PRIME AND PRIMARY IDEAL THEORIES IN NONASSOCIATIVE ALGEBRAS*

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(Received October 13, 1975)

1. Introduction and definitions.

Several definitions of a prime ideal in an arbitrary nonassociative ring have been introduced. The recent one of *Myung* [12], based on a *-operation in the family of ideals, effectively extended the results of *Tsai* [19] for Jordan rings to a class of nonassociative rings which include both Jordan and alternative rings.

The purpose of this paper is to establish a primary and tertiary ideal theory for arbitrary nonassociative rings which is based on the notion of prime ideal given by *Myung* [12]. Throughout the paper we will deal with an arbitrary nonassociative algebra R over a commutative associative ring Φ with identity rather than a nonassociative ring. We first recall some definitions in [12].

Definition 1.1. Let R be an arbitrary nonassociative algebra and $I(R)$ be the set of ideals in R. A *-operation on R is a function from $\mathscr{I}(R)\times\mathscr{I}(R)$ into the set of submodules of R such that, for ideals A, B, C, D,

(*1) if $A\subseteq C$ and $B\subseteq D$ then $A*B\subseteq C*D$,

(*2) $(0)*A=B*(0)=(0)$,

(*3) if $\bar{R}$ is a homomorphic image of R then $\overline{A*B}=\bar{A}*\bar{B}$.

It is shown in [12, Lemma 1.1] that in the presence of (*3) Condition (*2) is equivalent to

(*2′) $A*B\subseteq A\cap B$.

Various examples of *-operation for nonassociative rings are given in [12] and [13], and for most of which $A*B$ happens to be an ideal of R for ideals A, B. In fact, we put.

Definition 1.2. A *-operation is said to be strong if * is a binary operation on $\mathscr{I}(R)$.

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* The author gratefully acknowledges that this research was supported by the University of Northern Iowa under the 1974 Summer Faculty Research Fellowships. This paper incorporates the results presented to the International Congress of Mathematics at Vancouver, Canada, 1974.

In Section 2 we will construct various strong *-operations for most of the well-known nonassociative algebras, which satisfy the following additional condition.

Definition 1.3. A *-operation is said to be left-additive if $(A+B)*C \subseteq A*C+B*C$ for all $A, B, C \in \mathscr{I}(R)$. A right-additive *-operation is similarly defined.

This definition was introducted in [13] to generalize the results of *Murata, Kurata* and *Marubayashi* [11] for associative rings to nonassociative rings. For an arbitrary nonassociative algebra R, one can easily construct a left-additive, strong *-operation in R (Section 2).

Specifically, this paper is divided into two parts. In the first part, we revisit the results of *Myung* [12] to include some larger classes of rings in the present theory and then introduce the Baer lower radical into arbitrary algebras by using a *-operation. An important application of this is that if R is an s-algebra then there exists a strong left-additive *-operation in R which virtually leads to the same prime radical as in *Zwier* [21]. The most well-known s-algebras are 2- and 3-algebras, and the largest known class of 3-algebras is the class of weakly W-admissible algebras of *Thedy* [17] (Section 2). In the second part, using a strong left-additive *-operation, we shall give a definition and basic properties for primary ideals in an arbitrary nonassociative algebra which is based on the concept of prime ideal disscussed in the first part. We then give a necessary and sufficient condition in terms of *-operation for those algebras in which the Lasker-Noether decomposition theorem holds. Finally, a *-operation-analog of the tertiary ideal is introduced, and it is shown that, in a noetherian algebra, any ideal can be represented as a finite intersection of tertiary ideals.

A remarkable advantage of the present theory is that a particular choice of *-operation for an individual class of algebras yields a primary and tertiary ideal theory in that class which seems to be in the best analogy with the classical theory for associative rings and with the known theory for the Jordan case by *Tsai* and *Foster* [20]. If R is an s-algebra then one can find a *-operation which yields a primary ideal theory for s-algebras. In particular, if R is a weakly W-admissible algebra (a 3-algebra) and we set $A*B=AB^2+B^2A+(AB)B+B(AB)+(BA)B+B(BA)$ for ideals A, B in R, then it is shown that * is strong and left-additive. Finally, we introduce a *-operation in a class of modules with multi-operators under appropriate notion of ideal, so that the present theory can be carried out to this class. A special case of this yields the primary ideal theory of *Tsai* and *Foster* [20] for quadratic Jordan algebras.

Definition 1.4. For any algebra R, f is defined to be function of R into $\mathscr{I}(R)$ such that for every a in R

(f1) $a \in f(a)$,

(f2) if $x \in f(a)$ then $f(x) \subseteq f(a)$,

(f3) if $\bar{R}$ is a homomorphic image of R then $\overline{f(a)} = f(\bar{a})$.

