RECURRENT TENSORS ON A PSEUDO-RIEMANNIAN MANIFOLD

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1. Introduction

By differentiability, we shall mean that of class C^∞. In this paper, M will denote an n-dimensional connected differentiable manifold. The connectedness assumption on M implies that any two of its points can be joined by a sectionally smooth curve. All functions, tensor fields, etc., are assumed to be differentiable. All indices, both Latin and Greek, have range $1, 2, \cdots, n$. Summation over repeated indices, unless stated otherwise, is implied.

Let Γ be a linear connection on M. The covariant derivative with respect to Γ of a tensor S is denoted by ∇S. In a coordinate neighbourhood U with coordinate functions u^i, covariant differentiation and partial differentiation are denoted by a comma and a dot respectively. Thus if S has components S_k^{ij} in (U, u^i), ∇S has components

$$S^{ij}_{k,l} = S^{ij}_{k\cdot l} + \Gamma^i_{lm} S^{mj}_k + \Gamma^j_{lm} S^{im}_k - \Gamma^m_{lk} S^{ij}_m ,$$

where Γ^i_{jk} are the connection coefficients. The components of the curvature tensor R and the torsion tensor T of Γ in (U, u^i) are given respectively by

$$R^i_{jkl} = \Gamma^i_{lj\cdot k} - \Gamma^i_{kj\cdot l} + \Gamma^i_{kh}\Gamma^h_{lj} - \Gamma^i_{lh}\Gamma^h_{kj} ,$$
$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} .$$

A tensor S on M is said to be *recurrent* with respect to Γ if $S \not\equiv 0$ (i.e., not everywhere zero) and $\nabla S = \xi \otimes S$ for some covector ξ. We call ξ the *recurrence covector* of S. If $\xi = 0$, we say that S is *covariantly constant*. It can be proved that if $S \not\equiv 0$ satisfies $\nabla S = \xi \otimes S$, then $S \neq 0$ (i,e., nowhere zero). Thus, a recurrent tensor is nowhere zero. Recurrent tensors on a manifold with a linear connection has been considered in great detail in Wong [9] and [10]. On the other hand, Riemannian connection with recurrent curvature on a pseudo-Riemannian manifold has been studied by Walker in [7] where he proved the following

Theorem. *Let Γ be the Riemannian connection of a pseudo-Riemannian manifold. If the curvature tensor of Γ is recurrent, then its recurrence covector is locally a gradient.*

In this paper, we shall prove a number of results concerning recurrent tensors on a pseudo-Riemannian manifold M that are closely related to the above theorem. In § 2, we shall briefly review the theory of connection on a principal fibre bundle and state some results of Wong on recurrent tensors that are needed later. In § 3, we first consider recurrent tensors which are co-directional with covariantly constant tensors. Then, we express the condition for the recurrence covector of a recurrent tensor on M to be globally a gradient in terms of the bundle $L(M)$ of linear frames in M. In § 4, we develop some results on the bundle $O(M)$ of orthonormal frames that are useful in the study of recurrent tensors on a pseudo-Riemannian manifold, and obtain a characterization of recurrent tensors in terms of $O(M)$. In § 5, we define the norm of tensors on M, and show that a recurrent tensor on M is either everywhere null or everywhere non-null. In § 6, we prove that the recurrence covector of a non-null recurrent tensor is globally a gradient, and obtain a geometric interpretation of the recurrence covector in terms of the norm of the recurrent tensor.

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2. Connection on a fibre bundle

In this section, we shall briefly review the theory of connections on a principal fibre bundle. We shall also state some results of Wong on recurrent tensors that are needed later. All the results are well known, and can be found in [3] and [9].

A *frame* $z(u)$ at $u \in M$ is a basis $(X_1, \cdots, X_n)$ of the tangent space $T_u(M)$ at u. Let $L(M)$ be the collection of all frames at all points of M. The map π: $L(M) \to M$ defined by $z(u) \to u$ is called the *natural projection*. Each coordinate system (U, u^i) in M induces a coordinate system in $L(M)$ as follows. Since the vector X_α of the frame $z(u) \in \pi^{-1}(U)$ can be uniquely expressed as $X_\alpha = X^i_\alpha(\partial/\partial u^i)$, $\{\pi^{-1}(U), (u^i, X^i_\alpha)\}$ is a coordinate system in $L(M)$. In this way, $L(M)$ becomes an $(n+n^2)$-dimensional differentiable manifold. We shall use $[X^\alpha_i]$ to denote the inverse of the $n \times n$ matrix $[X^i_\alpha]$. The general linear group $GL(n, R)$ acts on $L(M)$ to the right in the sense that $A = [A^\alpha_{\alpha*}] \in GL(n, R)$ takes the frame $z = (X_\alpha)$ to the frame $zA = (X_\alpha A^\alpha_{\alpha*})$ and we call such an action on $L(M)$ a right translation. Then $L(M)$ is a principal fibre bundle over M with $GL(n, R)$ as the structure group. We call it the *bundle of linear frames* over M.

