A NET CHARACTERISATION OF HERRLICH'S k-COMPACTNESS

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H. Herrlich, has defined in [1], a Tychonoff space X to be k-compact for an infinite cardinal k, if each ultrafilter of zero sets in X such that the intersection of every collection of less than k members of it is non-void, is fixed. (cf. also [2] (130)). In this paper, we give a net characterisation of the same. That this is not got by a direct translation of 'filters' into 'nets', is evident from the generality of our definition of Z-nets.

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Our proof depends on a characterisation of k-compactness due to *Hušek* ([3]), which is given below. We recall that a space X is E-compact (E being any space) if X can be embedded homeomorphically as a closed subspace of a power of E.

0. Result. (*M. Hušek*) ([3]) (Theorems 1 and 2).

(a) Let k be a limit cardinal number $>\aleph_0$; and let M be a cofinal set of infinite cardinal numbers less than k. Assume that for each $m \in M$, there is a topological space E_m such that the E_m-compact spaces are just m-compact spaces. Then $(\prod_{m \in M} E_m)$-compact spaces are just k-compact spaces.

(b) Let $E_{k^+} = I^k - (p)$, where $I = [0, 1]$ with the usual topology, k^+ the cardinal number next larger to the infinite cardinal number k and $p \in I^k$. Then E_{k^+}-compact spaces are exactly k^+-compact spaces.

1. Definitions. A net S in a Tychonoff space X is said to be a Z-net if given any zero-set E, the net S is either eventually in E or eventually in the complement of E. Also S is said to be k-directed for an infinite cardinal k, if the directed set D on which S is defined, is k-directed; that is, if given any $\{d_i\}_{i \in J}$ a subset of D such that $|J| < k$, there exists an element $d \in D$ such that $d \geq d_i$ for every $i \in J$.

2. Theorem. *A Tychonoff space X is k-compact for an infinite cardinal k, if and only if it has the following property (P_k).*

(P_k) = *every k-directed Z-net in X is convergent.*

Proof. (*Sufficiency*). Suppose X has (P_k). Let $\mathscr{F}$ be an ultrafilter of zero-sets in X such that the intersection of every family of less than k members of it is non-empty. Let $\mathscr{G}=\{A\subset X \mid A=\bigcap_{\alpha\in\Delta} F_\alpha,\ F_\alpha\in\mathscr{F},\ |\Delta|<k\}$. Then clearly $\mathscr{F}\subset\mathscr{G}$. For each $G\in\mathscr{G}$, choose a point $g\in G$ and define $S(G)=g$. Then S is a net on G which is k-directed; and S is eventually in each $G\in\mathscr{G}$. Also S is a Z-net since given any zero-set A, either $A\in\mathscr{F}\subset\mathscr{G}$, or there exists $B\in\mathscr{F}\subset\mathscr{G}$ such that $A\subset cB$, since $\mathscr{F}$ is a Z-ultrafilter. So, by hypothesis, S converges to (say) $p\in X$.

Let $E\in\mathscr{F}$. Now E is a closed set, S is eventually in E and S converges to p imply $p\in E$. Hence $p\in\bigcap_{E\in\mathscr{F}} E$ and hence $\mathscr{F}$ is fixed. Thus sufficiency is proved.

(*Necessity*). Step (i). First we note that spaces having property (P_k) form a closed-hereditary productive family. Let X be a space having property (P_k) and let A be a closed subset of X. Let S be a k-directed Z-net in A. Then it is clearly a Z-net in X and so converges to (say) p in X. But since A is closed $p\in A$. So A is also having (P_k).

Now let $(X_\alpha)_{\alpha\in J}$ be a family of spaces each having (P_k). Let S be a k-directed Z-net in $\prod_{\alpha\in J} X_\alpha$. If π_α is the projection onto X_α, then $\pi_\alpha\circ S$ is a k-directed Z-net in X_α (it is a Z-net since if U_α is an open set in X_α, then $\pi_\alpha^{-1}(U_\alpha)$ is an open set in $\prod_{\alpha\in J} X_\alpha$). So $\pi_\alpha\circ S$ is convergent to (say) $p_\alpha\in X_\alpha$. Now S converges to $p=(p_\alpha)_{\alpha\in J}$. Thus $\prod_{\alpha\in J} X_\alpha$ has (P_k).

Thus Step (i).

Step (ii). If a Tychonoff space X has an α-base, that is a base $\mathscr{B}$ for its open sets such that $|\mathscr{B}|\le\alpha$, then X is α-Lindelöf, that is, every open cover of X has a subcover of cardinality $\le\alpha$, and so it can be easily proved that X has (P_{α^+}).

In particular, I^k; where $I=[0,1]$ with the usual topology has a base of cardinality k and hence is hereditarily k-Lindelöf, so that $I^k-(p)$ as described in Result 0, is k-Lindelöf and so has (P_{k^+}). Now by Step (i), and using Result 0 (b), it follows that if X is k^+-compact, then X has (P_{k^+}).

Step (iii). Let $k=\aleph_0$. Then k-compactness coincides with compactness. That compactness implies $(P_{\aleph_0})$ follows easily.

Step (iv). Let k be a limit cardinal $>\aleph_0$. Then we can find a cofinal set M in the set of infinite cardinal numbers less than k such that each $m\in M$ is a non-limit cardinal, so that we can write $m=m_1^+$ for some infinite cardinal m_1. Now let X be k-compact. Then by Result 0(a), X is $(\prod_{m\in M} E_m)$-compact where E_m is m_1^+-compact, and so by Step (ii), E_m has $(P_{m_1^+})$, that is (P_m). But it is easy to see that (P_m) implies (P_k) since $m<k$. So for each $m\in M$, E_m has (P_k). Now

by using Step (i), it follows that X has (P_k).

This completes the proof of the theorem.

References

[1] H. Herrlich: *Fortsetzbarkeit stetiger Abbildungen und Kompaktheitsgrad Topologischer Räume*, Math. Zeit. 96 (1967) 64–72.

[2] H. Herrlich: *Topologische Reflexionen und Coreflexionen*, Lecture Notes (Springer-Verlag) 78 (1968).

[3] M. Hušek: *The class of k-compact spaces is simple*, Math. Zeit. 110 (1969) 123–126.

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