NOTES ON THE STABILITY OF MINIMAL SUBMANIFOLDS OF RIEMANNIAN MANIFOLDS

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1. Introduction. In a recent work [5], *J. Spruck* studied the stability of minimal submanifolds of n-dimensional Euclidean space. In this note, we study similar results in general ambient space. The author wishes to express his hearty thanks to Prof. *H. Kitahara* for his kind advices and to Mr. *M. Maeda* for his valuable suggestions.

2. Statement of results. Let $M=M^m$ be an m-dimensional compact orientable C^∞ manifold with boundary ∂M and let $x: M\to\bar{M}^n$ be a minimal immersion of M into an n-dimensional Riemannian manifold $\bar{M}^n$. It is well known that if E is a vector field on $\bar{M}^n$ such that $E|M$ is normal and $E|\partial M=0$, and if ϕ_t denotes the flow generated by E in a neighborhood of M in $\bar{M}^n$, then setting $A(t)=$ volume $\phi_t(M)$, $dA/dt|_{t=0}=0$. We say that M is (infinitesimally) stable if $d^2A/dt^2|_{t=0}>0$, i.e., volume M is a strict minimum for all such variations.

Theorem 1. *Let $\bar{M}^n(n\geqq 4)$ be a complete simply connected Riemannian manifold with sectional curvature $-b^2\leqq\bar{K}_\sigma\leqq-\delta b^2$ for some constants b and δ such that $b\geqq 0$, $0<\delta\leqq 1$. Let $x: M=M^m(m\geqq 3)\to\bar{M}^n$ be a minimal immersion with the second fundamental form B, and let $f: M\to R$ be the function defined as $f(p):=\mathrm{Max}\,\{||B||^2(p)-m\delta b^2\}$ for $p\in M$. Then, there is a constant $c_1(m)>0$ depending only on m such that if $\left(\int_M f^{m/2}dV\right)^{1/m}<c_1(m)$, then (M, x) is stable.*

Theorem 2. *Let $\bar{M}(n\geqq 4)$ be a Riemannian manifold with sectional curvature $0<\bar{K}_\sigma\leqq b^2$ and injectivity radius $\bar{R}(\bar{M})\geqq b^{-1}\pi$ for some constant $b>0$. Let $x: M=M^m(m\geqq 3)\to\bar{M}^n$ be a minimal immersion with the second fundamental form B. Then, there is a constant $c_2(m)>0$ depending only on m such that if $\left(\int_M(||B||^2+mb^2)^{m/2}dV\right)^{1/m}<c_2(m)$, then (M, x) is stable.*

Remark. If $b=0$ in Theorem 1, then our Theorem reduces to the Theorem 2 in [5]. We note that the condition on injectivity radius $\bar{R}(\bar{M})$ of $\bar{M}$ in Theorem 2 holds for complete simply connected n-dimensional manifolds with $(1/4)b^2<$

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$\bar{K}_\sigma \leqq b^2$ (e.g., n-dimensional sphere of radius b^{-1}), and for complete simply connected Riemannian symmetric spaces with $0<\bar{K}_\sigma \leqq b^2$, and for compact even-dimensional orientable manifolds with $0<\bar{K}_\sigma \leqq b^2$.

3. The second variation. Let $\bar{M}=\bar{M}^n$ be an n-dimensional Riemannian manifold with metric $\langle , \rangle$ and connection $\bar{\nabla}$. Let $x: M=M^m \to \bar{M}$ be a minimal immersion. We denote by $\langle , \rangle$ and ∇ the induced Riemannian metric and the induced Riemannian connection on M respectively. The tangent and normal bundle of M are denoted by TM and NM respectively, and X^T, X^N denote the projection of a vector field X along the mapping x onto TM, NM respectively. The second fundamental form $B: TM \times TM \to NM$ of the immersion x is given by

$$B(X, Y)=\bar{\nabla}_X Y-(\bar{\nabla}_X Y)^T=(\bar{\nabla}_X Y)^N .$$

Let $\{e_1, \cdots, e_m\}$ be an orthonormal basis of TM_p and let $E \in NM_p$, where TM_p and NM_p are the tangent and normal vector space of M at $p \in M$ respectively. We define

