A RENEWAL THEOREM FOR PROCESSES WITH STATIONARY INCREMENTS

By

TOSHIO MORI

(Received April 13, 1976)

1. Introduction. This note is a supplement to the previous paper [4] and is concerned with a renewal theorem for processes with stationary increments. Let V be the reduced second moment measure of a stationary random measure P on the real line and let P^t be the shift of the Palm measure P^0 of P. The main theorem of the above paper asserts that P^t converges weakly at $t\to\infty$ iff the shift of V converges vaguely to a scalar multiple of the Lebesgue measure. In the present note we apply this theorem to a process ξ with stationary increments and obtain a weak convergence result for "renewal process" induced from ξ. This is a continuous time version of results in § 6 of [4]. The result is applied to transient nonarithmetic Lévy processes.

2. Lemmas. Let $\mathscr{R}$ denote the σ-algebra of Borel sets of the real line R and let λ denote the Lebesgue measure on $\mathscr{R}$. The set of all locally finite nonnegative measures φ on $\mathscr{R}$ is denoted by M. Endowed with the vague topology M is a Polish space. The σ-algebra of Borel sets of M is denoted by $\mathscr{M}$. This is the smallest σ-algebra with respect to which every mapping $\varphi\to\varphi(A)$, $A\in\mathscr{R}$, is measurable. Let T_t, $t\in R$, be an automorphism of M defined by

$$(T_t\varphi)(A)=\varphi(A+t)\ ,\quad A\in\mathscr{R}\ ,\quad \varphi\in M\ .$$

The mapping $(t,\varphi)\to T_t\varphi$ is $\mathscr{R}\times\mathscr{M}/\mathscr{M}$-measurable. A measure Q on $\mathscr{M}$ is called stationary if $QT_t^{-1}=Q$ for $t\in R$.

Let X be the space of all right-continuous functions x: $R\to R$ having left hand limits satisfying $x(0)=0$ and $\lambda x^{-1}\in M$, where λx^{-1} is the measure induced from λ by the mapping x. The smallest σ-algebra of subsets of X with respect to which every coordinate mapping $x\to x(t)$, $t\in R$, is measurable is denoted by $\mathscr{X}$. Let θ_t, $t\in R$, be an automorphism of X defined by

$$(\theta_t x)(u)=x(t+u)-x(t)\ ,\quad x\in X\ ,\quad u\in R\ .$$

A measure P on $\mathscr{X}$ is called stationary if $P\theta_t^{-1}=P$ for $t\in R$.

Define a mapping τ from X to M by $\tau(x)=\lambda x^{-1}$. Then τ is $\mathscr{X}/\mathscr{M}$-measurable. In fact the mapping $x \to \int_{-\infty}^{\infty} f(x(u))du$ is $\mathscr{X}$ measurable for every continuous function f having compact support.

Lemma 1. *If P is a σ-finite stationary measure on $(X, \mathscr{X})$ then there exists a unique σ-finite stationary measure Q on $(M, \mathscr{M})$ such that $Q(\{0\})=0$ and the Palm measure for Q coincides with $P\tau^{-1}$.*

Proof. By Satz 2.5 of Mecke [3] it suffices to show that for every $\mathscr{R} \times \mathscr{M}$-measurable $v \geq 0$

$$\iint v(-t, T_t\varphi)\varphi(dt)P\tau^{-1}(d\varphi) = \iint v(t, \varphi)\varphi(dt)P\tau^{-1}(d\varphi) . \tag{1}$$

The integral on the left of (1) is written as

$$\iint v(-t, T_t\tau x)\lambda x^{-1}(dt)P(dx) = \iint v(-x(s),\ T_{x(s)}\tau x)dsP(dx) .$$

For $A \in \mathscr{R}$ we have $(T_{x(s)}\tau x)(A) = \lambda(x^{-1}(A+x(s))) = \lambda((\theta_s x)^{-1}(A))$. Hence

$$T_{x(s)}\tau x = \tau\theta_s x . \tag{2}$$

Thus by putting $y=\theta_s x$ it is seen that the above integral is equal to

$$\iint v(-x(s),\ \tau\theta_s x)P(dx)ds = \iint v(y(-s),\ \tau y)P(dy)ds$$
$$= \iint v(y(s), \tau y)dsP(dy) = \iint v(t, \tau y)\lambda y^{-1}(dt)P(dy) .$$

Lemma 2. *In Lemma* 1 *if P is ergodic with respect to θ_t then Q is ergodic with respect to T_t.*