Let Γ be a linear connection on M and Γ^i_{jk} the connection coefficients. Put

$r^i_k = \Gamma^i_{jk} du^j$. Then the local 1-forms

$$\theta^\alpha = X^\alpha_i du^i ,$$
$$\omega^\lambda_\mu = X^\lambda_i dX^i_\mu + X^\lambda_i r^i_j X^j_\mu ,$$

piece together to form $n+n^2$ global 1-forms on $L(M)$. These $n+n^2$ 1-forms are everywhere linearly independent. Furthermore, the equations $\theta^\alpha=0$ (resp. $\omega^\lambda_\mu=0$) define a field of n^2-planes (resp. n-planes) on $L(M)$, which we call the field of *vertical* (resp. *horizontal*) *planes* on $L(M)$. The field of horizontal planes is invariant under the right translations. The field of vertical planes is completely determined by the differential structure of M alone, whereas the field of horizontal planes is determined by, and also determines, the linear connection on M.

The determination of a linear connection on M by assigning to $L(M)$ a field of horizontal planes invariant under the right translations can be generalized to other principal fibre bundles. In § 4, we shall consider connections on the bundle of orthonormal frames. For the moment, we state the following

Theorem 2.1. ([3, p. 79]) *Let f: $B'(M', G') \to B(M, G)$ be a homomorphism of principal fibre bundles with corresponding homomorphism f: $G' \to G$ and induced diffeomorphism f: $M' \to M$. If Γ' is any connection on $B'(M', G')$, then there exists a unique connection Γ on $B(M, G)$ such that the horizontal planes of Γ' are mapped by f into horizontal planes of Γ.*

We now state some notions and results required in the characterization of recurrent tensors by $L(M)$. In what follows, theorems will usually be stated and proved for a tensor of type (2, 1) although they are also true for tensors of arbitrary type.

Theorem 2.2. (*Chern* [1, p. 78], *Wong* [9, p. 330]) *To a tensor S of type (2, 1) on M, there corresponds a set of n^3 functions $S^{\alpha\beta}_\gamma$ on $L(M)$ such that for any $z \in L(M)$ and any $A \in GL(n, R)$,*

$$S^{\alpha^*\beta^*}_{\gamma^*}(zA) = S^{\alpha\beta}_\gamma(z) A^{\alpha^*}_\alpha A^{\beta^*}_\beta A^\gamma_{\gamma^*} ,$$

where $A = [A^\gamma_{\gamma^*}]$ and $A^{-1} = [A^{\alpha^*}_\alpha]$. Conversely, to any such set of n^3 functions on $L(M)$, there corresponds a tensor of type (2, 1) on M. This correspondence is one-to-one.

If $\{\pi^{-1}(U), (u^i, X^i_\alpha)\}$ is the local coordinate system in $L(M)$ induced by the local coordinate system (U, u^i) in M, and S^{ij}_k are the components in (U, u^i) of the tensor S, then the above correspondence is defined locally by

$$S_{\gamma}^{\alpha\beta}=S_{k}^{ij}X_{i}^{\alpha}X_{j}^{\beta}X_{\gamma}^{k} .$$

From now on, we shall refer to $S_{\gamma}^{\alpha\beta}$ as the *functions on* $L(M)$ *corresponding to the tensor* S on M.

When a linear connection on M (or equivalently, a connection in $L(M)$) is given, we can define the concept of a holonomy bundle as follows. A curve $z(t)$ in $L(M)$ is called *horizontal* if all its velocity vectors lie in the horizontal planes. It can be proved that $z(t)$ is horizontal iff $z(t)$, regarded as a field of frames along the curve $\pi z(t)$ in M, is a parallel field of frames. For two points $z_0, z \in L(M)$, we shall write $z_0 \sim z$ if they can be joined by a (piecewise differentiable) horizontal curve. Then $\sim$ is an equivalence relation in $L(M)$ and we denote the equivalence class containing z_0 by $B[z_0]$. It can be proved that $B[z_0]$ is a regular submanifold and a reduced bundle of $L(M)$. We shall call it the *holonomy bundle* through z_0. The following results are well-known.

