SINGULAR SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

By

MASAKO and TAKUO FUKUDA

(Received April 12, 1976)

0. Introduction

Consider a first order ordinary differential equation

$$f(x, y, y')=0\ , \tag{1}$$

where f is a C^∞-function, y is the unknown function of x and $y'=dy/dx$. We know classically that if the equation (1) admits a general solution of the form

$$G(x, y;\ c)=0 \tag{2}$$

and if (1) has a singular solution, then the singular solution is the envelope of the family of curves defined by the general solution (2) moving the parameter c, and the envelope is obtained by elimination of the variable y' from

$$f(x, y, y')=0 \quad \text{and} \quad \partial f/\partial y'(x, y, y')=0\ . \tag{3}$$

On the other hand, we know by experience, generically equation (1) has no singular solutions. Here, we are interested in how the solutions of (1) and the discriminant set D obtained by elimination of y' from (3) are geometrically related.

In [9], Thom showed the following;

Theorem (*R. Thom*). *Almost every equation* (1) *has no singular solutions and the discriminant set D is the "pseudoenvelope" of the general solutions; that is D consists of the singular points of the general solutions.*

Here we say a property holds generically or for almost every equation (1) if it holds for every elements f of some open and dense set of the space of all C^∞-functions with the Whitney C^∞-topology.

The purpose of this paper is to show that we are exactly in a similar situation also for differential equations of arbitrary n-th order

$$f(x, y, y', y'', \cdots, y^{(n)})=0 \text{ with } y^{(i)}=d^iy/dx^i\ , \tag{4}$$

that is, *almost every equation* (4) *has no singular solutions and the discriminant set D obtained by elimination of $y^{(n)}$ from* (4) *and $\partial f/\partial y^{(n)}=0$ consists* *of the singular points of the general solutions.*

For the precise definition of *singular points of a general solution*, see § 1 Definitions 1.3 and 1.5.

Moreover, we can describe more precisely how the solutions of (4) and the discriminant set D are geometrically related. We state here a result only for the simplest type called "fold type" of the equation (4). For the other cases, the result is more complicated in its expression and it needs some new definitions, so we do not state it here. (See the final section.)

Let

$$f(x, y_0, y_1, \cdots, y_{n-1}, z)=0 \tag{5}$$

be an ordinary differential equation, where $y_0=y$ is the unknown function of x, $y_i=d^iy/dx^i$ and $z=d^ny/dx^n$. The hypersurface $f^{-1}(0)\subset R^{n+2}$ is denoted by S and the canonical projection of R^{n+2} onto R^{n+1}, $(x, y_0, y_1, \cdots, y_{n-1}, z)\to(x, y_0, y_1, \cdots, y_{n-1})$, is denoted by π. Let D be the set of critical values of $\pi|S$: $S\to R^{n+1}$. Then for almost C^∞ function f D is obtained by elimination of z from the equations $f=0$ and $\partial f/\partial z=0$ (see § 1 Proposition 1.6).

For a solution $y=\gamma(x)$ of equation (5), consider the curve in R^{n+1} defined by

$$\tilde{\gamma}(x)=\left(x, \gamma(x), \frac{d\gamma}{dx}(x), \cdots, \frac{d^{n-1}\gamma}{dx^{n-1}}(x)\right).$$

We call the curve $\tilde{\gamma}$ in R^{n+1} also a solution of (5). Now we can state a part of our results. Equation (5) is called of *fold type* if $\partial f/\partial z(0)=0$, $\partial^2 f/\partial z^2(0)\neq 0$ and $0\in S=f^{-1}(0)$.

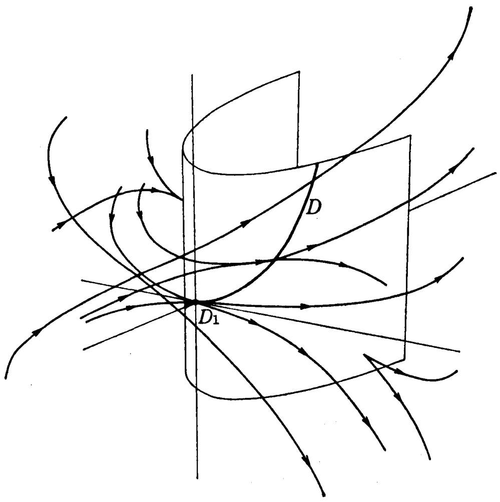

Fig. (0)

A special case of the main theorem. For almost every equation (5) *of fold type, the following properties hold:*

(i) *D is a submanifold of R^{n+1} of codimension* 1.

(ii) *The set D_1 of points $q\in D$ such that there is a solution of* (5) *which passes through q and is tangent to D at q is a submanifold of D of codimension* 1.

(iii) *Inductively, the set D_i of points $q\in D_{i-1}$ such that there is a solution of* (5) *which passes through q and tangent to*

D_{i-1} at q is a submanifold of D_{i-1} of condimension 1.

The following example will help for us to understand the situation mentioned above.

Example. $y'^2-x^3y''+y=0$. (See Fig. (0))

Table of contents

1. General solutions and singular solutions.

In this section, "general solutions", "singular solutions" and "singular points of general solutions" are defined.

Definition 1.1. A family of solutions of n-th order differential equation (5) is called a *general solution* of (5) if it is defined by an equation of the form

$$G(x, y, c_1, \cdots, c_n)=0\ , \tag{6}$$

$c=(c_1, \cdots, c_n)$ being n arbitrary constants and G being a smooth function defined on an open subset U of $R^2\times R^n$, and if it is never defined by any equation with $(n-1)$ arbitrary constants.

Definition 1.2. A solution of (5) contained in a general solution is called a *regular* (or a *special*) *solution* of (5). A solution of (5) which is never contained in any general solution is called a *singular solution*. A singular solution $y=\gamma(x)$ defined on an interval (a, b) is called a *singular solution in the strict sense* if for any subinterval (c, d) of (a, b), the restriction $y=\gamma|_{(c,d)}(x)$ is not a regular solution.

Definition 1.3. Let $y=\gamma(x)$ be a regular solution of (5) defined on an interval (a, b), and let Γ: $(a, b)\to R\times R$ be the map defined by $\Gamma(x)=(x, \gamma(x))$. A point of $\overline{\Gamma([(a+b)/2, b))}-\Gamma([(a+b)/2, b))$ (resp. a point of $\overline{\Gamma((a, (a+b)/2]))}-\Gamma((a, (a+b)/2]))$ is called a *singular point of a regular* (or *a general*) *solution* of (5) if $y=\gamma(x)$ cannot be extended to a regular solution defined on an interval $(a, b+\varepsilon)$ (resp. $(a-\varepsilon, b)$) for any $\varepsilon>0$.

