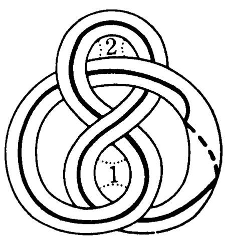
We shall construct two Heegaard splittings of genus 2 and show that different knots are obtained from them.

Moreover from the fact that equivalent links are obtained from equivalent Heegaard splittings in the sense of Waldhausen, it follows that the above-mentioned two Heegaard splittings are not equivalent. Of course these splittings are minimal as M does not have Heegaard splittings of genus 1, M being a homology sphere.

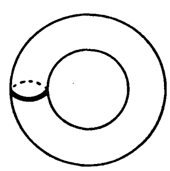
Let M be the 3-manifold obtained from S^3 by removing a tubular neighbourhood of figure-eight knot (1-2, 1-3) and sewing it back so that the bold lines coincide.



(1-1)



(1-2)

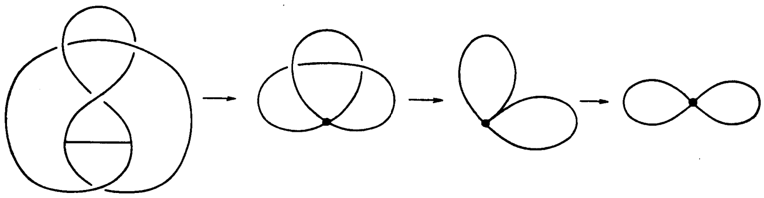


(1-3)

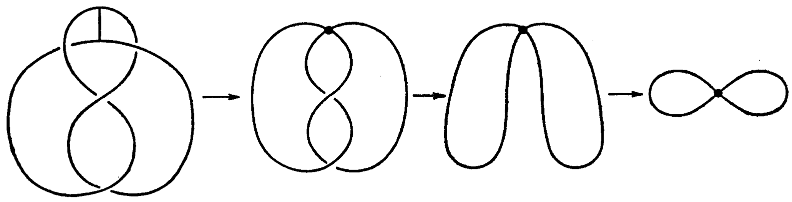
Let us construct two Heegaard splittings of genus 2 of this manifold M. First remove the handle body shown as the dotted line 1 on figure (1-2). Then

* The same result is independently obtained in [4].

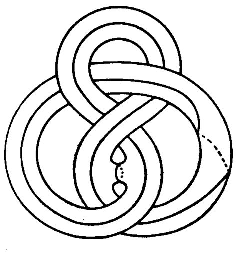
the remaining part (complement from S^3) is a solid torus M_1 of genus 2 (c.f. the following figure (2)).



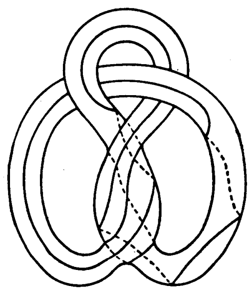
(2)



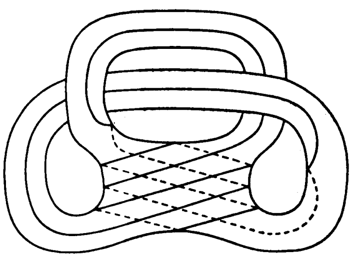
(3)



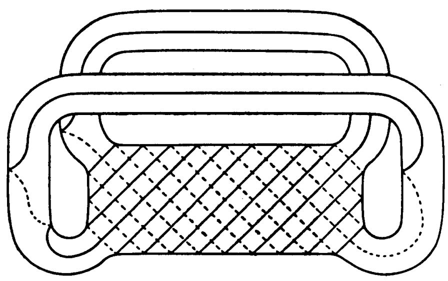
(4-1)



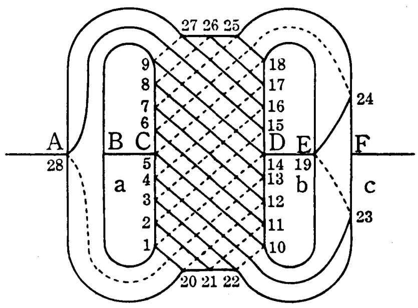
(4-2)



(4-3)



(4-4)