$$||B(E)||^2:=\sum_{i,j} (\langle B(e_i, e_j), E\rangle)^2 .$$

It is easy to see that $||B(E)||$ is independent of the choice of an orthonormal basis of TM_p and that if E is unit, $||B(E)||^2$ is the squares of the principal curvatures of M at p with respect to the unit normal direction E. Let $\{E_1, \cdots, E_{n-m}\}$ be an orthonormal basis of NM_p. Then the quantity

$$||B||^2:=\sum_k ||B(E_k)||^2$$

is the square of the length of the second fundamental form.

We next define the Laplace operator $\Delta: \Gamma(NM) \to \Gamma(NM)$, where $\Gamma(NM)$ denotes the space of C^∞ normal vector fields on M. We define the connection $\nabla^\perp_X$ in NM as $\nabla^\perp_X \nu=(\bar{\nabla}_X \nu)^N$ for $X \in \Gamma(TM)$, $\nu \in \Gamma(NM)$, where $\Gamma(TM)$ denotes the space of C^∞ tangent vector fields on M. In terms of this connection

$$(\Delta\nu)(p):=\sum_j \{\nabla^\perp_{e_j}\nabla^\perp_{e_j}\nu-(\nabla^\perp_{\nabla_{e_j}e_j})\nu\}(p)$$

where $\{e_1, \cdots, e_m\}$ is an orthonormal basis of TM_p.

Let us denote by $\bar{R}$ the curvature tensor field of $\bar{M}$. And set

$$\bar{R}(X)=\sum_j \bar{R}_{e_j, X}e_j$$

for each vector $X \in T\bar{M}_p$, where $\{e_1, \cdots, e_m\}$ is an orthonormal basis of TM_p.

We can state the second variation formula (cf. [3]):

$$(1)\qquad \left.\frac{d^2A}{dt^2}\right|_{t=0}=-\int_M\{\langle E,\, \Delta E\rangle+||B(E)||^2-\langle \bar{R}(E),\, E\rangle\}dV$$

for a variation vector field $E\in\Gamma(NM)$, $E|\partial M=0$, where dV is the volume element of M.

We analyze this formula further by writing $E=u\nu$, where ν is a unit normal vector field on M and u is a C^∞ function on M vanishing on ∂M. Then

$$\begin{aligned}\Delta E&=\sum_j\{\nabla^\perp_{e_j}\nabla^\perp_{e_j}(u\nu)-\nabla^\perp_{\nabla_{e_j}e_j}(u\nu)\}\\&=\sum_j\{\nabla^\perp_{e_j}(e_j(u)\nu+u\nabla^\perp_{e_j}\nu)-(\nabla_{e_j}e_j)(u)\nu-u(\nabla^\perp_{\nabla_{e_j}e_j})\nu\}\\&=\sum_j[\{e_j(e_j(u))-(\nabla_{e_j}e_j)u\}\nu+u\{\nabla^\perp_{e_j}\nabla^\perp_{e_j}\nu-(\nabla^\perp_{\Delta_{e_j}e_j})\nu+2e_j(u)\nabla^\perp_{e_j}\nu\}\\&=(\Delta u)\nu+u\Delta\nu+2\Sigma_j e_j(u)\nabla^\perp_{e_j}\nu\end{aligned}$$

where Δu is the Laplacian of u. Hence

$$(2)\qquad \langle E,\, \Delta E\rangle=u\Delta u+u^2\langle\nu,\, \Delta\nu\rangle$$

since $\langle\nu,\, \nabla^\perp_{e_j}\nu\rangle=0$.
Combining (1) and (2) we have the formula

$$(3)\qquad \left.\frac{d^2A}{dt^2}\right|_{t=0}=-\int_M\{u\Delta u+(\langle\nu,\, \Delta\nu\rangle+||B(\nu)||^2-\langle\bar{R}(\nu),\, \nu\rangle)u^2\}dV$$

In what follows, we consider only the case where $\bar{M}$ is assumed to satisfy the curvature condition of Theorem 2, namely $0<\bar{K}_\sigma\leqq b^2$, where b is a positive real number, the other case is similar.