Example 1.4. Consider the following two equations;

$$\text{(i)}\quad y-xy'+\frac{1}{2}y'^2=0 \quad \text{(Fig. (1.1))}$$

and

$$\text{(ii)}\quad x-y'^3/3+yy'=0\,. \quad \text{(Fig. (1.2))}$$

The first equation is known as of Clairaut type and its general solution is given by $y=cx-(1/2)c^2$. It has a singular solution in the strict sense which is given

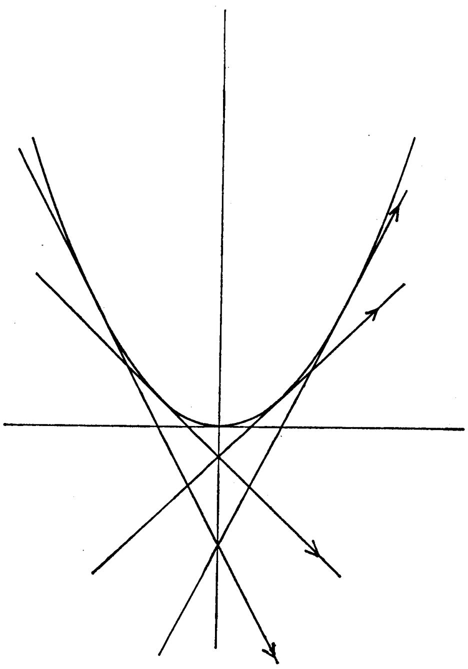

Fig. (1.1)

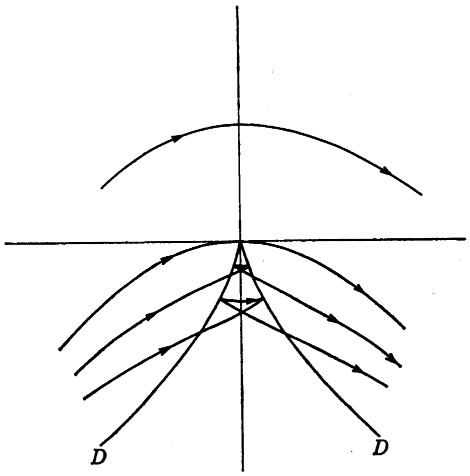

Fig. (1.2)

by elimination of y' from $y-xy'+(1/2)y'^2=0$ and $-x+y'=0$: the singular solution is $y=(1/2)x^2$. And the equation (i) has no singular points of the general solution.

On the other hand, the equation (ii) has no singular solutions and has many singular points of regular solutions. The set of singular points of the regular solutions is the discriminant set D given by elimination of y' from $x-y'^3/3+yy'=0$ and $-y'^2+y=0$: $D=\{9x^2=4y^3\}$.

It is convenient to define for a curve in $(x, y_0, y_1, \cdots, y_{n-1})$-plane to be a solution of (5).

Definition 1.5. A curve $\tilde{\gamma}$ in R^{n+1} is called a *solution* of (5) if $\tilde{\gamma}$ has the form

$$\tilde{\gamma}(x)=\left(x, \gamma(x), \frac{d\gamma}{dx}(x), \cdots, \frac{d^{n-1}\gamma}{dx^{n-1}}(x)\right) \tag{7}$$

for some solution $y=\gamma(x)$ of (5).

A family of solutions of (5) in $(x, y_0, y_1, \cdots, y_{n-1})$-plane is called *a general solution* of (5) if the corresponding family of solutions in (x, y)-plane is a general solution of (5).

For solutions of (5) in $(x, y_0, y_1, \cdots, y_{n-1})$-plane, *regular solutions* and *singular solutions* are similarly defined. Let $\tilde{\gamma}(x)=(x, \gamma(x), d\gamma/dx(x), \cdots, d^{n-1}\gamma/dx^{n-1}(x))$ be a regular solution of (5) defined on an interval (a, b). A point of $\overline{\gamma(a, b)}$ is *a singular point* of the regular solution $\tilde{\gamma}$ if the corresponding point in (x, y)-plane is a singular point of the corresponding regular solution $y=\gamma(x)$.

Let $S=f^{-1}(0)$ be the hypersurface of zeros of the function f in (5). We may assume $f(0)=0$ without loss of generality. By *Thom's* transversality theorem, for almost every equation, S is a smooth hypersurface. Let π: $R^{n+2}\to R^{n+1}$ be the canonical projection defined by $\pi(x, y_0, \cdots, y_{n-1}, z)=(x, y_0, y_1, \cdots, y_{n-1})$. Let C be the set of critical points of $\pi|S: S\to R^{n+1}$ and $D=\pi(C)$.

Then generically we may assume that for some $k>0$, we have $\partial f/\partial z(0)=\cdots=\partial^{k-1}f/\partial z^{k-1}(0)=0$ and $\partial^k f/\partial z^k(0)\neq 0$. Then by the Malgrange preparation f is of the form

$f(x, y, z)=(z^k+a_1(x, y)z^{k-1}+\cdots+a_k(x, y))Q(x, y, z)$ with $Q(0)\neq 0$. Then D is obtained by elimination of z from the equations $f=0$ and $\partial f/\partial z=0$, where y is an abbreviation of $(y_0, y_1, \cdots, y_{n-1})$. And from the fact that the roots of

$$z^k+a_1(x, y)z^{k-1}+\cdots+a_k(x, y)=0$$

are continuous functions of the coefficients, therefore of variables x and y, we know that $\pi|S: S\to R^{n+1}$ is a proper mapping.

Consequently, we have;

Proposition 1.6. *For almost every C^∞-function f, the following properties hold:*

(i) $S=f^{-1}(0)$ *is a smooth submanifold,*

(ii) D *is obtained by elimination of z from the equations $f=0$ and $\partial f/\partial z=0$,*

(iii) $\pi|S: S\to R^{n+1}$ *is a proper mapping,*

(iv) $\pi|(S-\pi^{-1}(D)): (S-\pi^{-1}(D))\to\pi(S)-D$ *is a covering map.*

Since $\pi|S: S\to R^{n+1}$ is proper, $f^{-1}(y)$ is a finite set for each point $y\in R^{n+1}$. Note that the degree of the covering may be different according to connected components of $\pi(S)-D$.

Proposition 1.7. *Let $q_0=(x^0, y_0^0, y_1^0, \cdots, y_{n-1}^0)\in R^{n+1}$ be a point of one of the* *connected components of* $\pi(S)-D$, say U. *Then there exist the same number of solutions of* (5) *passing through* q_0 *as the degree of the covering* $\pi|S\cap\pi^{-1}(U)$: $S\cap\pi^{-1}(U)\to U$. *Such solutions are all regular solutions.*

Proof. Since the covering map $\pi|S\cap\pi^{-1}(U)$: $S\cap\pi^{-1}(U)\to U$ is trivial, the restricted map $\pi|V$: $V\to U$ is a diffeomorphism for each connected component V of $S\cap\pi^{-1}(U)$. Therefore, for such a component V we can define a C^∞ vector field ξ_V on U by

$$\xi_V(p)=\partial/\partial x+\sum_{i=1}^{n=1} y_i\partial/\partial y_{i-1}+z((\pi|V)^{-1}(p))\partial/\partial y_{n-1}\,, \quad p=(x, y_0, y_1, \cdots, y_{n-1})\in U\,. \tag{8}$$

By the fundamental theorem of ordinary differential equations, there is unique integral curve of ξ_V passing through q_0, which is easily checked to be a regular solution of (5). So there exist a greater or equal number of regular solutions passing through q_0 than the number of the coverings of $\pi|S\cap\pi^{-1}(U)$: $S\cap\pi^{-1}(U)\to U$.