Theorem 2.3. (*Wong* [9, p. 336]) *Let Γ be a linear connection and S a tensor on M. If S is recurrent with respect to Γ, then the restrictions of its corresponding functions to any $B[z_0]$ of $L(M)$ have no common zero and are proportional to a set of constants. Conversely, if this condition is satisfied on any $B[z_0]$, then S is recurrent.*

Theorem 2.4. (*Wong* [9, p. 338]) *Let Γ be a linear connection and S a tensor on M. If S is covariantly constant with respect to Γ, then the restrictions of its corresponding functions to any $B[z_0]$ of $L(M)$ are constants, not all zero. Conversely, if this condition is satisfied on any $B[z_0]$, then S is covariantly constant.*

Before leaving this section, we wish to remark that concepts such as horizontal curves and holonomy bundles for a connection on $L(M)$ can similarly be defined for a connection on any principal fibre bundle.

3. The recurrence covector

In this section, we first consider a class of recurrent tensors which bears a special relationship to the covariantly constant tensors. Then, we formulate the condition for the recurrence covector to be globally a gradient in terms of the bundle of linear frames.

Two tensor fields S and T on M are said to be *co-directional* if there exists a nowhere zero function g on M such that $T=gS$. We begin with

Theorem 3.1. *Let Γ be a linear connection on M.*

(a) *Suppose S is a recurrent tensor whose recurrence covector is globally a gradient, say ∇W, where W is a global function on M. Then $e^{-W}S$ is a covariantly constant tensor, and the most general covariantly constant tensor co-directional with S is of the form $A\,e^{-W}S$, where A is a nonzero real number.*

(b) *Suppose T is a covariantly constant tensor. Then any tensor co-directional with T is recurrent and its recurrence covector is globally a gradient. If W is any global function on M, any tensor co-directional with T having ∇W as the recurrence covector is of the form $A\,e^{W}T$, where A is a nonzero real number.*

Proof. (a) That $e^{-W}S$ is covariantly constant if S is recurrent can be verified by straightforward checking. Now, assume that f is a nowhere zero function and that the tensor fS is covariantly constant. Then,

$$0=\nabla(fS)=(\nabla f)\otimes S+f(\nabla W)\otimes S\,.$$

Since S is nowhere zero, it follows from the above equation that $\nabla \log|f|=-\nabla W$. Therefore, $f=A\,e^{-W}$ where A is a nonzero real number.

(b) The proof is similar to that of (a) and is hence omitted.

It follows from this theorem that a tensor is recurrent with a recurrence covector which is globally a gradient iff it is co-directional with a covariantly constant tensor. The next theorem gives a characterization of such tensors in terms of its corresponding functions on $L(M)$. The result is analogous to Theorem 2.3 of Wong.

Theorem 3.2. *Let Γ be a linear connection on M, S a tensor of type* (2, 1) *and $S_\gamma^{\alpha\beta}$ its corresponding functions on $L(M)$. If S is recurrent and its recurrence covector is globally a gradient, then on any $B[z_0]$,*

$$S_\gamma^{\alpha\beta}(z)=f(\pi z)c_\gamma^{\alpha\beta}\,, \tag{3.1}$$

where f is a nowhere zero C^∞ function on M and the constants $c_\gamma^{\alpha\beta}$ are not all zero. Conversely, if condition (3.1) *is satisfied on any $B[z_0]$, then S is recurrent and its recurrence covector is globally a gradient.*

Proof. Assume that S is recurrent with recurrence covector ∇W, where W is a global function on M. By Theorem 3.1, $e^{-W}S$ is a covariantly constant tensor. The functions on $L(M)$ corresponding to $e^{-W}S$ are $e^{-W}S_\gamma^{\alpha\beta}$, and by Theorem 2.4, we have

$$e^{-W(\pi z)}S_\gamma^{\alpha\beta}(z)=c_\gamma^{\alpha\beta} \quad \text{on} \quad B[z_0]\,,$$

where $c_\gamma^{\alpha\beta}$ are not all zero. Thus $S_\gamma^{\alpha\beta}(z)=e^{W(\pi z)}c_\gamma^{\alpha\beta}$ on $B[z_0]$, and condition (3.1) is satisfied. Conversely, assume that condition (3.1) is satisfied on a $B[z_0]$. Let $g=1/f$. Then g is a nowhere zero function on M, and we have

$$g(\pi z)S_\gamma^{\alpha\beta}(z)=c_\gamma^{\alpha\beta} .$$

Therefore by Theorem 2.4, gS is covariantly constant. Hence by Theorem 3.1(b), S is recurrent and its recurrence covector is globally a gradient.

The next theorem is a modified version of Theorem 3.2. The emphasis is on the condition for a recurrence covector to be globally a gradient formulated in terms of $L(M)$.