Lemma 1. *Assume* $0<\bar{K}_\sigma\leqq b^2$. *Then*

$$\begin{aligned}\left.\frac{d^2A}{dt^2}\right|_{t=0}&\geqq-\int_M\{u\Delta u+(||B||^2+mb^2)u^2\}dV\\&=\int_M\{|\nabla u|^2-(||B||^2+mb^2)u^2\}dV\,,\end{aligned}$$

where ∇u is the gradient vector field of u.

Proof. Since $||B(\nu)||^2\leqq||B||^2$ and $\langle\bar{R}(E),\, E\rangle=u^2\langle\bar{R}(\nu),\, \nu\rangle\geqq-mb^2u^2$ we need only show $\langle\nu,\, \Delta\nu\rangle\leqq0$. For $\langle\nu,\, \Delta\nu\rangle=\sum_j\langle\nu,\, \nabla^\perp_{e_j}\nabla^\perp_{e_j}\nu-\nabla^\perp_{\nabla_{e_j}e_j}\nu\rangle=\sum_j\langle\nu,\, \nabla^\perp_{e_j}\nabla^\perp_{e_j}\nu\rangle=-\sum_j|\nabla^\perp_{e_j}\nu|^2$. Here we have used the identities $\langle\nu,\, \nabla^\perp_X\nu\rangle=0$ and $0=X\langle\nu,\, \nabla^\perp_X\nu\rangle=|\nabla^\perp_X\nu|^2+\langle\nu,\, \nabla^\perp_X\nabla^\perp_X\nu\rangle$ for each vector field X on M. Substitution in (3) completes the proof.

From this Lemma we see that M is stable if we can show that

$$(4)\qquad \int_M(||B||^2+mb^2)u^2dV<\int_M|\nabla u|^2dV$$

for all $u \not\equiv 0$ in the Sobolev space $\mathring{H}_1(M)$.

4. Proof of Theorem 2. At first, applying the Sobolev inequality of *D. Hoffman* and *J. Spruck* [2] we get the following Lemma.

Lemma 2. *Assume* $0<\bar{K}_\sigma \leqq b^2$, *and let* $x: M \to \bar{M}$ *be a minimal immersion and* h *be a non-negative* C^1 *function on* M *vanishing on* ∂M. *Then*

$$\left(\int_M h^{2m/(m-2)} dV\right)^{(m-2)/2m} \leqq c_2(m)^{-1} \left(\int_M |\nabla h|^2 dV\right)^{1/2}$$

provided

$$b(1-\alpha)^{-1/m}(\omega_m^{-1} \operatorname{Vol}(\operatorname{supp} h))^{1/m} \leqq \frac{1}{2}$$

and

$$2\rho \leqq \bar{R}(\bar{M}) \text{ (the injectivity radius of } \bar{M})$$

where

$$\rho = b^{-1} \sin^{-1} [2b(1-\alpha)^{-1/m}(\omega_m^{-1} \operatorname{Vol}(\operatorname{supp} h))^{1/m}] .$$

Here α *is a free parameter* $0<\alpha<1$ *and*

$$c_2(m) = c_2(m, \alpha) = (1/\pi) 2^{1-m} \alpha (1-\alpha)^{1/m} ((m-2)/m) \omega_m^{1/m}$$
$$(= \text{the same constant in Theorem 2}) .$$

Proof. At first we notice that there is a gap in the proof of Lemma 4.2. in [2]. But we can supply the gap and then have the Sobolev inequality in the following form:

Under the same assumptions and the same notations of this Lemma we have

$$\left(\int_M h^{m(m-1)} dV\right)^{(m-1)/m} \leqq c(m) \int_M |\nabla h| dV$$

because of M is minimal, where $c(m) = c(m, \alpha) = \pi 2^{m-2} \alpha^{-1} (1-\alpha)^{-1/m} (m/(m-1)) \omega_m^{-1/m}$. So, replacing h by $h^{2(m-1)/(m-2)}$ in the Sobolev inequality gives

$$\left(\int_M h^{2m/(m-2)} dV\right)^{(m-1)/m} \leqq c(m) \int_M \frac{2(m-1)}{m-2} h^{m/(m-2)} |\nabla h| dV .$$