On the other hand, solutions of (5) passing through q_0 are all integral curves of one of such ξ_V's. For, consider a solution $\tilde{\gamma}(x)=(x, \gamma(x), d\gamma/dx(x), \cdots, d^{n-1}\gamma/dx^{n-1}(x))$ of (5) passing through q_0. Then the curve $\tilde{\tilde{\gamma}}(x)=(x, \gamma(x), d\gamma/dx(x), \cdots, d^{n-1}\gamma/dx^{n-1}(x), d^n\gamma/dx^n(x))$ is contained in some connected component V of $S\cap\pi^{-1}(U)$. We have $\pi(\tilde{\tilde{\gamma}})=\tilde{\gamma}$ and we see that $\tilde{\gamma}$ is an integral curve of ξ_V. So it is a regular solution of (5). Q.E.D.

As Corollary of Proposition 1.7, we have;

Corollary 1.8. *If* (5) *has a singular solution* (*in* R^{n+1}) *in the strict sense, then it is contained in* D.

Proof. Let $\tilde{\gamma}$: $(a, b)\to R^{n+1}$ be a singular solution in the strict sense, and suppose that $\tilde{\gamma}(a, b)\not\subset D$. Then for some $x_0\in(a, b)$ $q_0=\tilde{\gamma}(x_0)\notin D$, and so for some $\varepsilon>0$, $\tilde{\gamma}((x_0-\varepsilon, x_0+\varepsilon))\subset\pi(S)-D$. Then by Proposition 1.7, $\tilde{\gamma}(x_0-\varepsilon, x_0+\varepsilon)$ is a regular solution. Therefore, by definition, $\tilde{\gamma}$ is not a singular solution in the strict sense, which contradicts the hypotheses.

2. Characterizations of critical points.

Notation 2.1. Let $J^k(M, N)$ denote the k-jet space from a smooth manifold M to another manifold N. For a C^∞ map g: $M\to N$, let $j^k(g)$: $M\to J^k(M, N)$ denote the k-extension of g. For a non-negative integer i, let $\Sigma^i\subset J^k(M, N)$ denote the set of k-jets having rank exactly $m-i$, where $m=\dim. M$. More generally for a sequence $I=(i_1, \cdots, i_r)$ of non-negative integers, Σ^I denotes the Thom-Boardman singularity subset of type I in $J^k(M, N)$, $k \geq r$.

For simplicity, we abbreviate the symbol $\Sigma^{\overbrace{1,\cdots,1}^{r}}$ to Σ^{1^r} and $\Sigma^{\overbrace{1,\cdots,1}^{r},0}$ to $\Sigma^{1^r,0}$. Similarly, for a map g: $M \to N$, we abbreviate $j^k(g)^{-1}(\Sigma^{1^r})$ to $\Sigma^{1^r}(g)$ and $j^k(g)^{-1}(\Sigma^{1^r,0})$ to $\Sigma^{1^r,0}(g)$.

Definition 2.2. Define subsets T_{1^r} and $T_{1^r,0}$ in $J^k(R^{n+2}, R)$, $k>r\geq 0$, as follows: For a function g: $R^{n+2} \to R$ and a point $p \in R^{n+2}$, $j^k(g)(p) \in T_{1^r}$ if and only if $g(p)=\partial g/\partial z(p)=\cdots=\partial^r g/\partial z^r(p)=0$; $j^k(g)(p) \in T_{1^r,0}$ if and only if $j^k(g)(p) \in T_{1^r}$ and $\partial^{r+1}g/\partial z^{r+1}(p) \neq 0$.

By Definition, $T_{1^r,0} \subset T_{1^r}$.

Now let us return to equation (5). Let F: $R^{n+2} \to R^{n+2}$ be the map defined by

$$F(x, y, z)=(x, y, f(x, y, z)),$$

where f is the function in (5) and y is an abbreviation of $(y_0, \cdots, y_{n-1})$. Note that, by the definition of F, every singular point of F is of type Σ^1.

Proposition 2.3. *For a point p in R^{n+2}, the following conditions are equivalent;*

(2.3.1) *$p \in \Sigma^{1^r,0}(F) \cap S$ and $j^k(F)$ is transversal to $\Sigma^{1^r,0} \cap \pi_2^{-1}(R^{n+1} \times 0)$, where π_2: $J^k(R^{n+2}, R^{n+2}) \to R^{n+2}$ is the projection to the target space and $k>r$.*

(2.3.2) *$f(p)=\partial f/\partial z(p)=\cdots=\partial^r f/\partial z^r(p)=0$, $\partial^{r+1}f/\partial z^{r+1}(p) \neq 0$ and $(f, \partial f/\partial z, \cdots, \partial^r f/\partial z^r)$: $R^{n+2} \to R^{r+1}$ has rank $(r+1)$ at p.*

(2.3.3) *$p \in (j^k f)^{-1}(T_{1^r,0})$ and $j^k f$ is transversal to $T_{1^r,0}$ at p for $k>r$.*

Proof. The equivalence of (2.3.2) and (2.3.3) is obvious by definition 2.2. For the equivalence of (2.3.1) and (2.3.2), see for instance *Morin* [8].

Proposition 2.4. *If the k-extension $j^k f$ of a function f: $R^{n+2} \to R$ is transversal to every $T_{1^i,0}$, $0 \leqq i \leqq k$, then we have $\Sigma^{1^i,0}(\pi|S)=\Sigma^{1^i,0}(F) \cap S=(j^k f)^{-1}(T_{1^{i+1},0})$.*

Proof. We see first that $\Sigma^{1^i}(\pi|S)=(j^k f)^{-1}(T_{1^i})$ for each $i \leqq k$. We do it by induction. Since $j^k f$ is transversal to every $T_{1^i,0}$, $(j^k f)^{-1}(T_{1^i})$ is a submanifold of S of condimension i. As the first step of the induction, we have $\Sigma^1(\pi|S)=\{p \in S|$ rank $(\pi|S)=\dim. S-1\}=\{p \in S|\partial f/\partial z(p)=0\}$. Supposing $\Sigma^{1^i}(\pi|S)=(j^k f)^{-1}(T_{1^i})$, we show next that $\Sigma^{1^{i+1}}(\pi|S)=(j^k f)^{-1}(T_{1^{i+1}})$. Since $\Sigma^{1^i}(\pi|S)=(j^k f)^{-1}(T_{1^i})$ is a submanifold by the hypotheses of the induction, we have $\Sigma^{1^{i+1}}(\pi|S)=\{p \in \Sigma^{1^i}(\pi|S)|$ corank $\pi|\Sigma^{1^i}(\pi|S)=1\}$ by the definition of the Thom-Boardman singularities. So, we have

$$\Sigma^{1^{i+1}}(\pi|S)=\{p\in S|\partial f/\partial z(p)=\partial^2 f/\partial z^2(p)=\cdots=\partial^i f/\partial z^i(p)=\partial^{i+1}f/\partial z^{i+1}(p)=0\}$$
$$=(j^k f)^{-1}(T_{1^{i+1}})\ .$$

On the other hand, by Proposition 2.3 we have

$(j^k f)^{-1}(T_{1^{i+1},0})=\Sigma^{1^i,0}(F)\cap S$. Hence $\Sigma^{1^i,0}(\pi|S)=(j^k f)^{-1}(T_{1^{i+1},0})=\Sigma^{1^i,0}(F)\cap S$.

