NOTE ON THE LAW OF THE ITERATED LOGARITHM FOR MULTI-DIMENSIONAL STATIONARY PROCESSES SATISFYING MIXING CONDITIONS

By

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1. Introduction and result

In [3, Lemma 2], Finkelstein proved the law of the iterated logarithm for sequences of independent identically distributed random vectors. Analogical result for m-dependent random vectors was obtained by *Yokoyama* [8, Lemma 3]. The proof of Lemma 3 in [8] is based on the result of [7], the law of the iterated logarithm for one-dimensional ϕ-mixing stationary processes. From this point of view, we can easily see that the law of the iterated logarithm for multi-dimensional mixing processes is induced from the law for one-dimensional cases.

Let $\{X_n, -\infty<n<\infty\}$ be a sequence of random variables defined on some probability space $(\Omega, \mathscr{B}, P)$ which is strictly stationary and satisfies one of the following conditions:

(I) $$\sup |P(A\cap B)-P(A)P(B)|/P(A)=\phi(n)\downarrow 0 \quad (n\to\infty)$$

(the ϕ-mixing condition) and

(II) $$\sup |P(A\cap B)-P(A)P(B)|=\alpha(n)\downarrow 0 \quad (n\to\infty)$$

(the strong mixing condition).

Here the supremum is taken over all $A\in\mathscr{M}^{k}_{-\infty}$ and $B\in\mathscr{M}^{\infty}_{k+n}$, and $\mathscr{M}^{b}_{a}(-\infty\leq a<b\leq\infty)$ denotes the σ-field generated by $X_a, \cdots, X_b$.

Let $\{Z_n=(Z_{n1}, \cdots, Z_{np}), -\infty<n<\infty\}$ be a strictly stationary sequence of random vectors defined on $(\Omega, \mathscr{B}, P)$ with values in p-dimensional Euclidean space $R^p (p\geq 1)$. For $\{Z_n\}$, mixing conditions (I) and (II) are defined in the same manner, here $\mathscr{M}^{b}_{a}$ is generated by $Z_a, \cdots, Z_b$. If $\{X_a, \cdots, X_b\}$ and $\{Z_a, \cdots, Z_b\}$ generate the same σ-field $\mathscr{M}^{b}_{a}$ for $-\infty\leq a<b\leq\infty$, then $\{X_n\}$ and $\{Z_n\}$ have the same mixing coefficients $\phi(n)$ or $\alpha(n)$. If $\{X_n\}$ has the same mixing coefficients as $\{Z_n\}$, then we denote $\{X_n\}$ by $\{X_n(Z)\}$. In what follows, we assume that $EX_0=0$, $EZ_0=0$ and $0<\sigma^2=EX_1^2+2\sum_{j=2}^{\infty} EX_1X_j<\infty$.

Let K be the p-th order symmetric positive definite matrix $\{K_{ij}\}$ with inverse $\{K^{ij}\}$, and let $H(K)$ be the reproducing kernel space with reproducing kernel K, that is, $H(K)$ consists of all p-dimensional vectors $x=(x_1, \cdots, x_p)$ with inner product

$$(1) \qquad \langle x, y\rangle_K=\sum_{i,j=1}^{p} x_i K^{ij} y_j \quad \text{for} \quad x, y \in H(K) .$$

Put

$$(2) \qquad \Delta_{ij}=EZ_{1i}Z_{1j}+\sum_{k=2}^{\infty} EZ_{1j}Z_{kj}+\sum_{k=2}^{\infty} EZ_{ki}Z_{1j} , \qquad i, j=1, \cdots, p ,$$

if these series converge. Define the p-th order matrix Δ by

$$(3) \qquad \Delta=\{\Delta_{ij}\} .$$

Further, when Δ is positive definite, let B_p denote the unit ball of $H(\Delta)$, i.e.,

$$(4) \qquad B_p=\{x \in R^p: ||x||_\Delta \leq 1\}$$

where $||\cdot||_\Delta$ denotes the norm of $H(\Delta)$ which is defined as $||x||_\Delta^2=\langle x, x\rangle_\Delta$ for $x \in H(\Delta)$.

Theorem 1. *Suppose that the sequence* $\{Z_n=(Z_{n1}, \cdots, Z_{np}), -\infty<n<\infty\}$ *satisfies* (I) *or* (II) *and that the matrix* Δ *is positive definite. Suppose further that the sequences* $\{X_n(Z)\}$ *obey the law of the iterated logarithm, then with probability* 1, *the sequence*

$$(5) \qquad \Sigma_n=\frac{\Sigma_{i=1}^n Z_i}{(2n \log \log n)^{1/2}} , \qquad n=3, 4, \cdots$$

is relatively compact and the set of its limit points is B_p.

Let $\hat{R}^p$ be the conjugate space of R^p and let

$$(6) \qquad \sigma_T^2=E\{(TZ_1)^2\}+2\sum_{j=2}^{\infty} E\{(TZ_1)(TZ_j)\} \quad \text{for} \quad T \in \hat{R}^p .$$

We easily show that if the series in (2) converge for $i, j=1, \cdots, p$ and Δ is positive definite, then $\sigma_T^2=||T||_{\Delta^{-1}}^2$ and $\sigma_T^2=0$ if and only if $T=0$. The rest of the proof of Theorem 1 is obtained in the same line as the proof of Lemma 3 in [8] by using Lemma 3 in [2, p. 172].

2. Applications

Reznik [7], *Oodaira-Yoshihara* [6] and many authors have shown that the law of the iterated logarithm for mixing processes holds under the suitable conditions for decays of mixing coefficients. In view of Theorem 1, we shall show their multivariate versions.

(C-1) (see [4, Corollary 3]) $\{Z_n\}$ satisfies (I) with

1. $E||Z_n||^{2+\delta}<\infty$ for some $\delta>0$ ($||\cdot||$ is the usual Euclidean norm);
2. $\Sigma\{\phi(n)\}^{(1+\delta)/(2+\delta)}<\infty$.

(C-2) (see [6, Theorem 4]) $\{Z_n\}$ satisfies (II) with

1. $||Z_n||<C<\infty$ with probability 1;
2. $\alpha(n)=O(1/n^{1+\varepsilon})$ for some $\varepsilon>0$.

(C-3) (see [6, Theorem 5]) $\{Z_n\}$ satisfies (II) with

1. $E||Z_n||^{2+\delta}<\infty$ for some $\delta>0$;
2. $\Sigma\{\alpha(n)\}^{\delta'/(2+\delta')}<\infty$ for some $\delta'(0<\delta'<\delta)$.

Theorem 2. *Suppose that the matrix Δ is positive definite and that one of the conditions* (C-1), (C-2) *and* (C-3) *is satisfied, then with probability* 1, *the sequence*

$$\Sigma_n=\frac{\Sigma_{i=1}^n Z_i}{(2n\log\log n)^{1/2}}, \quad n=3, 4, \cdots$$

is relatively compact and the set of its limit points is B_p.

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