ON THE TRANSVERSALITY CONDITIONS IN ELECTRICAL CIRCUITS

By

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(Received August 25, 1976)

1. Introduction.

In the formulation of electrical circuit theory given by S. Smale ([1]) and its generalization by T. Matsumoto ([2]), the transversality condition of the characteristic submanifold and the Kirchhoff space is a standing hypothesis. According to Thom's transversality theorem we can make them transverse by an arbitrarily small perturbation of the characteristic submanifold. But, in effect of the drift of the temperature or of the pressure and so on, the characteristic submanifold is always perturbed and hence the transversality may be destroyed.

Therefore, the following problem proposed by T. Matsumoto is of importance.

Problem. *By adding small capacitances in parallel to the given circuit and small inductances in series, can we make the characteristic manifold and the Kirchhoff space transverse?*

In this note, we will give an affirmative answer to the above problem. The author thanks T. Matsumoto, H. Kawakami and S. Matsumoto for thir enlightening discussions.

2. Statement of result.

As in [1], we assume that the electrical circuit is represented by an oriented graph G. Let $C_j=C_j(G)(C^j=C^j(G))$ be the real j-chains (j-cochains) of G, $j=0, 1$. Then the currents and the voltages in the branches of the circuit can be thought as elements of C_1 and of C^1 respectively.

The characteristic submanifold $\Lambda=\Lambda(G)$ representing the characteristics of the elements (possibly including non-linear coupled resistors and so on (see [2])) is a $(2b-\rho)$-dimensional smooth submanifold of $C_1\times C^1$, where b is the number of the branches and ρ corresponds to the number of the resistors. The currents (voltages) of the branches are denoted by $\boldsymbol{i}=(i_1, i_2, \cdots, i_b)\in C_1$, $(\boldsymbol{v}=(v_1, v_2, \cdots v_b)\in C^1)$.

The Kirchhoff's law restricts the possible states to a b-dimensional linear subspace of $C_1\times C^1$ called the Kirchhoff space, $K=\operatorname{Ker}\partial\times\operatorname{Im}\partial^*$, where $\partial(\partial^*)$ is the boundary (coboundary) operator ∂: $C_1\to C_0$, (∂^*: $C^0\to C^1$).

The result of this paper is the following.

Theorem. *By adding (small) capacitances in parallel and (small) inductaces in series to appropriate branches of G, we can get a new circuit G' for which the characteristic submanifold $\Lambda(G')$ and the Kirchhoff space $K(G')$ are transverse. Furthermore, any perturbation of the characteristic submanifold does not affect the transversality.*

3. Preliminaries from circuit theory.

We recall what we need from the circuit theory (c.f. Rohrer [3]). Let G be the (oriented) graph of the given circuit. For simplicity, we assume that G is connected. We take a (maximal) tree T of G, and put $G=T\cup L$. Here L denotes the subcomplex of G consisting of branches which are not contained in T and we call L the link of T. Now, we get the following natural direct sum decompositions.

$$C_1(G)=C_1(T)+C_1(L), \qquad C^1(G)=C^1(T)+C^1(L).$$

Since T is a (maximal) tree, the following properties hold.

(1) Each node of G is a node of T.

(2) T is connected.

(3) T contains no loop.

If we remove a branch of T, T is disconnected into two parts, and hence the set of nodes of G is also partitioned into two disjoint sets. (This makes a fundamental cut-set.) Note that each pair of nodes of G is connected uniquely through only branches of T. Therefore a branch β of L determines a loop l_β of G which contains no branch of L other than β and the direction of l_β is consistent with β. By this correspondence, we define the following injective linear map,

$$\iota\colon C_1(L)\to C_1(G).$$

In fact, ι is an isomorphism into $\operatorname{Ker}\partial$. Let $\iota^*\colon C^1(G)\to C^1(L)$ be the dual map of ι. Then the matrix B which represents the map ι^* with respect to the natural basis of $C^1(G)$ is called "the fundamental loop matrix".

