ON THE PRODUCT OF MEASURES

By

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1. For each $i=1, 2$ let S_i be a set, $\mathfrak{S}_i$ a σ-ring of subsets of S_i, X a topological ring and μ_i: $\mathfrak{S}_i \to X$ a measure. If μ is the function defined on the measurable rectangles $E \times F$, $E \in \mathfrak{S}_1$, $F \in \mathfrak{S}_2$, by $\mu(E \times F) = \mu_1(E)\mu_2(F)$, it is well known that μ is generally not countably additive on the ring generated by the measurable rectangles (*R. M. Dudley* [7] Proposition). In this paper we shall discuss the countable additivity of μ.

2. In this section we shall consider the product of topological ring-valued measures.

For each $i=1, 2$ let S_i be a set, $\mathfrak{S}_i$ a σ-ring of subsets of S_i, X a topological ring (that is, associative ring which is Hausdorff space such that the mappings $(a, b) \to a-b$ and $(a, b) \to ab$ are continuous functions of a and b) and $\mathfrak{U}$ a base for neighborhoods of 0 in X, consisting of closed, symmetric sets. N denotes the set of all positive integers.

Definition 1. A set function μ_i: $\mathfrak{S}_i \to X$ is called a measure if for every sequence $\{E_n\}$ of mutually disjoint sets of $\mathfrak{S}_i$ holds $\mu_i(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu_i(E_i)$.

For a set $V \subset X$ and $k \in N$ put $kV = \{\sum_{i=1}^{k} y_i : y_i \in V \ i=1, 2, \cdots, k\}$.

Definition 2. A set $K \subset X$ is called bounded if for every V in $\mathfrak{U}$ there exists a $k \in N$ such that $K \subset kV$.

A set function μ_i is called bounded if its range is a bounded set. Let μ_i: $\mathfrak{S}_i \to X$ be a measure $(i=1, 2)$.

Proposition 1. *If singleton sets in X are bounded, then μ_i is bounded.*

Proof. See *T. Traynor* [12] Theorem 3.2.1.

Example. Let $X=N$ be the topological ring of all integers with the discrete topology. Then singleton set $\{1\}$ is not bounded and hence the point measure $\mu_{\{1\}}$ is not bounded.

For a set $F \in \mathfrak{S}_2$ put $\mu_2((F)) = \{\mu_2(F') : F' \in \mathfrak{S}_2, F' \subset F\}$.

Proposition 2. *If $\{F_n\}$ is a decreasing sequence in $\mathfrak{S}_2$ with $\bigcap_{n=1}^{\infty} F_n = \phi$, then for every V in $\mathfrak{U}$ there exists an $n \in N$ such that $\mu_2((F_n)) \subset V$.*

Proof. See *M. Takahashi* [11] Proposition 3.

For a set $A\in\mathfrak{Z}_1$ and a set $B\subset X$ put $\mu_1((A))B=\{\sum_{i=1}^{n}\mu_1(A\cap A_i)b_i: A_i\in\mathfrak{Z}_1$ and $b_i\in B (i=1,2,\cdots,n)$ and $A_i\cap A_j=\emptyset$ $(i\neq j), n\in N\}$.

Definition 3. μ_1 is called Daniell continuous (D-continuous) if for every decreasing sequence $\{E_n\}$ in $\mathfrak{Z}_1$ with $\bigcap_{n=1}^{\infty}E_n=\emptyset$, every V in $\mathfrak{U}$ and every bounded set $B\subset X$ there exists a $k\in N$ such that $\mu_1((E_k))B\subset V$.

Proposition 3. *If μ_1 is D-continuous, then for every sequence $\{E_n\}$ of mutually disjoint sets in $\mathfrak{Z}_i$, every bounded set $B\subset X$ and every V in $\mathfrak{U}$ there exists a $k\in N$ such that $\mu_1((E_n))B\subset V$ for all $n\geqq k$.*

Proof. For every $n\in N$ put $E'_n=\bigcup_{k=n}^{\infty}E_k$. Then $\{E'_n\}$ is a decreasing sequence with $\bigcap_{n=1}^{\infty}E'_n=\emptyset$. Since μ_1 is D-continuous, there exists a $k\in N$ such that $\mu_1((E_k))B\subset V$. Since $E_n\subset E'_k$ for all $n\geqq k$, we have $\mu_1((E_n))B\subset V$ for all $n\geqq k$.

Corollary. *If μ_1 is D-continuous, then for every monotone sequence $\{E_n\}$ in $\mathfrak{Z}_1$, every bounded set $B\subset X$ and every $V\in\mathfrak{U}$ there exists a $k\in N$ such that $\mu_1((E_m\triangle E_n))B\subset V$ holds for every $m, n\in N$ $(m, n\geqq k)$.*

Proof. It is obvious.