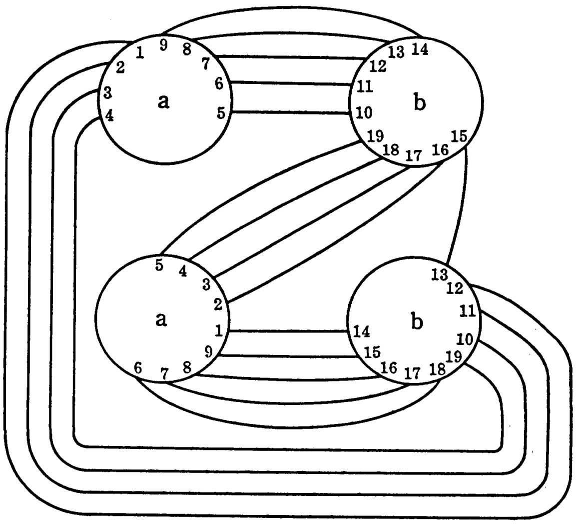
(4-5)

On the other hand the removed handle body is attached to the solid torus (1–3) and we have another solid torus M_2 of genus 2. These two solid tori M_1 and M_2 of genus 2 constitute a Heegaard splitting (Heegaard splitting I).

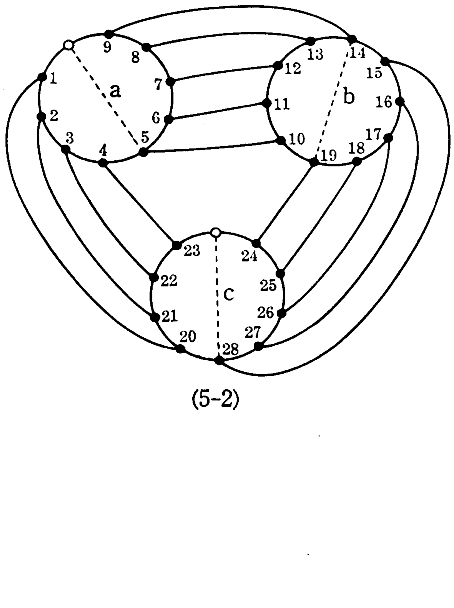
Similarly we obtain another Heegaard splitting of genus 2 by removing from (1–2) the handle body shown in figure (1–2) as the dotted line 2, instead of the dotted line 1 (c.f. the figure (3)). Let M_3 and M_4 constitute this Heegaard splitting Heegaard splitting II).

Now we shall construct the diagram of meridian loops (Heegaard diagram) for each of these Heegaard splittings.

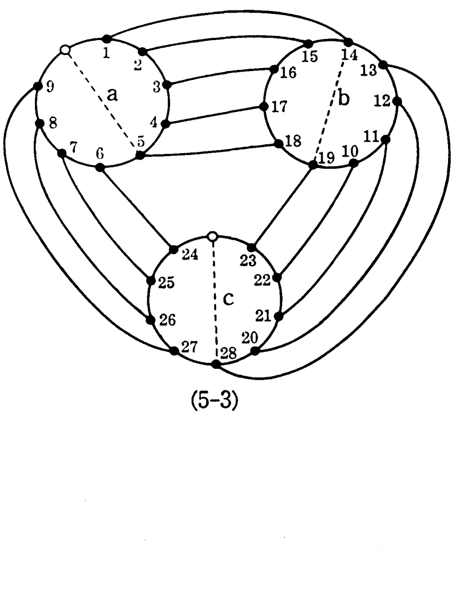
If we draw the meridian loops of solid torus M_2 on the boundary of a tubular neighbourhood of figure-eight knot, we have figure (4–1). By isotopic transformation of it we successively have (4–2), (4–3), (4–4) and (4–5). We shall explain it in detail.



(5-1)



(5-2)



(5-3)

[Explanation of transformation]

$$(4\text{–}1) \Rightarrow (4\text{–}2)\ .$$

If we disregard the meridian loops drawn on the surface it is obvious that (4–2) is obtained from (4–1) by expanding the portion attached.

In this case it is not convenient if meridian loops are on the attached part, since then they get inside.

So from the beginning we transform them isotopically so that they do not pass through the attached part and then expand the attached part.