Theorem 3.3. *Let Γ be a linear connection on M, S a recurrent tensor of type* (2,1) *and $S_\gamma^{\alpha\beta}$ its corresponding functions on $L(M)$. Then* (*by Theorem* 2.3) *on any holonomy bundle* $B[z_0]$,

$$S_\gamma^{\alpha\beta}(z)=f(z)c_\gamma^{\alpha\beta} . \tag{3.2}$$

The recurrence covector is globally a gradient iff for each $u\in M$, the function f appearing in the above equation is constant on $\pi^{-1}(u)\cap B[z_0]$.

Proof. To prove the "sufficiency", we first note that the function f in (3.2) is a C^∞ function on $B[z_0]$. Since local cross-section exists on $B[z_0]$, the assumption on $f(z)$ implies that f induces a C^∞ function on M which we denote also by f. The "sufficiency" then follows from Theorem 3.2. To prove the "necessity", assume that the recurrence covector is globally a gradient. By Theorem 3.2, we have, on $B[z_0]$,

$$S_\gamma^{\alpha\beta}(z)=h(\pi z)d_\gamma^{\alpha\beta} ,$$

where h is a function on M. Comparision of this with (3.2) shows that

$$f(z)c_\gamma^{\alpha\beta}=h(\pi z)d_\gamma^{\alpha\beta} .$$

Thus for any nonzero constant $c_\gamma^{\alpha\beta}$, we have

$$f(z)=h(\pi z)d_\gamma^{\alpha\beta}/c_\gamma^{\alpha\beta} .$$

Form this it follows that f is constant on $\pi^{-1}(u)\cap B[z_0]$ for every $u\in M$.

We have thus obtained, in our Theorem 3.3, a necessary and sufficient condition for the recurrence covector to be globally a gradient. In one of his papers, Ludden [4, Corollary 3.2] asserts that the same condition is a necessary and sufficient condition for the recurrence covector to be locally a gradient. However, we believe that the "necessity" of his Corollary 3.2 does not hold. In fact, in proving the "necessity" in his Theorem 3.1 (on which his Corollary 3.2 is based), he seemed to have assumed that the holonomy bundle $B_U[z_0]$ in $\pi^{-1}(U)$ through a point $z_0 \in \pi^{-1}(U)$ is the same as $\pi^{-1}(U) \cap B[z_0]$ which is not necessarily true as pointed out to the author by Professor Y. C. Wong.

The recurrence covector of a recurrent tensor which is locally a gradient is not necessarily globally a gradient, as the following example shows.

Example. Let R^2 be the Euclidean 2-space with a rectangular coordinate system (x, y), and $M = R^2 \setminus \{(0,0)\}$. If $\xi_1 = -y/(x^2+y^2)$ and $\xi_2 = x/(x^2+y^2)$, then the covector field $\xi = \xi_1 dx + \xi_2 dy$ on M is locally, but not globally, a gradient (see [6, p. 93]). Let Γ be the linear connection on M defined by $\Gamma^i_{jk} = \xi_j \delta^i_k$. Then, it is easy to verify that the vector $X = A(\partial/\partial x) + B(\partial/\partial y)$ where A, B are constants not both zero, is a recurrent vector on (M, Γ) with ξ as the recurrence covector.

4. The bundle of orthonormal frames

From now on, we shall consider recurrent tensors on an n-dimensional pseudo-Riemannian manifold M. For such M, we know that there is a special class of linear connections on M closely related to the Riemannian metric of M. They are the metric connections (to be defined later). In this case, it is important to consider, instead of the bundle $L(M)$ of linear frames, the bundle of orthonormal frames.

Let g be the metric tensor of the pseudo-Riemannian manifold M, so that g is a symmetric tensor of type $(0, 2)$ on M whose component matrix $[g_{ij}]$ is everywhere non-singular. At each point $u \in M$, the signature of the matrix $[g_{ij}]$ is independent of the coordinate system used, and is called the signature of the tensor g at u. It is known, and can be proved by a continuity argument, that for a connected manifold M, the signature of g is the same everywhere. The following is another proof of this fact.

Lemma 4.1. *Let g be a symmetric tensor of type $(0, 2)$ on a connected manifold M and $[g_{ij}]$ its component matrix. If $[g_{ij}]$ is everywhere non-singular, then the signature of g is the same everywhere on M.*

Proof. The tensor g given in the Lemma is a Riemannian metric on M. Let Γ be the Levi-Civita connection of g. Then g is covariantly constant with respect to Γ. According to Theorem 2.4, the functions $g_{\alpha\beta} = X^i_\alpha g_{ij} X^j_\beta$ on $L(M)$ are constant on any $B[z_0] \subset L(M)$. Let u be an arbitrary point of M. Since $\pi^{-1}(u) \cap B[z_0] \neq \phi$, we see that the signature of $[g_{ij}]$ at u is the same as the signature of the matrix $[g_{\alpha\beta}]$ of constants.