Let X be a metric ring with the metric $||\cdot||$. For every set $E\in\mathfrak{Z}_1$ put $||\mu_1||(E)=\sup\{||\sum_{i\in I}\mu_1(E_i)x_i||$, where the supremum is taken for all finite families $\{E_i\}_{i\in I}$ of mutually disjoint sets of $\mathfrak{Z}_1$ contained in E and for all the finite families $\{x_i\}_{i\in I}$ of elements of X such that $||x_i||\leqq 1$ for all $i\in I$. Then $||\mu_1||$ is a monotone and σ-subadditive set function on $\mathfrak{Z}_1$. We say that μ_1 has Bartle property ((B)-property) if for every decreasing sequence $\{E_n\}$ in $\mathfrak{Z}_1$ with $\bigcap_{n=1}^{\infty}E_n=\emptyset$ holds $\lim_{n\to\infty}||\mu_1||(E_n)=0$.

Proposition 4. *If X is a metric ring and if μ_1 has (B)-property, then μ_1 is D-continuous.*

Proof. Since $V=\{x: ||x||\leqq 1\}$ is a neighborhood of 0 in X, there exists a $k\in N$ such that $B\subset kV$. For every $\varepsilon>0$ there exists an $n\in N$ such that $||\mu_1||(E_n)<\varepsilon/k$. Let $\{E_{ni}\}_{i\in I}$ be any finite family of mutually disjoint sets in $\mathfrak{Z}_1$ contained in E_n and $\{b_i\}_{i\in I}$ any finite family of elements of B. Since $b_i=\sum_{j=1}^{k}x_{ij}$, $x_{ij}\in V$ $(i\in I, j=1,2,\cdots,k)$, holds $||\sum_{i\in I}\mu_1(E_{ni})b_i||\leqq\sum_{j=1}^{k}||\sum_{i\in I}\mu_1(E_{ni})x_{ij}||\leqq k||\mu_1||(E_n)<\varepsilon$.

Corollary. *If X is a normed ring, then the D-continuity implies (B)-property.*

Proof. Since $B\subset X$ is a bounded set, $\sup\{||x||: x\in B\}<\infty$ (see *S. Berberian* [2] Lemma (26.3)). The proof follows.

Proposition 5. *If X is a metric ring, then the following statements are equivalent.*

(1) *μ_1 has (B)-property.*

(2) *$||\mu_1||$ is s-bounded, that is, if $E_i\cap E_j=\emptyset$ $(i\neq j)$, then $\lim_{n\to\infty}||\mu_1||(E_n)=0$.*

(3) *For every monotone sequence $\{E_n\}$ in $\mathfrak{J}_1$ follows $\lim_{m,n\to\infty}||\mu_1||(E_m\triangle E_n)=0$.*

Proof. See *L. Drewnowski* [4] Theorem 5.3.

Definition 4. μ_1 is called D-bounded if for every $E\in\mathfrak{J}_1$ and every $V\in\mathfrak{U}$ there exists a $W\in\mathfrak{U}$ such that $\mu_1((E))W\subset V$.

Proposition 6. *If μ_1 is D-bounded, then for every $E\in\mathfrak{J}_1$ and every bounded set $B\subset X$ $\mu_1((E))B$ is bounded.*

Proof. By the hypotheses for every $V\in\mathfrak{U}$ there exists a $W\in\mathfrak{U}$ such that $\mu_1((E))W\subset V$. Since B is bounded, there exists a $k\in N$ such that $B\subset kW$. For every $E_i\in\mathfrak{J}_1$ and every $b_i\in B$ $i=1,2,\cdots,n$ and $E_i\cap E_j=\emptyset$ $(i\neq j)$ holds $\sum_{i=1}^{n}\mu_1(E_i)b_i=\sum_{i=1}^{n}\mu_1(E_i)\sum_{j=1}^{k}b_{ij}=\sum_{j=1}^{k}\sum_{i=1}^{n}\mu_1(E_i)b_{ij}\subset kV$, as $b_i=\sum_{j=1}^{k}b_{ij}$, $b_{ij}\in W$, $i=1,2,\cdots,n$, $j=1,2,\cdots,k$. Hence $\mu_1((E))B$ is bounded.

Let R be the ring generated by the measurable rectangles $\{E\times F: E\in\mathfrak{J}_1, F\in\mathfrak{J}_2\}$. Then it is well known that

(1) R coincides with the class of all finite disjoint unions $K=\bigcup_{i=1}^{n}E_i\times F_i$, where $E_i\in\mathfrak{J}_1$ and $F_i\in\mathfrak{J}_2 (i=1,2,\cdots,n)$.