Q.E.D.

By the transversality theorem, for almost every map f: $R^{n+2}\to R$, $j^k f$ is transversal to every $T_{1^i,0}(k>i\geqq 0)$. Taking $k>n+1$, we have the following.

Proposition 2.5. *For almost every ordinary differential equation* (5), *there exist Thom-Boardman stratifications* $\{S_{i,0}\}$ *of* C *satisfying the following conditions;*

(2.5.1) $C=\bigcup_{i=1}^{n+1} S_{i,0}$*(disjoint sum), where* $S_{i,0}=\Sigma^{1^i,0}(\pi|S)=\Sigma^{1^i,0}(F)\cap S$,

(2.5.2) $\pi|S_{i,0}$: $S_{i,0}\to R^{n+1}$ *is an immersion.*

Remark 2.6.1 In general, for a map f: $M\to N$ the decomposition of M into the *Thom-Boardman* singularities $\Sigma^I(f)$ does not give a *Whitney* stratification of M. However, the decomposition of C into $S_{i,0}$ is a *Whitney* stratification (see the *Morin*'s canonical forms [8]).

Remark 2.6.2 For a point p of $S_{1,0}$ it is known that there is a neighborhood U of p such that $\pi|S\cap U$: $S\cap U\to R^{n+1}$ is of the form

$$\begin{cases} v_j=u_j & 1\leq j\leq n \\ v_{n+1}=u^2_{n+1} & \end{cases}$$

under some local coordinate system $(u_1,\cdots,u_{n+1})$ on $S\cap U$ and some local coordinate system $(v_1,\cdots,v_{n+1})$ on a neighborhood of $\pi(p)$ (see for instance Morin [8]). Hence $\pi|S_{1,0}\cap U$ is an immersion, $\pi(S\cap U, S_{1,0}\cap U)$ is diffeomorphic to $(R^n\times[0,\infty), R^n)$ and as a consequence, the image $\pi(p)$ can not be an interior point of $\pi(S\cap U)$

3. Conditions necessary for a solution to be singular.

Let $C=\bigcup_{i=1}^{n+1} S_{i,0}$ be the Thom-Boardman stratification of C obtained by Proposition 2.5. Set $S_i=\Sigma^{1^i}(\pi|S)$. Then $S_i=\overline{S_{i,0}}$, $j^k f(S_{i,0})\subset T_{1^i,0}$ and $S_{i,0}$ is a submanifold of S_{i-1} of codimension 1, where as a convention we set $S_0=S$.

Let p_0 be a point of $S_{i,0}$. Since $\pi|S_{i,0}$: $S_{i,0}\to R^{n+1}$ is an immersion, there is a neighborhood U of p_0 in R^{n+2} such that $\pi|U\cap S_{i,0}$ is an embedding. Then $\pi(U\cap S_{i,0})$ is a submanifold of R^{n+1}.

Definition 3.1. Let $\tilde{\tilde{\xi}}$ be the vector field on R^{n+2} defined by

$$\tilde{\tilde{\xi}}(x, y_0, y_1, \cdots, y_{n-1}, z)=\partial/\partial x+y_1\partial/\partial y_0+\cdots+y_{n-1}\partial/\partial y_{n-2}+z\partial/\partial y_{n-1}\,. \tag{9}$$

And let $\tilde{\xi}$ be the vector field on $\pi(S_{i,0}\cap U)$ defined by

$$\tilde{\xi}(\pi(p))=(d\pi)_p(\tilde{\tilde{\xi}}(p))=\partial/\partial x+y_1\partial/\partial y_0+\cdots+y_{n-1}\partial/\partial y_{n-2}+z\partial/\partial y_{n-1} \tag{10}$$

where $p=(x, y_0, \cdots, y_{n-1}, z)\in S_{i,0}\cap U$.

Remark 3.2. In general $\tilde{\xi}$ is not tangent to $\pi(S_{i,0}\cap U)$.

Since $\pi|S_{i,0}$: $S_{i,0}\to R^{n+1}$ is an immersion, we have $(\partial/\partial z)_p\notin T_p(S_{i,0})$ for each point $p\in S_{i,0}$. On the other hand, since $\pi|S_{i-1}$: $S_{i-1}\to R^{n+1}$ is not regular at every point of $S_{i,0}$, we have $(\partial/\partial z)_p\in T_p(S_{i-1})$ for every point p of $S_{i,0}$. From these facts and the fact that $S_{i,0}$ is a submanifold of S_{i-1}, there is a normal bundle

$$\nu\colon N\cap S_{i-1}\to S_{i,0}$$

such that each fibre contains the z-direction, that is $(\partial/\partial z)_p\in T_p(\nu^{-1}(p))$ for each point $p\in S_{i,0}$, where N is a neighborhood of $S_{i,0}$ in R^{n+2}.

Proposition 3.3. *Let V be a submanifold of $S_{i,0}\cap U$. Then for a point p of V, the following conditions are equivalent:*

(3.3.1) *$\tilde{\xi}$ is tangent to $\pi(V)$ at (p).*
(3.3.2) *$\tilde{\tilde{\xi}}$ is tangent to $\nu^{-1}(V)$ at p.*

Proof. From the fact that $\pi|V$: $V\to\pi(V)$ is a diffeomorphism and $\tilde{\xi}(\pi(p))=(d\pi)_p(\tilde{\tilde{\xi}}(p))$, (3.3.1) is obviously equivalent to the following condition:

(3.3.3) $\tilde{\tilde{\xi}}+\alpha\cdot\partial/\partial z$ is tangent to V at p for some real number α.

On the other hand, since $(\partial/\partial z)_p\in T_p(S_{i-1})-T_p(S_{i,0})$, $T_p(\nu^{-1}(V))$ is spanned by $T_p(V)$ and $(\partial/\partial z)_p$, from which the equivalence between (3.3.2) and (3.3.3) follows.

Q.E.D.