In what follows, we shall assume that the given Riemannian metric on M is of signature r $(0 \leq r \leq n)$. The case $r=n$ (resp. $r=0$) corresponds to a positive (resp. negative) definite metric. All other cases correspond to indefinite metrics. Let

$$e_\alpha = \begin{cases} 1 & \text{if } \alpha=1, \cdots, r, \\ -1 & \text{if } \alpha=r+1, \cdots, n. \end{cases}$$

We denote by $O(r, n-r)$ the subgroup of $GL(n, R)$ consisting of all those matrices leaving the bilinear form

$$f(u, v) = \sum_\alpha e_\alpha u^\alpha v^\alpha$$

invariant. Thus, in particular, $O(n, 0)$ is the n-dimensional orthogonal group.

A frame (X_α) at M is said to be *orthonormal* if $g_{ij} X^i_\alpha X^j_\beta = e_\alpha \delta_{\alpha\beta}$ (not summed over α). Let $O(M)$ be the subset of $L(M)$ consisting of all the orthonormal frames of M. Then $O(M)$ is an $(1/2)n(n+1)$-dimensional submanifold of $L(M)$. It is also a reduced bundle of $L(M)$ over M with structure group $O(r, n-r)$. For details, we refer to [2] and [3, p. 158]. This manifold is sometimes difficult to handle, because convenient local coordinate systems are not available. However, by considering the restriction to $O(M)$ of functions on $L(M)$, we can obtain some interesting and useful results.

As an example, to each tensor S of type $(2, 1)$ on M there correspond n^3 functions $S^{\alpha}_{\gamma}{}^{\beta}$ on $L(M)$ and these functions have restrictions on $O(M)$. The following theorem is analogous to Theorem 2.2.

Theorem 4.2. *Let M be a pseudo-Riemannian manifold of signature r. To a tensor of type $(2, 1)$ on M, there corresponds a set of n^3 functions $S^{\alpha}_{\gamma}{}^{\beta}$ on $O(M)$ such that for any $z \in O(M)$ and any $A \in O(r, n-r)$,*

$$S^{\alpha^*}_{\gamma^*}{}^{\beta^*}(zA) = S^{\alpha}_{\gamma}{}^{\beta}(z) A^{\alpha^*}_{\alpha} A^{\beta^*}_{\beta} A^{\gamma}_{\gamma^*},$$

where $A=[A^{\gamma}_{\gamma^*}]$ and $A^{-1}=[A^{\alpha^*}_{\alpha}]$. Conversely, to any such set of n^3 functions on $O(M)$, there corresponds a tensor of type $(2, 1)$ on M. Moreover, this correspondence is one-to-one.

The proof of Theorem 4.2 is similar to that of Theorem 2.2 and is therefore omitted. As before, we shall refer to $S^{\alpha}_{\gamma}{}^{\beta}$ as the *functions on $O(M)$ corresponding to the tensor* S on M. Obviously, they are the restriction to $O(M)$ of the corresponding functions on $L(M)$.

Let $[g^{ik}]$ be the inverse matrix of $[g_{ij}]$. Then g^{ik} are the components of a global tensor field on M. The components g_{ij} and g^{ik} are frequently used to lower or raise the indices of tensors. Thus, the equation $S_i{}^j{}_k=g_{il}S^{lj}{}_k$ is equivalent to $S^{ji}{}_k=g^{il}S_l{}^j{}_k$. The advantage in considering the functions on $O(M)$ corresponding to tensors on M is that these functions remain essentially unchanged when indices are raised or lowered. We make this precise in

Theorem 4.3. *When an index of a function, say α, is lowered or raised, the function is multiplied by e_α. Thus, if $S^{\alpha\beta}{}_\gamma$ and $S_\alpha{}^\beta{}_\gamma$ are the functions on $O(M)$ corresponding to tensors on M having components $S^{ij}{}_k$ and $S_i{}^j{}_k=g_{il}S^{lj}{}_k$ respectively, then $S^{\alpha\beta}{}_\gamma=e_\alpha S_\alpha{}^\beta{}_\gamma$ (not summed over α).*