(2) for every set $K=\bigcup_{i=1}^{n}E_i\times F_i\in R$ there exist two families $\{E_1,E_2,\cdots,E_n\}$ $(E_i\in\mathfrak{J}_1)$ and $\{F_1,F_2,\cdots,F_n\}(F_i\in\mathfrak{J}_2)$ of sets with $E_i\cap E_j=\emptyset(i\neq j)$.

(3) for every decreasing sequence $\{K_n\}(K_n=\bigcup_{i=1}^{k_n}E_{ni}\times F_{ni})$ in R and every $n\in N$ there exist $E_{ni}\in\mathfrak{J}_1$ and $F_{ni}\in\mathfrak{J}_2$ such that $E_{ni}\cap E_{nj}=\emptyset(i\neq j)$ and any each E_{ni} is contained in some $E_{n-1,k}$.

Lemma. *Let $\{K_n\}$ be a decreasing sequence in R. For some $W\in\mathfrak{U}$ put $J_n=\{j_n$: for some set $F\in\mathfrak{J}_2$ with $F\subset F_{nj_n}$ holds $\mu_2(F)\notin W\}$ and $E'_n=\bigcup_{j_n\in J_n}E_{nj_n}$ $(n=1,2,\cdots)$. Then $\{E'_n\}$ is a decreasing sequence in $\mathfrak{J}_1$.*

Proof. Let $E_{nj_n}\times F_{nj_n}$ be any measurable rectangle with $j_n\in J_n$. Since $E_{nj_n}\times F_{nj_n}\subset K_n\subset K_{n-1}=\cup E_{n-1,i}\times F_{n-1,i}$, there exists a $k\in N$ such that $E_{nj_n}\times F_{nj_n}\subset E_{n-1,k}\times F_{n-1,k}$. Hence $F_{nj_n}\subset F_{n-1,k}$ and $k\in J_{n-1}$ and consequently $E_{nj_n}\subset E_{n-1,k}\subset E'_{n-1}$ and $E'_n\subset E'_{n-1}$.

For every set $K=\bigcup_{i=1}^{n}E_i\times F_i\in R$ $(\{E_i\times F_i\}$ is mutually disjoint) we put $\mu(K)=$

$\sum_{i=1}^{n}\mu_1(E_i)\mu_2(F_i)$. Then $\mu(K)$ is well defined and $\mu: R\to X$ is finitely additive.

Theorem 1. *Let X be a topological ring such that singleton sets in X are bounded. If μ_1 has the following properties* (1) *and* (2), *then μ is countably additive on R.*

(1) *μ_1 is D-continuous.*

(2) *μ_1 is D-bounded.*

Proof. We shall prove that if $\{K_n\}(K_n=\bigcup_{i=1}^{k_n}E_{ni}\times F_{ni})$ is a decreasing sequence in R such that $\mu(K_n)\notin V(n=1,2,\cdots)$ for some $V\in\mathfrak{U}$, then holds $\bigcap_{n=1}^{\infty}K_n\neq\emptyset$. Let $V_1\in\mathfrak{U}$ be a neighborhood of 0 in X such that $V_1+V_1\subset V$ and let $W\in\mathfrak{U}$ be a neighborhood which satisfies the condition (2) for V_1. Put $J_n=\{j_n$: for some $F\in\mathfrak{Z}_2$ with $F\subset F_{nj_n}$ holds $\mu_2(F)\notin W\}$, $J'_n=\{$other j_n's$\}$ for all $n\in N$ and $E'_n=\bigcup_{j_n\in J_n}E_{nj_n}$. Then $\{E'_n\}$ is a decreasing sequence. We shall show that $\bigcap_{n=1}^{\infty}E'_n\neq\emptyset$. If it were false, then there would exist an $n\in N$ such that $\mu_1((E_n))B\subset V$, B being the range of μ_2. Since in the equality $\mu(K_n)=\sum_{j_n\in J_n}\mu_1(E_{nj_n})\mu_2(F_{nj_n})+\sum_{j_n\in J'_n}\mu_1(E_{nj_n})\mu_2(F_{nj_n})$ holds $\sum_{j_n\in J_n}\mu_1(E_{nj_n})\mu_2(F_{nj_n})\in V_1$ and $\mu_2(F_{nj_n})\in W$ for all $j_n\in J'_n$, holds $\mu(K_n)\in V_1+V_1\subset V$, which is a contradiction. Consequently there exists a point $x_0\in S_1$ such that $x_0\in\bigcap_{n=1}^{\infty}E'_n$. Let j_n be positive integer with $x_0\in E_{nj_n}$. Since $\{K_n\}$ is decreasing, $\{E_{nj_n}\}$ also is decreasing. We shall show that $\bigcap_{n=1}^{\infty}F_{nj_n}\neq\emptyset$. If it were false, then by Proposition 2 there would exist an $n\in N$ such that $\mu_2((F_n))\subset W$ and hence $j_n\notin J_n$, which is a contradiction. Let y_0 be a point in S_2 with $y_0\in\bigcap_{n=1}^{\infty}F_{nj_n}$, Then we have $(x_0,y_0)\in K_n$ for all n. The proof is complete.