Then we get (4–2).

$$(4\text{–}2) \Rightarrow (4\text{–}3)\ .$$

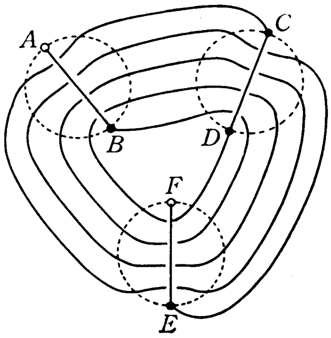
We simply stretch the attached portion horizontally.

$$(4\text{–}3) \Rightarrow (4\text{–}4)\ .$$

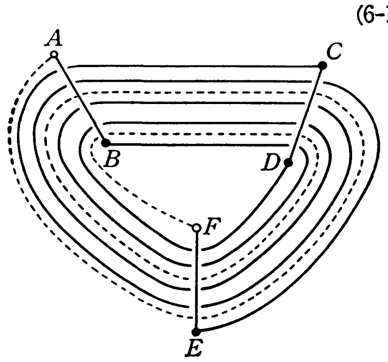
Twist the same portion in (4–3) once counterclockwise to get (4–4).

Each of two handle bodies in the upper part is then twisted once clockwise.

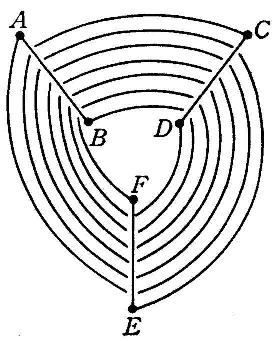
$$(4\text{–}4) \Rightarrow (4\text{–}5)\ .$$



(6-1)



(6-2)



(6-3)

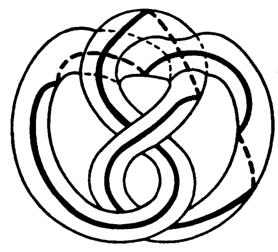
Bring down one of the handle bodies.

Now the outside of (4–5) is the solid torus M_1 of genus 2. As a system of meridian loops we can take the loop a (passing through 1, 2, 3, 4, 5, 6, 7, 8, 9, B in figure (4–5)) and the loop b (passing through 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,). The loop c outside is also a meridian loop.

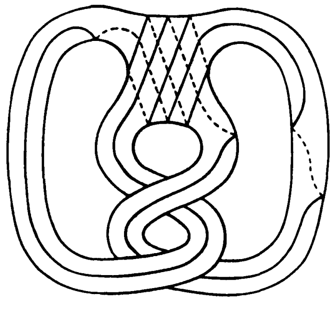
If we cut M_1 through the 2-discs with boundaries a and b respectively, we have a 3-disc. The figure of meridian loops drawn on the surface S^2 of this 3-disc is as shown in (5–1).

If, in addition, we cut this 3-disc through the 2-disc with boundary c, it is divided into two 3-discs and we have (5–2) and (5–3).

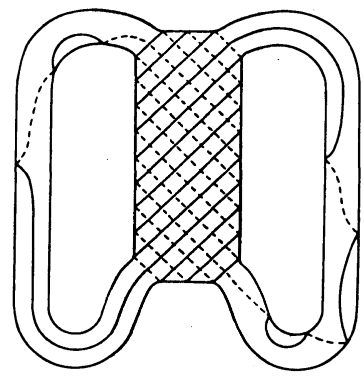
(7–1)



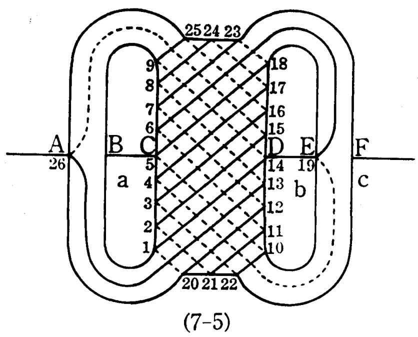
(7–2)



(7–3)



(7–4)



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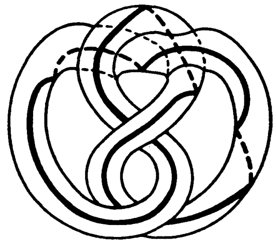
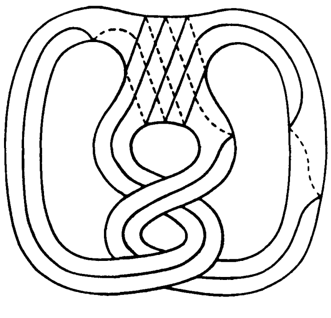
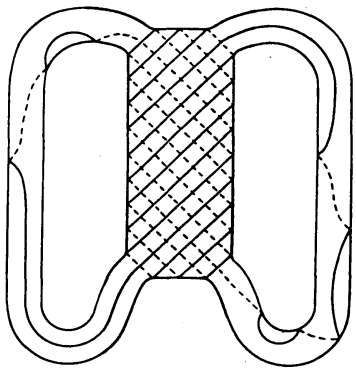


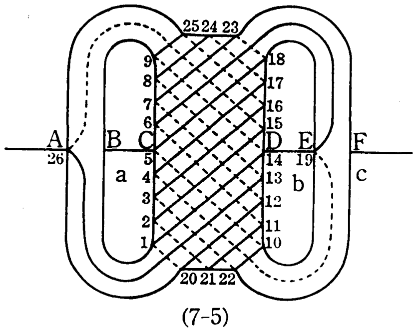
(7–2)



(7–3)



(7–4)



(7–5)

Now we can represent M as a 2-fold branched covering space over S^3 as follows (c.f. [6]).