By Hölder's inequality

$$\left(\int_M h^{2m/(m-2)} dv\right)^{(m-1)/m} \leqq c(m) \frac{2(m-1)}{m-2} \left(\int_M h^{2m/(m-2)} dV\right)^{1/2} \left(\int_M |\nabla h|^2 dV\right)^{1/2} .$$

Therefore by virture of $c(m) 2(m-1)/(m-2) = c_2(m)^{-1}$ we get

$$\left(\int_M h^{2m/(m-2)} dV\right)^{(m-2)/2m} \leqq c_2(m)^{-1}\left(\int_M |\nabla h|^2 dV\right)^{1/2}.$$

This completes the proof.

Remark. It is easy to see that $c_2(m)$ is maximum when $\alpha=m/(m+1)$ .From the definition of $\mathring{H}_1(M)$ we see that Lemma 2 holds for any $h\in\mathring{H}_1(M)$ which satisfies the conditions of this Lemma. And we can easily show that

$$\left(\int_M (||B||^2+mb^2)^{m/2} dV\right)^{1/m} < c_2(m)$$

implies $b(1-\alpha)^{-1/m}(\omega_m^{-1}\,\mathrm{Vol}\,M)^{1/m}<1/2$ and hence

$$\rho_0 := b^{-1}\sin^{-1}[2b(1-\alpha)^{-1/m}(\omega_m^{-1}\,\mathrm{Vol}\,M)^{1/m}] < \frac{\pi}{2}b^{-1}$$

implies $2\rho_0<\pi b^{-1}<\bar{R}(\bar{M})$. Therefore we see that the conditions of Theorem 2 imply the conditions of Lemma 2.

Now we are going to prove Theorem 2. As we have just remarked in the previous section, we need only show that the hypothesis $\left(\int_M (||B||^2+mb^2)^{m/2} dV\right)^{1/m} < c_2(m)$ implies (4). Suppose for contradiction that (4) is false, namely

$$\int_M |\nabla u|^2 dV \leqq \int_M (||B||^2+mb^2)u^2 dV \tag{5}$$

for some $u\in\mathring{H}_1(M)$, $u\not\equiv 0$. By the property of $\mathring{H}_1(M)$ we can assume that the function u in the inequality (5) satisfies $u\geqq 0$ on M. Then from Lemma 2 (or from Remark of Lemma 2)

$$\left(\int_M u^{2m/(m-2)} dV\right)^{(m-2)/2m} \leqq c_2(m)^{-1}\left(\int_M |\nabla u|^2 dV\right)^{1/2}. \tag{6}$$

Then from (5) we have

$$\left(\int_M u^{2m/(m-2)} dV\right)^{(m-2)/2m} \leqq c_2(m)^{-1}\left(\int_M (||B||^2+mb^2)u^2 dV\right)^{1/2}. \tag{7}$$

But by Hölder's inequality

$$\int_M (||B||^2+mb^2)u^2 dV \leqq \left(\int_M (||B||^2+mb^2)^{m/2} dV\right)^{2/m}\left(\int_M u^{2m/(m-2)} dV\right)^{(m-2)/m}. \tag{8}$$

Combining (7) and (8) gives

$$\left(\int_M u^{2/(m-2)} dV\right)^{(m-2)/2m} \leqq c_2(m)^{-1}\left(\int_M (||B||^2+mb^2)^{m/2} dV\right)^{1/m}\left(\int_M u^{2m/(m-2)} dV\right)^{(m-2)/2m}.$$

Hence $\left(\int_M (||B||^2+mb^2)^{m/2} dV\right)^{1/m} \geqq c_2(m)$ contradicting our assumption.