Each chain $c\in C_1(G)$ is decomposed into

$$c=c_T+c_L,\ c_T\in C_1(T),\ c_L\in C_1(L).$$

Then, it is easily seen that the chain $c-\iota(c_L)$ belongs to $C_1(T)$. Therefore c is decomposed into

$$c=c_1+c_2,\ c_1=\iota(c_L)\in\operatorname{Ker}\partial,\ c_2=c-\iota(c_L)\in C_1(T).$$

In fact, this gives the following direct sum decomposition,

$$C_1(G) = \operatorname{Ker} \partial + C_1(T) .$$

Let $p\colon C_1(G) \to C_1(T)$ be the projection along $\operatorname{Ker} \partial$, (i.e. $p(\operatorname{Ker} \partial)=0$) or $p(c)=c_2$. Then the matrix Q which represents the map p with respect to the natural basis of $C_1(G)$ is called "the fundamental cut-set matrix".

Now, we get the following sequences of maps.

$$\begin{array}{ccccccccc} 0 \to & C_1(L) & \xrightarrow{\iota} & C_1(G) & \underset{Q}{\xrightarrow{p}} & C_1(T) & \to 0 \\ & \| & & \| & & & \\ 0 \to & C_1(L) & \xrightarrow{\iota} & C_1(G) & \xrightarrow{\partial} & C_0(G) , & \end{array}$$

$$\begin{array}{ccccccccc} 0 \leftarrow & C^1(L) & \underset{B}{\xleftarrow{\iota^*}} & C^1(G) & \xleftarrow{p^*} & C^1(T) & \leftarrow 0 \\ & \| & & \| & & & \\ 0 \leftarrow & C^1(L) & \xleftarrow{\iota^*} & C^1(G) & \xleftarrow{\partial^*} & C^0(G) . & \end{array}$$

Since the row of the above diagrams are exact, Kirchhoff's law can be represented as follows,

$$\text{(KCL)} \quad i \in \operatorname{Ker} p , \qquad \text{(KVL)} \quad v \in \operatorname{Ker} \iota^* ,$$

or

$$\text{(KCL)} \quad Qi=0 , \qquad \text{(KVL)} \quad Bv=0 .$$

By the way, $QB^t=0$ and $BQ^t=0$, because $p \circ \iota=0$ and $\iota^* \circ p^*=0$.

The Kirchhoff space K is

$$\operatorname{Ker} Q \times \operatorname{Ker} B (= \operatorname{Ker} \iota^* \times \operatorname{Ker} p),$$

and the map

$$\iota \times p^*\colon C_1(L) \times C^1(T) \to K$$

is an isomorphism. Therefore, the currents of link branches and the voltages of tree branches $(i_L, v_T) \in C_1(L) \times C^1(T)$ can be thought as coordinates of K.

3. Proof of Theorem.

Let T be a tree of G and L denote the link of T. Add a small capacitor in parallel to each branch of T and a small inductor in series to each branch of L, and we obtain the new circuit G'.

Put $G' = G \cup \bar{T} \cup \bar{L}$, where $\bar{T}(\bar{L})$ is the subset of the branches of G' consisting of the small capacitances (inductances) added. Since the nodes of $\bar{T} \cup L$ include all the nodes of G' and $\bar{T} \cup L$ has no loop, $T' = \bar{T} \cup L$ is a tree of G' and $L' = T \cup \bar{L}$ is the link of T'. (See Example.) Let $B=[A; I]$ be the fundamental loop matrix for G. Then it is easily verified that the fundamental loop matrix B'

for G' has the following form.

$$\begin{array}{c} \quad T' \;\; L' \\ B'=[A';\; I]\; L' \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc}\bar{T} & L & T & \bar{L}\end{array} \\ = & \left[\begin{array}{cc|c} I & O & \\ & ; & I \\ A & I & \end{array}\right]\begin{array}{c} T \\ \\ \bar{L}\end{array} \end{array}$$

Since we added only independent capacitors and inductors, the charactristic submanifold for G' is essentially unchanged.

For $x'=(x, y)\in\Lambda(G')=\Lambda(G)\times R^{2b}$, there exist a neighborhood U of x in R^{2b} and a smooth map $F\colon U\to R^{\rho}$ such that the rank of F at x is ρ and $U\cap\Lambda=F^{-1}(0)$. Let $F'\colon U\times R^{2b}\to R^{\rho}$ be the smooth map defined by $F'(x, y)=F(x)$ for $(x, y)\in U\times R^{2b}$. Then clearly the rank of F' at (x, y) is ρ and $U\times R^{2b}\cap\Lambda(G')=F'^{-1}(0)$.