Proof. Suppose P is ergodic and $A \in \mathscr{M}$ is T_t-invariant. Then in view of (2) $\tau^{-1}A \in \mathscr{X}$ is θ_t-invariant. Hence either $P(\tau^{-1}A)=0$ or $P(X \backslash \tau^{-1}A)=0$. It follows from Lemma 1 and the definition of Palm measure [3] that

$$P(\tau^{-1}A) = \iint \chi_{(0,1]}(s)\chi_A(T_s\varphi)\varphi(ds)Q(d\varphi) = \int_A \varphi(0, 1]Q(d\varphi) ,$$

where χ_A is the indicater of A. The last equality follows from the T_t-invariance of A. Suppose $P(\tau^{-1}A)=0$. Then $\varphi(0,1]=0$ Q-a.e. on A and therefore by the stationarity of Q we have $Q(A \backslash \{0\})=0$. Since $Q(\{0\})=0$ we have $Q(A)=0$. Similarly $P(X \backslash \tau^{-1}A)=0$ implies $Q(M \backslash A)=0$. This shows that Q is ergodic.

3. Renewal theorem. Let $\xi = \{\xi(t),\ t \in R\}$ be a stochastic process with stationary increments defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. Suppose $\xi(0)=0$

and almost every sample path of ξ is in X. Then ξ is an X-valued random element and the distribution $P=\mathbf{P}\xi^{-1}$ of ξ is a stationary probability measure on $(X, \mathscr{X})$. Define M-valued random elements Φ and Φ_t, $t \in R$, by $\Phi=\tau(\xi)$ and $\Phi_t=T_t\Phi$. By definition $\Phi(A)=\lambda\{s;\ \xi(s) \in A\}$ and $\Phi_t(A)=\Phi(A+t)$, $A \in \mathscr{R}$. By Lemma 1 the distribution $\mathbf{P}\Phi^{-1}=P\tau^{-1}$ of Φ is the Palm measure for a stationary probability measure Q on $(M, \mathscr{M})$. Let V denote the reduced second moment measure of Q:

$$V(A)=\mathbf{E}\Phi(A)=\int_{-\infty}^{\infty} \mu_t(A)dt$$

where $A \in \mathscr{R}$ and $\mu_t(A)=\mathbf{P}\{\xi(t) \in A\}$ (see [4]).

Let us define a measure V_β, $\beta>0$, by

$$V_\beta(A)=\int_{-\infty}^{\infty} e^{-\beta|s|}\mu_s(A)ds\ , \quad A \in \mathscr{R}\,.$$

Throughout the rest suppose $V \in M$. Then V is a positive, positive definite measure (see [6]). Let $\hat{V}$ and $\hat{V}_\beta$ denote the Fourier transforms of V and V_β resp. Then $\hat{V}_\beta$ is a locally finite signed measure. In fact

$$\hat{V}_\beta(A)=\int_A \hat{v}_\beta(t)dt\ , \quad A \in \mathscr{R}\,,$$

where

$$\hat{v}_\beta(t)=\int_{-\infty}^{\infty} \varphi_s(t)e^{-\beta|s|}ds= 2\int_0^{\infty} \mathrm{Re}\, \varphi_s(t)e^{-\beta s}ds\ , \tag{3}$$

and

$$\varphi_s(t)=\int_{-\infty}^{\infty} e^{itu}\mu_s(du)\,.$$

Since $V=\lim_{\beta\to+0} V_\beta$ vaguely we have $\hat{V}=\lim_{\beta\to+0} \hat{V}_\beta$ vaguely. Therefore if the family of functions $\{\hat{v}_\beta, \beta>0\}$ converges a.e. as $\beta\to+0$ and if it is uniformly bounded on every compact interval excluding the origin then $\hat{V}$ is absolutely continuous except for a possible atom at the origin. Thus we have from Theorem 3.2 and Theorem 4.1 of [4] the following result which is a continuous time version of Theorem 6.1 and Theorem 6.2 of [4].

Theorem 1. *Let $\xi=\{\xi(t),\ t \in R\}$ be a process with stationary increments having right-continuous paths with left limits and satisfying $\xi(0)=0$. Suppose $V \in M$. If the family of functions $\{\hat{v}_\beta, \beta>0\}$ converges a.e. as $\beta\to+0$*

and is uniformly bounded on every compact interval excluding the origin then

$$\lim_{t\to\infty} \mathrm{E}\Phi_t(I) = \lim_{t\to\infty} V(I+t) = \hat{V}(\{0\})\lambda(I) \tag{4}$$

for every bounded interval I. Moreover Φ_t converges in distribution to a stationary probability measure Q^∞ as $t\to\infty$. If in addition P is ergodic then Q^∞ is equal to either $Q(M)^{-1}Q$ or δ_0 according as $\mathrm{E}|\xi(1)|$ is finite or not, where δ_0 is the probability measure concentrated on $0\in M$.