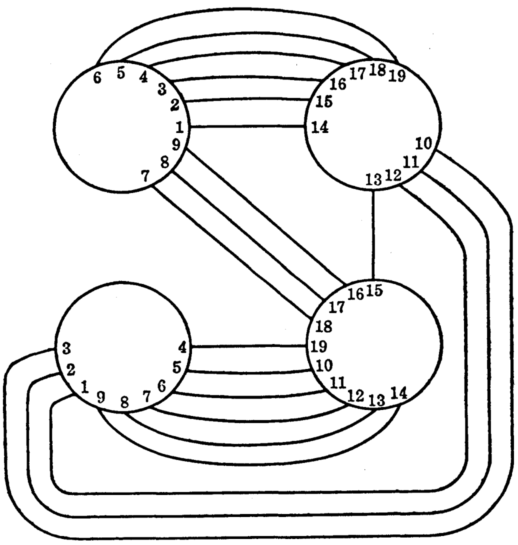
First we construct symmetric axis (the dotted lines in (5–2) and (5–3)) and connect the symmetric points with respect to these lines. Then we have (6–1).

We can regard this chart as a simple arc in S^3. It is composed of 5 parts, i.e. the upper path AB, the lower path BC the upper path CD, the lower path DE and the upper path EF.

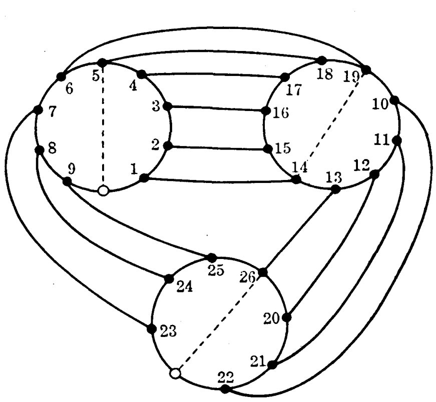
If we connect F with A by a suitable lower path, we get a knot K_1 ((6–2), (6–3)). This is the torus knot of type (7, 3).

M is homeomorphic with the 2-fold branched covering space over S^3 with K_1 as its branching line.

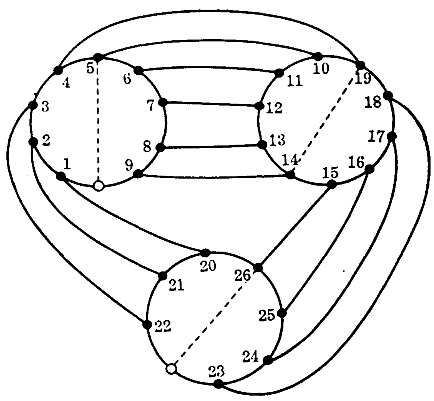
Alexander polynomial of K_1 is



(8-1)



(8-2)



(8-3)

$$\Delta_1 = \frac{(x^{21}-1)(x-1)}{(x^7-1)(x^3-1)}$$
$$= x^{12} - x^{11} + x^9 - x^8 + x^6 - x^4 + x^3 - x + 1 \ .$$

We shall also do the same thing with Heegaard splitting II. If we draw the meridian loops of M_4 on the surface of M_3 we have (7–1). We transform it to (7–5).

$$(7\text{–}1) \Rightarrow (7\text{–}2) \ .$$

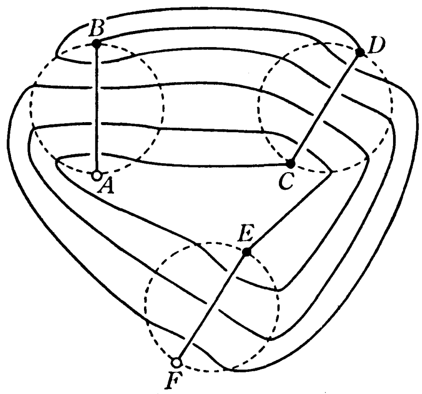
As before we extend the attached portion, transforming the loops isotopically so as to avoid this portion.