Proof. Let $z=(X_\alpha)$ be an orthonormal frame, and (X^β) its dual frame. Then

$$S_\alpha{}^\beta{}_\gamma(z)=S_i{}^j{}_k X^i_\alpha X^k_\gamma X^\beta_j=g_{il}S^{lj}{}_k X^i_\alpha X^k_\gamma X^\beta_j\ .$$

Since (X_α) is orthonormal,

$$g_{il}X^i_\alpha X^l_\varepsilon=e_\alpha\delta_{\alpha\varepsilon}\quad \text{(not summed over } \alpha\text{)},$$

i.e.,

$$g_{il}X^i_\alpha=e_\alpha\delta_{\alpha\varepsilon}X^\varepsilon_l=e_\alpha X^\alpha_l\quad \text{(not summed over } \alpha\text{)}.$$

Therefore,

$$S_\alpha{}^\beta{}_\gamma(z)=S^{lj}{}_k e_\alpha X^\alpha_l X^k_\gamma X^\beta_j=e_\alpha S^{\alpha\beta}{}_\gamma(z)\quad \text{(not summed over } \alpha\text{)}.$$

For the rest of this section, we shall study recurrent tensors with respect to the metric connections on a pseudo-Riemannian manifold. We first explain what a metric connection is.

Suppose we are given a connection in the bundle $O(M)$ of orthonormal frames (in the sense of connection on a principal fibre bundle). The inclusion map $O(M)\to L(M)$ is a bundle homomorphism and so the connection on $O(M)$ extends to a unique connection on $L(M)$ according to Theorem 2.1. This connection on $L(M)$ is then a linear connection on M. A linear connection on M induced from a connection on $O(M)$ as above is called a *metric connection*. We note that the definition of $O(M)$, and consequently the definition of a metric connection, involves the Riemannian metric g of M. It can be proved ([3, p. 158]) that a linear connection on M is a metric connection on M iff $\nabla g=0$. The familiar Riemannian (Levi-Civita) connection determined by g is the unique metric connection on M with zero torsion.

Associated with a metric connection are two fields of horizontal n-planes. The first is the field of horizontal n-planes in $O(M)$, the horizontal n-plane at a point $z\in O(M)$ being a linear subspace of $T_z(O(M))$. The second is the field of horizontal n-planes in $L(M)$, the horizontal n-plane at a point $z\in L(M)$ is a linear subspace of $T_z(L(M))$. If for each $z \in O(M)$, we regard $T_z(O(M))$ as a subspace of $T_z(L(M))$, the two horizontal n-planes at z are identical. This observation has implications on the holonomy bundles.

Containing any $z_0 \in O(M)$, there are two holonomy bundles. The first, $C[z_0]$, is the holonomy bundle of the connection on $O(M)$. The second, $B[z_0]$, is the holonomy bundle of the associated metric connection on $L(M)$. The next proposition says that we do not have to distinguish between the two.

Proposition 4.4. *Let a connection on $O(M)$ extend to a metric connection on $L(M)$, and $z_0 \in O(M)$, $z_1 \in L(M)$. Then,*

(a) *A curve $z(t)$ in $L(M)$ joining z_0, z_1 is a horizontal curve in $L(M)$ iff it is a horizontal curve in $O(M)$;*

(b) *The holonomy bundle $B[z_0]$ in $L(M)$ containing z_0 is the same as the holonomy bundle $C[z_0]$ in $O(M)$ containing z_0.*

Proof. (a) Let $u(t)=\pi z(t)$ be the projection of $z(t)$ in M. If $z(t)$ is a horizontal curve in $L(M)$, it is a parallel field of frames along the curve $u(t)$ with respect to the metric connection. Since $\nabla g=0$ for a metric connection, length and orthogonality of vectors in M are preserved by parallel transport. As $z(0)=z_0$ is an orthonormal frame, so is each $z(t)$. Therefore $z(t)$ is a curve in $O(M)$. Since the horizontal planes in $O(M)$ and $L(M)$ are identical, $z(t)$ is a horizontal curve in $O(M)$. The converse follows from the same observation.

(b) This follows easily from (a) and from the definition of $B[z_0]$ and $C[z_0]$.

Recurrent tensors with respect to the metric connections on a pseudo-Riemannian manifold will be called simply *recurrent tensors on a pseudo-Riemannian manifold*. The next two theorems are easy consequences of Theorems 2.3, 2.4 and Proposition 4.4.