If ξ is a transient Lévy process on R then ξ satisfies preceding assumptions. In particular $V\in M$. In this case

$$\varphi_s(t) = e^{sg(t)}\ , \quad s\geq 0\ ,$$

where

$$g(t) = i\alpha t - \frac{\sigma^2}{2}t^2 + \int_{R\setminus\{0\}}\left[e^{i\lambda t} - 1 - \frac{i\lambda t}{1+\lambda^2}\right]\nu(d\lambda)$$

with $\alpha\in R$, $\sigma^2\geq 0$, and ν being a Borel measure on $R\setminus\{0\}$ satisfying

$$\int_{R\setminus\{0\}} \frac{\lambda^2}{1+\lambda^2}\nu(d\lambda) < \infty\ .$$

Therefore (3) is written as

$$\hat{v}_\beta(t) = 2\,\mathrm{Re}\,\frac{1}{\beta+g(t)}\ .$$

A Lévy process ξ is called nonarithmetic if either ν is not arithmetic (i.e. is not supported by a proper closed subgroup of R) or $\sigma^2\neq 0$ or $\alpha - \int_{R\setminus\{0\}} \lambda/(1+\lambda^2)\nu(d\lambda)\neq 0$. If ξ is nonarithmetic then the continuous function g satisfies $g(t)\neq 0$ for $t\neq 0$ and therefore $\{\hat{v}_\beta, \beta>0\}$ satisfies the conditions in Theorem 1. By Lemma 2 the ergodicity of P implies that of Q. Thus from the first part of Theorem 1 we have (4) for bounded interval I. This result is included in the well-known renewal theorem for Lévy processes [1, 5]. From the second part of Theorem 1 we have the following corollary.

Corollary 1. *If $\{\xi(t),\ t\in R\}$ is a transient nonarithmetic Lévy process then as $t\to\infty$ Φ_t converges in distribution to either $(\mathrm{E}|\xi(1)|)^{-1}Q$ or δ_0 according as $\mathrm{E}|\xi(1)|$ is finite or not.*

A Lévy process ξ^+ on $[0,\infty)$ is obviously extended to a Lévy process ξ on R. Define Φ^+ and Φ_t^+ by $\Phi^+(A) = \lambda\{s;\ s\geq 0,\ \xi^+(s)\in A\}$ and $\Phi_t^+(A) = \Phi^+(A+t)$. Then we have the following corollary. This is proved using the renewal theorem for Lévy processes and the argument in the proof of Corollary 6.2 in [4]. In the case of subordinators this result was essentially obtained by Horowitz [2] in a different context.

Corollary 2. *If* $\{\xi^+(t),\ t\geq 0\}$ *is a transient nonarithmetic Lévy process then as* $t\to\infty$ Φ_t^+ *converges in distribution to either* $(\mathrm{E}\xi(1))^{-1}Q$ *or* δ_0 *according as* $0<\mathrm{E}\xi(1)<\infty$ *or not.*

References

[1] Herz, O. S.: *Les théorèmes de renouvellement.* Ann. Inst. Fourier (Grenoble) **15** (1), 169–188 (1965).

[2] Horowitz, J.: *Semilinear Markov Processes, subordinators and renewal theory.* Z. Wahrscheinlichkeitstheorie verw. Geb. **24**, 167–193 (1972).

[3] Mecke, J.: *Stationäre zufällige Masse auf lokalkompakten Abelschen Gruppen.* Z. Wahrscheinlichkeitstheorie verw. Geb. **9**, 36–58 (1967).

[4] Mori, T.: *Stationary random measures and renewal theory.* Yokohama Math. J. **23**, 31–54 (1975).

[5] Port, S. C., Stone, C. J.: *Infinitely divisible processes and their potential theory,* I. Ann. Inst. Fourier (Grenoble) **21** (2), 157–275 (1971).

[6] Vere-Jones, D.: *An elementary approach to the spectral theory of stationary random measures.* In: E. F. Harding and D. G. Kendall, eds., Stochastic Geometry. New York: Wiley 1974.

Department of Mathematics
Yokohama City University
4646 Mutsuura-cho, Kanazawa-ku
Yokohama 236, Japan