$$(7\text{–}2) \Rightarrow (7\text{–}3) \ .$$

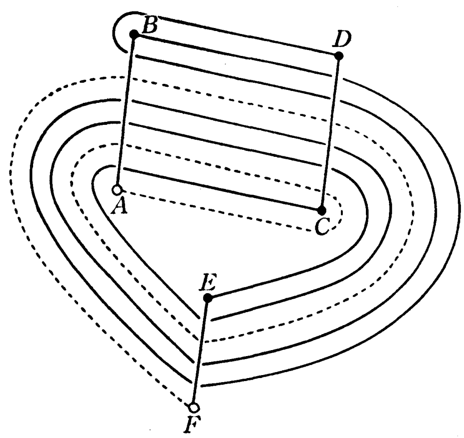
Stretch the upper part.

$$(7\text{–}3) \Rightarrow (7\text{–}4) \ .$$

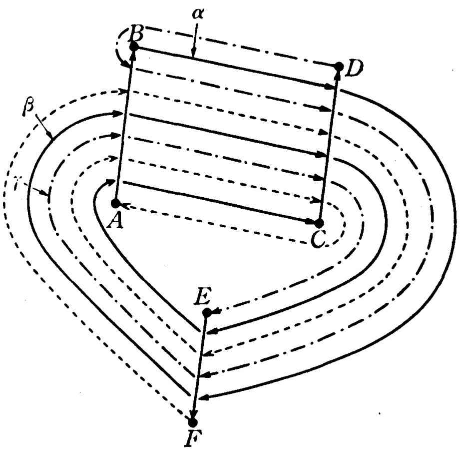
Twist the upper part vertically once clockwise to obtain (7–4). Two handles



(9-1)



(9-2)



(9-3)

are then twisted counterclockwise.

(7–4)⇒(7–5) .

As before transform the meridian loops isotopically.

(8–1) exhibits Heegaard diagram II. (8–2) and (8–3) are full Heegaard diagram.

We can construct the knot K_2 from this diagram as before. ((9-1), (9-2) and (9-3)) K_2 is not a torus knot. We compute Alexander polynomial directly using Fox's free differential calculus.

First the knot group of K_2 is presented as follows:

$\{\alpha, \beta, \gamma: \alpha\beta\gamma\alpha\beta\gamma\beta=\gamma\alpha\beta\gamma\alpha\beta\gamma,\ \beta\gamma\alpha\beta\gamma=\gamma\alpha\beta\gamma\alpha\}$. (i.e. the group generated by α, β, γ with indicated relations. These relations are read from the chart (9–3) of the knot K_2.)

We denote the abelianization by $*$ and put $\alpha^*=\beta^*=\gamma^*=x$.

$$\left\{\frac{\partial}{\partial\alpha}(\alpha\beta\gamma\alpha\beta\gamma\beta-\gamma\alpha\beta\gamma\alpha\beta\gamma)\right\}^*=1+x^3-x-x^4$$

$$\left\{\frac{\partial}{\partial\beta}(\alpha\beta\gamma\alpha\beta\gamma\beta-\gamma\alpha\beta\gamma\alpha\beta\gamma)\right\}^*=x+x^4+x^6-x^2-x^5$$

$$\left\{\frac{\partial}{\partial\alpha}(\beta\gamma\alpha\beta\gamma-\gamma\alpha\beta\gamma\alpha)\right\}^*=x^2-x-x^4$$

$$\left\{\frac{\partial}{\partial\beta}(\beta\gamma\alpha\beta\gamma-\gamma\alpha\beta\gamma\alpha)\right\}^*=1+x^3-x^2$$

$$\Delta_2=\begin{vmatrix}-x^4+x^3-x+1 & -x^4+x^2-x \\ x^6-x^5+x^4-x^2+x & x^3-x^2+1\end{vmatrix}$$

$$=x^{10}-x^9+x^7-x^6+x^5-x^4+x^3-x+1$$

We conclude that the knots K_1 and K_2 are different (inequivalent) as their Alexander polynomials are different. And M is homeomorphic to each of the 2-fold branched covering spaces over S^3 with K_1 or K_2 as their branch line.

The same holds for the 3-manifolds obtained by removing a tubular neighbourhood of figure-eight knot and sewing it back along an annulus that run through the knot $\pm n$ times.

The same also holds for the knot 6_2 instead of figure-eight knot.

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