We say that X has (G)-property if $\{a_n\}$ is any sequence in X such that $a_n\notin V$ $(n=1,2,\cdots)$ for some $V\in\mathfrak{U}$, then the set $\{\sum_i a_{ni}: \{a_{ni}\}$ is a finite subsequence of $\{a_n\}\}$ is not bounded.

For example, any H^*-algebra and the convolution algebra l^1 have (G)-property.

Theorem 2. *Under the hypotheses of Theorem* 1 *suppose X has* (G)*-property, then μ is s-bounded.*

Proof. It is easy to show that μ is bounded. Since X has (G)-property by *S. Ohba* [14] Theorem 3 μ is s-bounded.

Let $\mathfrak{Z}_1\times\mathfrak{Z}_2$ be the σ-ring generated R.

Corollary. *Under the hypothesis of Theorem* 2 *μ has the countably additive extension to $\mathfrak{Z}_1\otimes\mathfrak{Z}_2$ provided that X be complete metric ring.*

Proof. Since μ is s-bounded, the proof is obvious from *G. Fox* and *P. Morales* [8] Theorem 2.11.

3. In this section we shall consider the product of Banach-algebra-valued measures.

For each $i=1, 2$ let $\mathfrak{J}_i$ be a σ-ring of a set S_i, X a Banach algebra and μ_i: $\mathfrak{J}_i \to X$ a vector measure.

For every set $F \in \mathfrak{J}_2$ put $\tilde{\mu}_2(F) = \sup\{||\mu_2(A)||: A \subset F,\ A \in \mathfrak{J}_2\}$. Then $\sup\{\tilde{\mu}_2(F): F \in \mathfrak{J}_2\} < \infty$, because μ_2 is s-bounded. It is well known that if $\{F_n\}$ is a decreasing sequence in $\mathfrak{J}_2$ with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then $\lim_{n\to\infty} \tilde{\mu}_2(F_n) = 0$ holds (cf. Proposition 2).

For every set $E \in \mathfrak{J}_1$ we put $||\mu_1||(E) = \sup\{||\sum_{i \in I} \mu_1(E_i)x_i||\}$, where the supremum is taken for all the finite families $\{E_i\}_{i \in I}$ of mutually disjoint sets of $\mathfrak{J}_1$ contained in E and for all the finite families $\{x_i\}_{i \in I}$ of elements of X such that $||x_i|| \leqq 1$ for all $i \in I$.

Proposition 7. *The following statements are equivalent.*

(1) μ_1 *is D-continuous.*

(2) μ_1 *has (B)-property.*

(3) *There exists a bounded, non-negative measure* ν_1 *on* $\mathfrak{J}_1$ *such that* $\lim_{\nu_1(E)\to 0} ||\mu_1||(E) = 0$.

(4) *There exists a bounded, non-negative measure* ν_1 *on* $\mathfrak{J}_1$ *such that* $\nu_1(E) \to 0$ *if and only if* $||\mu_1||(E) \to 0$.

Proof. See *M. Duchoň* [6] Theorem 8 and Theorem 6.

Proposition 8. *If* μ_1 *is D-continuous, then* $\sup\{||\mu_1||(E): E \in \mathfrak{J}_1\} < \infty$.

Proof. See *M. Duchoň* [6] Theorem 2.

Corollary. *If* μ_1 *is D-continuous, then* μ_1 *is D-bounded.*

Proof. It is obvious.

For every set $K = \bigcup_{i=1}^{n} E_i \times F_i \in R$ put $\mu(K) = \sum_{i=1}^{n} \mu_1(E_i)\mu_2(F_i)$. Then holds the followings.

Theorem 3. *If* μ_1 *is D-continuous, then* μ *has the countable additive extension to* $\mathfrak{J}_1 \otimes \mathfrak{J}_2$

Proof. By Proposition 7 there exists a bounded, non-negative measure ν_1 on $\mathfrak{J}_1$ such that $\lim_{\nu_1(E)\to 0} ||\mu_1(E)|| = 0$. Since μ_2 is s-bounded, there exists a bounded, non-negative measure ν_2 on $\mathfrak{J}_2$ such that $\lim_{\nu_2(F)\to 0} ||\mu_2||(F) = 0$. Let $\nu_1 \times \nu_2$ be the product measure of ν_1 and ν_2. Then $\lim_{\nu_1 \times \nu_2(K)\to 0} ||\mu(K)|| = 0$ (see, for example, the proof of *I. Kluvanek* [9] Theorem). Consequently by *S. Ohba* [13] Theorem 3 μ has the countable additive extension to $\mathfrak{J}_1 \otimes \mathfrak{J}_2$.

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