For a solution $\tilde{\gamma}(x)=(x, \gamma(x), d\gamma/dx(x), \cdots, d^{n-1}\gamma/dx^{n-1}(x))$ of (5) in R^{n+1}, $\tilde{\tilde{\gamma}}$ denotes the curve in R^{n+2} defined by

$$\tilde{\tilde{\gamma}}(x)=\left(x, \gamma(x), \frac{d\gamma}{dx}(x), \cdots, \frac{d^n\gamma}{dx^n}(x)\right).$$

Corollary 3.4. *If $\tilde{\gamma}(x)$ is a solution of (5) such that the curve $\tilde{\tilde{\gamma}}$ is contained in $U\cap S_{i,0}$, then the restricted vector field $\tilde{\xi}|\tilde{\gamma}$ and $\tilde{\tilde{\xi}}|\tilde{\tilde{\gamma}}$ are tangent to $\tilde{\gamma}$ and to $\nu^{-1}(\tilde{\gamma})$ respectively.*

Proof. Since $\tilde{\gamma}$ is a solution such that $\tilde{\tilde{\gamma}}$ is in $U\cap S_{i,0}$, for each point

$q=\tilde{\gamma}(x)=(x, y_0, \cdots, y_{n-1})$ of $\tilde{\gamma}$, $\tilde{\xi}(q)$ is tangent to $\tilde{\gamma}$. Hence the corollary follows from Proposition 3.2.

Remark 3.5. If $\tilde{\gamma}(x)$ is a singular solution of (5) in the strict sense, then $\tilde{\tilde{\gamma}}$ is contained in C (see Corollary 1.8). So some non-void open subset of $\tilde{\tilde{\gamma}}$ must be contained in $S_{i,0}$ for some $i>0$, and this is the case of the corollary.

4. Thom's observation on first order differential equations.

In this section, we recall *Thom*'s observation on first order differential equations. Consider a first order differential equation

$$f(x, y, y')=0 . \tag{1}$$

We may assume $f(0)=0$ without loss of generality. We observe the generic behaviour of (1) near the origin. By the transversality theorem, generically we have only two types: they are

fold type, i.e. $\Sigma^{1,0}$-type: $f(0)=\partial f/\partial y'(0)=0$, $\partial^2 f/\partial y'^2\neq 0$,

cusp type, i.e. $\Sigma^{1,1,0}$-type: $f(0)=\partial f/\partial y'(0)=\partial^2 f/\partial y'^2(0)=0$, $\partial^3 f/\partial y'^3(0)\neq 0$.

By means of the transversality theorem, as fold types, we have the following two types:

fold type A. $f(0)=\partial f/\partial y'(0)=0$, $\partial f/\partial x(0)\neq 0$, $\partial^2 f/\partial y'^2(0)\neq 0$,

fold type B. $f(0)=\partial f/\partial y'(0)=\partial f/\partial x(0)=0$, $\partial^2 f/\partial y'^2(0)\neq 0$, $\partial f/\partial y(0)\neq 0$.

4.1. Fold type A.

Since $\partial f/\partial x(0)\neq 0$, by the Malgrange-preparation theorem, f has the following form:

$$f(x, y, y')=(x+a(y, y'))Q(x, y, y') \tag{11}$$

where a and Q are C^∞-functions with $Q(0)\neq 0$, $\partial a/\partial y'(0)=0$ and $\partial^2 a/\partial y'^2(0)\neq 0$. Hence, we may assume f has the following form:

$$f(x, y, y')=x+a(y, y') \tag{12}$$

with $a(0)=\partial a/\partial y'(0)=0$ and $\partial^2 a/\partial y'^2(0)\neq 0$.

Note that in the case of fold types, the discriminant set $D=\pi(C)$ is a smooth

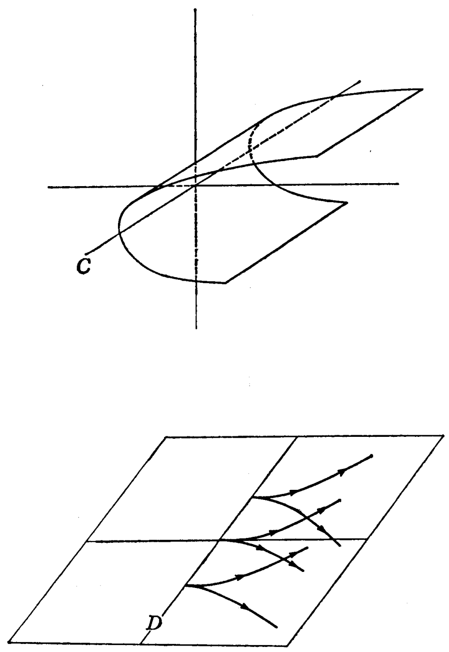

Fig. (4.1)

curve in $R^2(x, y)$. Now consider the vector field ξ and $\tilde{\tilde{\xi}}$ defined by (9) and (10) in §3. Then we have, by Proposition 3.2, that for a point $p \in C$, ξ is tangent to D at $\pi(p)$ if and only if $\tilde{\tilde{\xi}}$ is tangent to S at $p \in C$, hence, if and only if $\tilde{\tilde{\xi}}(p)(f)=(\partial/\partial x+y'\partial/\partial y)(x+a)=1+y'\partial a/\partial y(y, y')=0$. But this equation does not hold near $0 \in R^2$. That is to say, the vector field ξ is transversal to D near 0. Hence, from this fact and Remark 2.6.2, any solution $\tilde{\gamma}$ reaching to D can not be extended beyond D.

Moreover, since ξ is transversal to D, equations of fold type A have no singular solutions by Corollary 3.4 and Remark 3.5. Hence $D=\pi(C)$ consists of singular points of general solutions.

Examples. The local model of fold type A is given by the equation

$$x-y'^2=0 . \quad \text{(Fig. (4.1))}$$

4.2. Fold type B.

In this case, since $\partial f/\partial y(0)\neq 0$, using the Malgrange preparation theorem, we may assume f has the following form:

$$f(x, y, y')=y+a(x, y') \tag{13}$$

with $a(0)=\partial a/\partial y'(0)=\partial a/\partial x(0)=0$ and $\partial^2 a/\partial y'^2(0)\neq 0$.

Consider the vector field ξ and $\tilde{\tilde{\xi}}$. By proposition 3.2, we have that ξ is tangent to D at a point $\pi(p)$, $p \in C$, if and only if $\tilde{\tilde{\xi}}$ is tangent to S at $p \in C$, hence if and only if

$$\tilde{\tilde{\xi}}(p)(f)=y'+\partial a/\partial x(x, y')=0 .$$

On the other hand, by the transversality theorem, near the origin, this equation holds at no points of C but the origin. Consequently, near the origin, ξ is tangent to D at the origin and is transversal to D at other points of D. So, from Corollary 3.3 and Remark 3.4, equations of fold type B have no singular solutions. Moreover, from Remark 2.6.2 and the fact that ξ is transversal to D except at the origin, any special solution cannot be extended beyond D as a

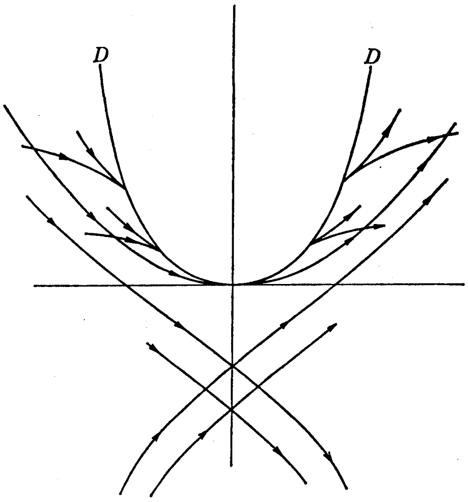

Fig. (4.2)

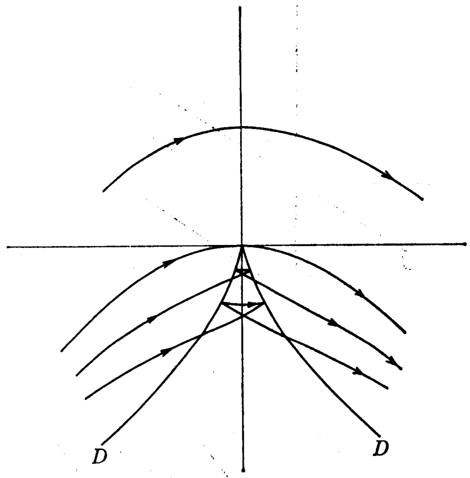

Fig. (4.3)

special solution. Hence D consists of singu- lar points of general solutions.