Corresponding to the decomposition $G'=T'\cup L'=\bar{T}\cup L\cup T\cup\bar{L}$, we can write

$$\begin{aligned}(i', v')&\in C_1(G')\times C^1(G')\\ &=(i(T'),\ i(L'),\ v(T'),\ v(L'))\in C_1(T')\times C_1(L')\times C^1(T')\times C^1(L')\\ &=(i(\bar{T}),\ i(L),\ i(T),\ i(\bar{L}),\ v(\bar{T}),\ v(L),\ v(T),\ v(\bar{L}))\\ &\quad\in C_1(\bar{T})\times C_1(L)\times C_1(T)\times C_1(\bar{L})\times C^1(\bar{T})\times C^1(L)\times C^1(T)\times C^1(\bar{L})\ ,\end{aligned}$$

and the Jacobian matrix JF' of F' is also partitioned as follows.

$$\begin{aligned}JF'&=[J_{i'}F',\ J_{v'}F']\\ &=[J_{i(\bar{T})}F',\ J_{i(L)}F',\ J_{i(T)}F',\ J_{i(\bar{L})}F',\ J_{v(\bar{T})}F',\ J_{v(L)}F', J_{v(T)}F',\ J_{v(\bar{L})}F']\\ &=[0,\ J_{i(L)}F,\ J_{i(T)}F,\ 0,\ 0,\ J_{v(L)}F,\ J_{v(T)}F,\ 0]\ .\end{aligned}$$

Noting that the map

$$\iota'\times p'^{*}\colon\ C_1(L')\times C^1(T')\to K'$$

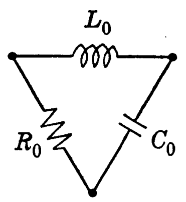
is a linear isomorphism, we get the following;

$$\begin{aligned}\operatorname{rank}(F'|_{K'})&=\operatorname{rank}(F'\circ(\iota'\times p'^{*}))\\ &=\operatorname{rank}(JF'\circ J(\iota'\times p'^{*}))\\ &=\operatorname{rank}\left([J_{i'}F',\ J_{v'}F']\begin{bmatrix}B'^{t} & 0\\ 0 & Q'^{t}\end{bmatrix}\right)\\ &=\operatorname{rank}\left([J_{i(T')}F',\ J_{i(L')}F',\ J_{v(T')}F',\ J_{v(L')}F']\begin{bmatrix} I & 0\\ A'^{t} & 0\\ 0 & -A\\ 0 & I\end{bmatrix}\right)\\ &=\operatorname{rank}\,[J_{i(T')}F'+(J_{i(L')}F')A'^{t},\ \ -(J_{v(T')}F')A+J_{v(L')}F']\end{aligned}$$

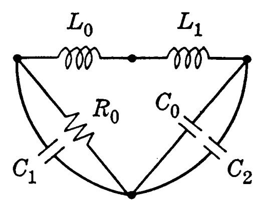
$$
\begin{aligned}
&= \operatorname{rank}\Big[(0, J_{i(L)}F) + (J_{i(T)}F, 0)\begin{bmatrix} I & 0 \\ A^t & I \end{bmatrix}, \\
&\qquad -(0, J_{v(L)}F)\begin{bmatrix} I & A^t \\ 0 & I \end{bmatrix} + (J_{v(T)}F, 0)\Big] \\
&= \operatorname{rank}\,[J_{i(T)}F, J_{i(L)}F, J_{v(T)}F, -J_{v(L)}F)] \\
&= \operatorname{rank}\,[J_iF, J_vF] \\
&= \operatorname{rank}\, JF \\
&= \rho\,.
\end{aligned}
$$

This proves the theorem.

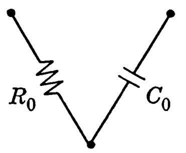
Example.



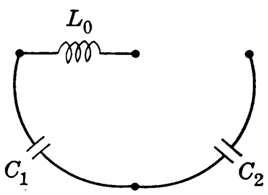
$G=\{R_0, L_0, C_0\}$



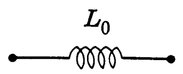
$G'=\{R_0, L_0, L_1, C_0, C_1, C_2\}$
$\bar{T}=\{C_1, C_2\}$, $\bar{L}=\{L_1\}$



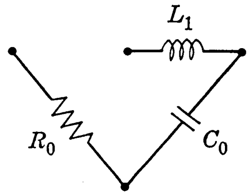
$T=\{R_0, C_0\}$



$T'=\bar{T}\cup L$



$L=\{L_0\}$



$L'=T\cup\bar{L}$

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