Theorem 4.5. *Let S be a tensor on a pseudo-Riemannian manifold M. Then S is recurrent iff the restriction of its corresponding functions on $O(M)$ to any $B[z_0]$ of $O(M)$ has no common zero and are proportional to a set of constants.*

Theorem 4.6. *Let S be a tensor on a pseudo-Riemannian manifold M. Then S is covariantly constant iff the restriction of its corresponding functions on $O(M)$ to any $B[z_0]$ of $O(M)$ are constants, not all zero.*

5. Norm of tensors in a pseudo-Riemannian manifold

In this section, we continue our study of recurrent tensors on a pseudo-

Riemannian manifold M. First, we define the norm of a tensor and generalize the notion of null and non-null vectors to tensors. As before, we shall confine our discussions to tensors of type (2, 1) although the same is true for tensors of arbitrary type.

Let $T_1^2(u)$ be the linear space of tensors of type (2, 1) at $u\in M$. The inner product in $T_u(M)$ defined by the metric $g=(g_{ij})$ induces an inner product $\langle\ ,\ \rangle_u$ in $T_1^2(u)$ as follows. For $S=(S_k^{ij})$ and $T=(T_k^{ij})$ in $T_1^2(u)$,

$$\langle S, T\rangle_u=g_{ip}g_{jq}g^{kr}S_k^{ij}T_r^{pq}\ . \tag{5.1}$$

The *norm* $\|S_u\|$ of a tensor S at u is defined as $|\langle S, S\rangle_u|^{1/2}$. A tensor S is said to be *null* at u if $S_u\neq 0$ and $\|S_u\|=0$. Otherwise, it is said to be *non-null*. Of course, null tensors only exist when the Riemannian metric is indefinite.

We shall find a formula for $\|S_u\|$ in terms of the functions $S_\gamma^{\alpha\beta}$ on $O(M)$ corresponding to S. Suppose $z=(X_1, \cdots, X_n)$ is an orthonormal frame at u and $(X^1, \cdots, X^n)$ its dual frame. Direct verification with definition (5.1) of the inner product will show that the set $\{X_\alpha\otimes X_\beta\otimes X^\gamma\colon \alpha, \beta, \gamma=1, \cdots, n\}$ is an orthonormal basis of $T_1^2(u)$. It can also be proved that for any orthonormal frame $z=(X_\alpha)$ at u,

$$\|S_u\|^2=\Big|\sum_{\alpha\beta\gamma} e_\alpha e_\beta e_\gamma(S_\gamma^{\alpha\beta}(z))^2\Big|\ . \tag{5.2}$$

The natural projection $\pi\colon L(M)\to M$ sends a frame at a point u to u. For simplicity, we shall denote the restriction of π to $O(M)$ by the same symbol, and again call it the natural projection. The following theorem is an immediate consequence of (5.2).

Theorem 5.1. *Let S be any tensor in M and $S_\gamma^{\alpha\beta}$ its corresponding functions on $O(M)$. Then the function $|\sum_{\alpha\beta\gamma} e_\alpha e_\beta e_\gamma(S_\gamma^{\alpha\beta}(z))^2|$, being equal to the square norm of S_u, is constant on $\pi^{-1}(u)$ for every $u\in M$.*

We now prove

Theorem 5.2. *A recurrent tensor S on a pseudo-Riemannian manifold is either everywhere null or everywhere non-null.*

Proof. Since S is recurrent, we may choose (cf. Theorem 4.5) a $B[z_0]$ lying in $O(M)$ so that for all $z\in B[z_0]$,

$$S_\gamma^{\alpha\beta}(z)=f(z)c_\gamma^{\alpha\beta}$$

where f is nowhere zero on $B[z_0]$ and the constants $c_\gamma^{\alpha\beta}$ are not all zero. Therefore, by (5.2),

$$(5.3)\qquad ||S_{\pi z}||^2=|\sum_{\alpha\beta\gamma} e_\alpha e_\beta e_\gamma (S_\gamma^{\alpha\beta}(z))^2|=(f(z))^2|\sum_{\alpha\beta\gamma} e_\alpha e_\beta e_\gamma (c_\gamma^{\alpha\beta})^2|\ .$$

Since f is nowhere zero on $B[z_0]$ and $\pi B[z_0]=M$, it follows from (5.3) that $||S_u||^2$ is either everywhere zero or everywhere nonzero in M.

It is seen from the above theorem that for a non-null recurrent tensor S, the norm $||S_u||>0$ for every $u\in M$. Consequently, the function $||S||: M\to R$ defined by $||S||(u)=||S_u||$ is differentiable on M. This function $||S||$ will be used in § 6.