5. Appendix. We shall show that in the $(m+1)$-dimensional unit sphere S^{m+1} with the canonical Riemannian metric structure there are many compact orientable, minimal submanifolds with boundary which satisfy the condition of Theorem 2. At first, we note that $\bar{M}:=S^{m+1}(n=m+1)$ satisfy the hypotheses of Theorem 2 for $b=1$, i.e., it is of constant sectional curvature $\bar{K}_\sigma=1$ and injectivity radius $\bar{R}(\bar{M})=\pi$.

For any α, $0<\alpha<1$, let

$$\Sigma_\alpha^m=\{x=(x^1, \cdots, x^{m+1}) \in S^m;\ x^{m+1} \geqq \alpha\}\ ,$$

where S^m is the m-dimensional unit sphere of R^{m+1}. Since S^m is an orientable, totally geodesic submanifold of S^{m+1}, $M:=\Sigma_\alpha^m$ is a compact orientable, totally geodesic submanifold of S^{m+1} and with boundary $\partial\Sigma_\alpha^m=\{x=(x^1, \cdots, x^{m+1}) \in S^m;\ x^{m+1}=\alpha\}$. On the other hand, volume Σ_α^m is a decreasing continuous function of α, $0<\alpha<1$, and volme $\Sigma_\alpha^m \to 0$ as $\alpha \to 1$. Thus have certainly a constant α satisfying the condition of Theorem 2, i.e.,

$$\left(\int_M (||B||^2+m)^{m/2} dV\right)^{1/m} = m^{1/2}(\mathrm{Vol}\ \Sigma_\alpha^m)^{1/m} < c_2(m)\ .$$

Therefore we have in S^{m+1} a compact orientable, totally geodesic (and hence minimal) submanifold Σ_α^m with boundary which is stable.

To get other compact orientable, minimal submanifolds with boundary which are stable, we consider R^{m+2}, the $(m+2)$-dimensional Euclidean space, as $R^{p+1}\times R^{q+1}$ let

$$\Sigma_\alpha^p(r)=\{x=(x^1, \cdots, x^{p+1}) \in S^p(r);\ x^{p+1} \geqq \alpha\}\ ,$$
$$\Sigma_\beta^q(s)=\{y=(y^1, \cdots, y^{q+1}) \in S^q(s);\ y^{q+1} \geqq \beta\}\ ,$$

where $p+q=m$ and $r, s>0$, $r^2+s^2=1$ and $0<\alpha, \beta<1$, and $S^p(r)$ (resp. $S^q(s)$) is the p-dimensional sphere of radius r in R^{p+1} (resp. the q-dimensional sphere of radius s in R^{q+1}). Then $\Sigma_\alpha^p(r)\times\Sigma_\beta^q(s)$ a compact orientable hypersurface of S^{m+1} and with boundary $(\partial\Sigma_\beta^p(r)\times\Sigma_\beta^q(s))\cup(\Sigma_\alpha^p(r)\times\partial\Sigma_\beta^q(s))$. It is easy to see that the second fundamental form B has eigen-values s/r of multiplicity p and $-r/s$ of multiplicity q. And hence $\Sigma_\alpha^p(r)\times\Sigma_\beta^q(s)$ is minimal if and only if $r=\sqrt{p/m}$, $s=\sqrt{q/m}$. Thus, when $M:=\Sigma_\alpha^p(r)\times\Sigma_\beta^q(s)$ is minimal, we have $||B||^2=m$. Then from the previous discussion we have certainly constants α, β satisfying the condition of Theorem 2, i.e.,

$$\left(\int_M (\|B\|^2+m)^{m/2} dV\right)^{1/m} = (2m)^{1/2} \operatorname{Vol} \Sigma_\alpha^p(\sqrt{p/m}) \operatorname{Vol} \Sigma_\beta^q(\sqrt{q/m}) < c_2(m) .$$

Therefore we have in S^{m+1} a compact orientable, minimal submanifold $\Sigma_\alpha^p(\sqrt{p/m}) \times \Sigma_\beta^q(\sqrt{q/m})$ with boundary which is stable, but not totally geodesic.

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