Examples. The local model of fold type B is given by the equation $y-x^2+y'^2=0$. (Fig. (4.2))

4.3. Cusp type.

In this case, we have $f(0)=\partial f/\partial y'(0)=\partial^2 f/\partial y'^2(0)=0$ and $\partial^3 f/\partial y'^3(0)\neq 0$. By the transversality theorem, we have generically $\partial f/\partial x(0)\neq 0$. Hence, using the Malgrange-preparation theorem, we may assume that f is of the form

$$f(x, y, y')=x+a(y, y') \tag{14}$$

with $a(0)=\partial a/\partial y'(0)=\partial^2 a/\partial y'^2(0)=0$ and $\partial^3 a/\partial y'^3(0)\neq 0$.

By the same argument as in 4.2, we see that the vector field $\tilde{\xi}$ does not tangent to $D-\{0\}$ at any point of $D-\{0\}$, and that equations of cusp types have no singular solutions and D consists of singular points of general solutions. The following example gives a picture of the generic situation for the cusp type.

Example.

$$x-y'^3/3+yy'=0\ . \quad \text{(Fig. (4.3))}$$

5. Fold type equations.

In this section, we consider equations (5) of fold types, that is, equations

$$f(x, y_0, y_1, \cdots, y_{n-1}, z)=0 \tag{15}$$

with $f(0)=\partial f/\partial z(0)=0$ and $\partial^2 f/\partial z^2(0)\neq 0$, which make clear what our theorem, stated in § 7, asserts.

Since $\partial f/\partial z(0)=0$ and $\partial^2 f/\partial z^2(0)\neq 0$, we can assume by the preparation theorem that f is of the form

(16) $$f(x, y, z)=z^2+a(x, y)z+b(x, y)$$

with $a(0)=b(0)=0$ and $db(0)\neq 0$, where $y=(y_0, y_1, \cdots, y_{n-1})$.

From (16), we have

$$C=\{(x, y, z)|f(x, y, z)=0 \text{ and } z=-1/2\cdot a(x, y)\} .$$

Eliminating the variable z from

(17) $$f(x, y, z)=0 \text{ and } \partial f/\partial z(x, y, z)=0 .$$

we have

$$D=\pi(C)=\{(x, y)\in R^{n+1}|4b(x, y)-a^2(x, y)=0\} .$$

We define inductively a sequence of submanifolds of $J^k(R^{n+1}, R^2)$

$$J^k(R^{n+1}, R^2)\supset T_1\supset T_2\supset\cdots\supset T_i\supset\cdots$$

as follows:

Definition 5.1. For a map $f=(f_1, f_2)$: $R^{n+1}\to R^2$ and a point $p\in R^{n+1}$, $j^k(f_1, f_2)(p)\in T_1$ if and only if $(4f_2-f_1^2)(p)=0$ and $\xi_f(4f_2-f_1^2)(p)=0$, where ξ_f is the vector field of R^{n+1} defined by

(18) $$\xi_f(x, y)=\partial/\partial x+y_1\partial/\partial y_0+\cdots+y_{n-1}\partial/\partial y_{n-2}-1/2\cdot f_1(x, y)\partial/\partial y_{n-1}$$

Then T_1 is a submanifold of $J^k(R^{n+1}, R^2)$ of codimension 2.

Definition 5.2. Suppose that we have already defined the submanifold T_i, then we define T_{i+1} by $j^k f(p)=j^k(f_1, f_2)(p)\in T_{i+1}$ if and only if

(19) $$j^k(f_1, f_2)(p)\in T_i$$
$$\xi_f^{i+1}(4f_2-f_1^2)(p)=\overbrace{\xi_f\xi_f(\cdots}^{i+1}(\xi_f(4f_2-f_1^2))\cdots)(p)=0$$

Then T_{i+1} is a submanifold of T_i of codimension 1.

Now assume that the k-extension of (f_1, f_2) is transversal to all T_i. Set $D_1=(j^k(f_1, f_2))^{-1}(T_1)$, then by the definition of T_1, we have $D_1=\emptyset$ or

(1) D_1 is a submanifold of $D=\pi(C)$ of codimension 1,

(2) D_1 coincides with the set of points of D at which the vector field ξ_f is tangent to D.

Inductively, set $D_i=(j^k(f_1, f_2))^{-1}(T_i)$, then we have $D_i=\emptyset$ or

(1) D_i is a submanifold of D_{i-1} of codimension 1,

(2) D_i coincides with the set of points of D_{i-1} at which the vector field $\tilde{\xi}_f$ is tangent to D_{i-1}.

By the preparation theorem and the transversality theorem, we see that almost every function f defining a fold type eqnation (5) has the form

$$f(x, y, z)=z^2+a(x, y)z+b(x, y)\ , \tag{16}$$

and for almost every function f, the pair (a, b) of the functions a and b in (16) is transversal to all T_i. Then from the above argument we have:

Proposition 5.3. (*A special case of our theorem*). *For almost every equation* (5) *of fold type, the following properties hold:*

(i) *The discriminant set D is a submanifold of R^{n+1} of codimension* 1.

(ii) *The set D_1 of points q of D such that there is a solution passing through q and tangent to D at q is a submanifold of D of codimension* 1, *or* $D_1=\emptyset$.

(iii) *Inductively, the set D_i of points q of D_{i-1} such that there is a solution of* (5) *passing through q and tangent to D_{i-1} at q is a submanifold of D_{i-1} of codimension* 1, *or* $D_i=\emptyset$.

Proof. The set C and D are nothing but $S_{1,0}$ and $\pi(S_{1,0})$ observed in §2 and §3. And the vector field $\tilde{\xi}_f|D$ is nothing but the vector field $\tilde{\xi}$ defined in Definition 3.1. Let $\tilde{\gamma}$ be a solution of (5) passing through a point q of D_{i-1}, that is $\tilde{\gamma}(x_0)=q$ for some $x_0\in R$, then we have $d\tilde{\gamma}/dt(x_0)=\tilde{\xi}(q)$. So $\tilde{\gamma}$ is tangent to D_{i-1} at q if and only if $\tilde{\xi}$ is tangent to D_{i-1} at q. Hence the proposition follows from the above arguments. Q.E.D.