6. The recurrence covector in a pseudo-Riemannian manifold

In this section, we shall study the recurrence covector of a non-null recurrent tensor on a pseudo-Riemannian manifold. We first prove

Theorem 6.1. *The recurrence covector of a non-null recurrent tensor on a pseudo-Riemannian manifold is globally a gradient.*

Proof. Let S be any non-null recurrent tensor on a pseudo-Riemannian manifold M. We shall show that S satisfies the condition stated in Theorem 3.3. As in the proof of Theorem 5.2, we choose a $B[z_0]$ lying in $O(M)$ and obtain, on $B[z_0]$

$$(5.3)\qquad ||S_{\pi z}||^2=|\sum_{\alpha\beta\gamma} e_\alpha e_\beta e_\gamma (S_\gamma^{\alpha\beta}(z))^2|=(f(z))^2|\sum_{\alpha\beta\gamma} e_\alpha e_\beta e_\gamma (c_\gamma^{\alpha\beta})^2|\ .$$

It follows from this and the fact that S is non-null that for each $u\in M$, $(f(z))^2$ is a nonzero constant on $\pi^{-1}(u)\cap B[z_0]$. Since f is a nowhere zero differentiable function on the arcwise connected $B[z_0]$, it has a constant sign on $B[z_0]$, and consequently, for each $u\in M$, f is constant on $\pi^{-1}(u)\cap B[z_0]$. Therefore by Theorem 3.3, the recurrence covector of S is globally a gradient.

It would be interesting to compare Theorem 6.1 with Walker's result quoted in § 1. While Walker's result asserts that if the curvature tensor of a Riemannian connection is recurrent, then no matter whether it is null or non-null, its recurrence covector is locally a gradient, our result asserts that for every non-null recurrent tensor, its recurrence covector is globally a gradient. Moreover, our proof is entirely different from Walker's proof, which involves an extremely delicate manipulation with indices.

Since every tensor is non-null with respect to a definite Riemannian metric, an immediate consequence of Theorem 6.1 is the rather obvious result that every recurrent tensor in a pseudo-Riemannian manifold with definite Riemannian metric has a globally gradient recurrence covector.

As to null recurrent tensors Theorem 6.1 is not true in general. In fact, Patterson [5, § 4] has given an example of a recurrent tensor on an indefinite Riemannian manifold whose recurrence covector is not even locally a gradient. However, we have the following easy consequences of Theorem 3.1.

Proposition 6.2. *Let S be a null recurrent tensor on a pseudo-Riemannian manifold M. Then*

(a) *The recurrence covector of S is globally a gradient iff S is co-directional to a null, parallel tensor field;*

(b) *The recurrence covector of S if locally a gradient iff in a suitably chosen neighbourhood of each point there exists a null, parallel tensor field with which S is co-directional.*

A problem in the theory of Riemannian connection with recurrent curvature on a pseudo-Riemannian manifold is to find a natural geometrical interpretation of the recurrence covector, see e.g., [8, p. 238]. We end this section with an interpretation of the recurrence covector of a non-null recurrent tensor. We recall that for a non-null recurrent tensor S, the norm $||S||$ of S is a differentiable function on M and is everywhere positive.

Theorem 6.3. *Let S be a non-null recurrent tensor on a pseudo-Riemannian manifold M, ξ the recurrence covector and $||S||$ the norm of S. Then $\xi=\nabla(\log||S||)$.*

Proof. Without loss of generality, we may assume that S is of type $(2,1)$. By Theorem 6.1, ξ is of the form ∇W for some function W on M. By Theorem 3.1, $e^{-W}S$ is covariantly constant. Choosing a $B[z_0]$ contained in $O(M)$, we have, on $B[z_0]$,

$$e^{-W(\pi z)}S_{\gamma}^{\alpha\beta}(z)=c_{\gamma}^{\alpha\beta} ,$$

where $S_{\gamma}^{\alpha\beta}$ are the functions on $L(M)$ corresponding to S and $c_{\gamma}^{\alpha\beta}$ are constants, not all zero. Then, using the notations in § 5,

$$||S_{\pi z}||^2=|\sum_{\alpha\beta\gamma} e_\alpha e_\beta e_\gamma(S_{\gamma}^{\alpha\beta}(z))^2|=e^{2W(\pi z)}|\sum_{\alpha\beta\gamma} e_\alpha e_\beta e_\gamma(c_{\gamma}^{\alpha\beta})^2| .$$

I.e.,

$$||S_u||^2=e^{2W(u)}|\sum_{\alpha\beta\gamma} e_\alpha e_\beta e_\gamma(c_{\gamma}^{\alpha\beta})^2|$$

for all $u\in M$. This means that

$$\log||S||=W+\frac{1}{2}\log|\sum_{\alpha\beta\gamma} e_\alpha e_\beta e_\gamma(c_{\gamma}^{\alpha\beta})^2| ,$$

and so

$$\xi = \nabla W = \nabla(\log ||S||) .$$

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