Remark. In the above proposition, by "solution passing through a point q", we mean a solution $\tilde{\gamma}$ from an interval I (not necessarily an open interval) into R^{n-1} satisfying $q=\tilde{\gamma}(x_0)$ for some point $x_0\in I$.

Proposition 5.4. *Generically, ordinary differential equations* (5) *of fold type have no singular solutions in the strict sense, and its discriminant set D consists of singular points of general solutions.*

Proof. If (5) has a singular solution $\tilde{\gamma}$, then $\tilde{\gamma}\subset D$ and $\tilde{\xi}=\tilde{\xi}_f$ must be tangent to $\tilde{\gamma}$ at every point $q=\tilde{\gamma}(x)$ (see Corollary 1.8 and Remark 3.4). But Proposition 5.3 asserts that generically there exist no curves in D to which $\tilde{\xi}_f=\tilde{\xi}$ is tangent. Hence, generically (5) has no singular solutions.

It remains to show that D consists of singular points of regular solutions. Consider the vector field

$$\tilde{\eta}(x, y_0, \cdots, y_{n-1})=\partial/\partial x+y_1\partial/\partial y_0+\cdots+y_{n-1}\partial/\partial y_{n-2}$$
$$+1/2\cdot(-a(x, y)+a^2(x, y)-4b(x, y))\partial/\partial y_{n-1} .$$

$\tilde{\eta}$ is defined on $\pi(S)$ and of C^∞. And we have $\tilde{\eta}|D=\xi_f|D$. Therefore, for any point $q=(x^0, y_0^0, \cdots, y_{n-1}^0)$ of D, there is an integral curve $\tilde{\gamma}$ passing through or reaching to q and it is a solution of (5). We may assume without loss of generality that $\tilde{\gamma}(x)=(x, \gamma(x), \cdots, d^{n-1}\gamma/dx^{n-1}(x))$ is defined on a half open interval $(x_0-\varepsilon, x_0]$. Since no solutions of (5) are contained in D as seen above, we may assume that $\tilde{\gamma}((x_0-\varepsilon, x_0))\subset\pi(S)-D$. Consequently, $\tilde{\gamma}|(x_0-\varepsilon, x_0)$ is a regular solution of (5), as seen in the proof of Proposition 1.7.

So, to see that q is a singular point of a regular solution, it is sufficient to see that $\tilde{\gamma}$ can not be extended to a regular solution defined on an open interval containing $(x_0-\varepsilon, x_0]$.

Suppose $q\in D-D_1$. Then $\tilde{\gamma}$ is not tangent to D at q, hence if we have an extension of the curve $\tilde{\gamma}$, it must go beyond D and go out of $\pi(S)$, but it is impossible for a solution of (5). So q is a singular point of $\tilde{\gamma}(x_0-\varepsilon, x_0)$.

Consider now the case $q\in D_1$. The general solution containing $\tilde{\gamma}|(x_0-\varepsilon, x_0)$ consists of integral curves of $\tilde{\eta}$. Now, arbitrarily near q there exist many points of $D-D_1$, which are singular points of regular solutions contained in this regular solution. Therefore $\tilde{\gamma}|(x_0-\varepsilon, x_0)$ can not be extended to a regular solution defined on an open interval containing $(x_0-\varepsilon, x_0]$.

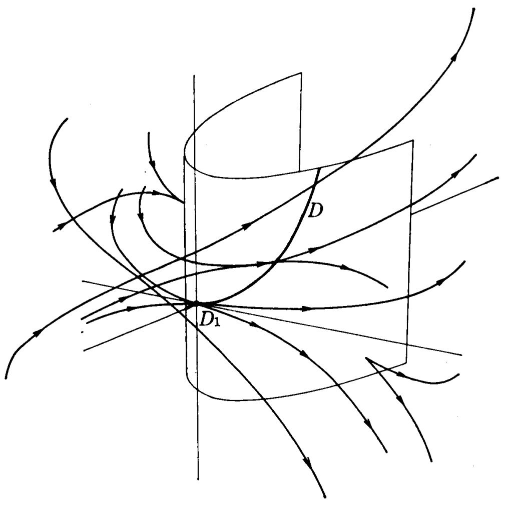

Fig. (5.1)

Q.E.D.

Example.

$$y'^2-x^3y''+y=0 . \quad \text{(Fig. (5.1))}$$

6. A decomposition of the jet space

Corollary 3.4 in § 3 and the argument in the previous section motivate to decompose $T_{1^t,0}$ into

$$J^\infty(R^{n+2}, R)\supset T_{1^t,0}\supset T_{1^t,1}\supset T_{1^t,2}\supset\cdots$$

where $T_{1^t,j}$ are defined inductively as follows:

Definition 6.1. Let g: $R^{n+2}\to R$ be a C^∞-function, and p be a point in R^{n+2}.

Let $j(\xi)$ be the tangent vector field on $J^\infty(R^{n+2}, R)$, defined by $j(\xi)(j^\infty(g)(p))= d(j^\infty(g))_p(\tilde{\tilde{\xi}}(p))$, where $\tilde{\tilde{\xi}}$ si the vector field on R^{n+2} defined in §3. Define $j^\infty(g)(p) \in T_1{}^i{}_{,1}$ if and only if $j^\infty(g)(p) \in T_1{}^i{}_{,0}$ and the vector field $j(\xi)$ on $J^\infty(R^{n+2}, R)$ is tangent to $T_1{}^{i-1}$ at $j^\infty(g)(p)$.

Remark 6.2. By definition 2.2,

$$T_1{}^{i-1}=\{j^\infty g(p) \in J^\infty(R^n, R) | g(p)=\partial g/\partial z(p)=\cdots=\partial^{i-1}g/\partial z^{i-1}(p)=0\}$$

and

$$T_1{}^i{}_{,0}=\{j^\infty g(p) \in T_1{}^{i-1} | \partial^i g/\partial z^i(p)=0 \text{ and } \partial^{i+1}g/\partial z^{i+1}(p)\neq 0\} .$$

So, we have a canonical tubular neighborhood $T_1{}^i{}_{,0}\times(-1,1)$ of $T_1{}^i{}_{,0}$ in $T_1{}^{i-1}$. The condition that "$j(\xi)$ is tangent to $T_1{}^{i-1}$ at $j^\infty g(p)$" is equivalent to the condition that "$j(\xi)$ is tangent to the tubular neighborhood $T_1{}^i{}_{,0}\times(-1,1)$ of $T_1{}^i{}_{,0}$ in $T_1{}^{i-1}$". Let τ: $T_1{}^i{}_{,0}\times(-1,1)\to T_1{}^i{}_{,0}$ denote the projection of the tubular neighborhood.

Since $T_1{}^{i-1}$ has codimension i in $J^\infty(R^{n+2}, R)$, we have the following:

Proposition 6.3. *$T_1{}^i{}_{,1}$ is a submanifold of $T_1{}^i{}_{,0}$ of condimension i.*

Definition 6.4. Suppose we have defined $T_1{}^i{}_{,0}\supset T_1{}^i{}_{,1}\supset\cdots\supset T_1{}^i{}_{,j-1}$ such that $T_1{}^i{}_{,k}$ is a submanifold of $T_1{}^i{}_{,k-1}$ with codimension i, $1\leq k\leq j-1$. We define $T_1{}^i{}_{,j}$ so that $j^\infty(g)(p) \in T_1{}^i{}_{,j}$ if and only if $j^\infty(g)(p) \in T_1{}^i{}_{,j-1}$ and $j(\xi)$ is tangent to $\tau^{-1}(T_1{}^i{}_{,j-1})$ at $j^\infty(g)(p)$.

By Definition, if $j^\infty(g)(p) \in T_1{}^i{}_{,j-1}$, then $j(\xi)$ is tangent to $\tau^{-1}(T_1{}^i{}_{,j-2})$ at $j^\infty(g)(p)$ and the codimension of $\tau^{-1}(T_1{}^i{}_{,j-1})$ in $\tau^{-1}(T_1{}^i{}_{,j-2})$ is i, we have the following:

Proposition 6.5. *$T_1{}^i{}_{,j}$ is a submanifold of $T_1{}^i{}_{,j-1}$ of codimension i.*

7. Main Theorem

By the transversality theorem, for almost every differentiable map f: $R^{n+2}\to R$, $j^\infty(f)$ is transversal to every $T_1{}^i{}_{,j}\subset J^\infty(R^{n+2}, R)$, $i\geqq 1$, $j\geqq 0$. For such a map f, we have a stratification $\{S_{i,0}\}$ of C as in Proposition 2.5. We set $S_{i,j}= j^\infty(f)^{-1}(T_1{}^i{}_{,j})$, then $S_{i,j}$ is a submanifold of $S_{i,j-1}$ of codimension i.

Let p_0 be a point of $S_{i,0}$ and U be a neighborhood of p_0 such that $\pi|U\cap S_{i,0}$ is an embedding. Denote $\pi(S_{i,j}\cap U)=D_{i,j}$. Then $S_{i,j}\cap U$ and $D_{i,j}$ are submanifolds of $S_{i,j-1}\cap U$ and of $D_{i,j-1}$ respectively or $S_{i,j}\cap U=D_{i,j}=\varnothing$.

By the definition of $S_{i,j}$ and the transversality theorem, a point p of $S_{i,j-1}$ is in $S_{i,j}$ if and only if $\tilde{\tilde{\xi}}$ is tangent to $\nu^{-1}(S_{i,j-1})$ at p, where ν: $U\cap S_{i-1}\to S_{i,0}$ is the projection of the normal bundle of $S_{i,0}$ in S_{i-1} defined in § 3.

Hence by Proposition 3.2, a point q of $D_{i,j-1}$ is in $D_{i,j}$ if and only if the vector field $\tilde{\xi}$ is tangent to $D_{i,j-1}$ at q.

Remark. $S_{i,j}=D_{i,j}=\emptyset$ for $0>n-i+1-ji$, that is, for $j>(n-i+1)/i$.

By the above construction, we have:

Main Theorem I. *For almost every ordinary differential equation*

(5) $$f(x, y_0, y_1, \cdots, y_{n-1}, z)=0$$

the following properties hold:

(1) *The discriminant set D is decomposed into a sum*

$$D=\bigcup_{i=1}^{n+1}\pi(S_{i,0}),$$

which is not necessarily a disjoint sum.

(2) *For each i, $1\leqq i\leqq n+1$, and for a point $p_0\in S_{i,0}$, there is a neighborhood U of p_0* such that $\pi|S_{i,0}\cap U$: $S_{i,0}\cap U\to R^{n+1}$ *is an embedding.*

(3) *The set $D_{i,1}$ of the points q of $D_{i,0}=\pi(S_{i,0}\cap U)$ such that there is a solution passing through q and tangent to D at q is a submanifold of $D_{i,0}$ of codimension i, or $D_{i,1}=\emptyset$.*

(4) *Inductively, the set $D_{i,j}$ of points q of $D_{i,j-1}$ such that there is a solution of* (5) *passing through q and tangent to $D_{i,j-1}$ at q is a submanifold of $D_{i,j-1}$ of codimension i, or $D_{i,j}=\emptyset$.*

Proof. Let q be a point of $D_{i,j-1}$ such that there is a solution $\tilde{\gamma}$ of (5) passing through q, that is $\tilde{\gamma}(x_0)=q$ for some $x_0\in R$ (for the definition "solutions passing through p", see Remark under Proposition 5.3). Then, we have $d\tilde{\gamma}/dt(x_0)=\tilde{\xi}(q)$. So, $\tilde{\gamma}$ is tangent to $D_{i,j-1}$ at q. Hence the theorem follows from the above argument. Q.E.D.

We can prove the following theorem similarly to Proposition 5.4.

Main Theorem II. *Generically, ordinary differential equations* (5) *have no singular solutions in the strict sense, and its discriminant set D consists of singular points of general solutions.*

Bibliographical notes. We have used, without explaining so much, many notions, e.g. (multi)-jet spaces, the transversality theorem, the Thom-Boardman singularities, the Malgrange-preparation theorem, and stratifications, which are very familiar to those who work on the singularities of differential mappings.

For jet spaces and the transversality theorem, refer, for instance [3], [1] or [6], for the *Thom-Boardman* singularities, refer [1] and for the *Malgrange*-preparation theorem, refer [4], [5] or the many articles in the "*Proceedings of Liverpool Singularities-Symposium*", Lecture Notes in Math. 1972, Springer. For stratified sets, refer [7], [10] or [2].

REFERENCES

[1] J. M. Boardman, *Singularities of differentiable maps*, Publ. Math. I. H. E. S., 33, 21–57 (1967).

[2] T. Fukuda, *Types topologiques des polynomes*, to appear in Publ. Math. I. H. E. S.

[3] H. E. Levine, *Singularities of differential mappings*, Proc. Liverpool Singularities-Symposium I, Lecture Notes in Math. 192, Springer-Verlag (1971).

[4] B. Malgrange, *Ideal of differentiable functions*, Oxford University Press.

[5] J. N. Mather, *Stability of C^∞ mappings I, the division theorem*, Ann. of Math. 87, 89–104 (1968).

[6] J. N. Mather, *Stability of C^∞ mappings V, Transversality*, Advances in Math. 4, 301–336 (1970).

[7] J. N. Mather, *Notes on topological stability*, mimeographed, Harvard University.

[8] B. Morin, *Formes canoniques des singularités d'une application différentiable*, Comptes Rendus Acad. Sci. Paris, 260, 5662–5665, 6503–6506 (1965).

[9] R. Thom, *Sur les équations différentièlles multiformes et leurs intégrales singulières*, Boletim da Sociedade Brasileira de Matemàtica, 3, No. 3, 1–11 (1972).

[10] R. Thom, *Ensembles et morphismes stratifiés*, Bull. A. M. S., 75 240–284 (1969).

Tokyo Institute of Technology
and Chiba University