The function f is similar to one given in [11] and was introduced by *Myung* [12] to generalize the notion of prime ideal in two ways by combining with *-operation. The basic example of $f(a)$ is the ideal (a, S) generated by a and a fixed subset S of R. The most efficient example of $f(a)$ is the principal ideal (a) generated by a in R, to which we will restrict our attention for the second part of this paper. If A is an ideal of R, we denote the ideal $\sum_{a \in A} f(a)$ by $f(A)$. We then note that $A \subseteq f(A)$ and $f(A) \subseteq f(B)$ if $A \subseteq B$, and also $f((a)) = (a)$. Let $\mathscr{I}_f(R) = \{f(A) \mid A \in \mathscr{I}(R)\}$. Then $\mathscr{I}_f(R) \subseteq \mathscr{I}(R)$ and, in particular, if $f(a) = (a)$ for all $a \in R$, then $f(A) = A$ and so $\mathscr{I}_f(R) = \mathscr{I}(R)$.

2. Construction of the *-operation.

Let $X = \{x_1, x_2, x_3, \cdots\}$ be a fixed set of indeterminates and $\Phi\langle X \rangle$ be the free nonassociative algebra over Φ generated by X. Let R be an arbitrary nonassociative algebra over Φ. If $m(x_1, \cdots, x_n) \in \Phi\langle X \rangle$ is a monomial and $A_1, \cdots, A_n$ are submodules of R, we denote by $m(A_1, \cdots, A_n)$ the submodule of R generated by the elements $m(a_1, \cdots, a_n)$, $a_i \in A_i$. Let $p(x_1, \cdots, x_n)$ be an element in $\Phi\langle X \rangle$ and $p(x_1, \cdots, x_n) = \sum m_i(x_1, \cdots, x_n)$ be the sum of distinct monomials occuring in p. Then $p(A_1, \cdots, A_n)$ means the submodule $\sum m_i(A_1, \cdots, A_n)$ (note $p(A_1, \cdots, A_n)$ does not necessarily equal the submodule generated by $p(a_1, \cdots, a_n)$, $a_i \in A_i$). Let $w(x_1, \cdots, x_n)$ denote the sum of all nonassociative words in $x_1, \cdots, x_n$ whose x_i-degrees are exactly 1 for all $i = 1, \cdots, n$. As usual, if A is a submodule of R, we set $w(A, \cdots, A) = A^n$. Then an easy induction on n shows

$$A^n = A^{n-1}A + A^{n-2}A^2 + \cdots + AA^{n-1} \quad \text{with} \quad A^1 = A .$$

An algebra R over Φ is called an *s-algebra* if A^s is an ideal of R for every ideal A in R.

Example 2.1. Let $k \geq 1$ and $n > k$, and let $p(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n)$ be an element in $\Phi\langle X \rangle$ such that each monomial in p has x_i-degree ≥ 1 for some $1 \leq i \leq k$ and for some $k \leq i \leq n$. Setting $A*B = p(A, \cdots, A, B, \cdots, B)$ for ideals A, B in R gives rise to a *-operation in R. In particular, if we define $A*B$ to be the ideal generated by $p(A, \cdots, A, B, \cdots, B)$ in R, we obtain a strong *-operation, and furthermore, if $k=1$ and each monomial in p has x_1-degree 1, the *-operation is left-additive. Setting $p(x_1, x_2)=\alpha x_1x_2+\beta x_2x_1$ for $\alpha, \beta \in \Phi$, we obtain a left- and right-additive *-operation in R.

Example 2.2. Let R be an s-algebra and let $w(x_1, \cdots, x_s)$ be the same as above. For ideals A, B in R, if we set $A*B=w(A, B, \cdots, B)$, we get a left-additive *-operation in R such that $A*A=A^s$. Moreover if we define $A*B$ to be the ideal generated by $w(A, B, \cdots, B)$ then * is strong and $A*A=A^s$ still holds.

There is some occasion where $w(A, B, \cdots, B)$ in Example 2.2 becomes an ideal. The largest known class of s-algebras for this case is the class of weakly W-admissible algebras of *Thedy* [17]. Recall that an algebra R is called a noncommutative Jordan algebra if the flexible law $(x, y, x)=0$ and the Jordan identity $(x^2, y, x)=0$ hold in R where $(x, y, z)=(xy)z-x(yz)$. A noncommutative Jordan algebra R is called a *weakly W-admissible* algebra if R satisfies the identities

(1) $$[(a, b, c), c]-([a, c], c, b)=0 ,$$

(2) $$([a, b], d, c)+([b, c], d, a)+([c, a], d, b) =\rho[(a, b, c), d]+\sigma[S(a, b, c), d]+\tau[d, [b, [a, c]]]$$

for some elements $\rho, \sigma, \tau \in \Phi$ such that either the mapping $x \to (3+2\rho+6\sigma-4\tau)x$ or $x \to (\rho+4\tau)x$ induces a bijection on any submodule of R, where $[a, b]=ab-ba$ and $S(a, b, c)=(a, b, c)+(b, c, a)+(c, a, b)$. The latter condition would be the case when $3+2\rho+6\sigma-4\tau$ or $\rho+4\tau$ is invertible in Φ. *Thedy* [17] calls a noncommutative Jordan algebra over a field *W-admissible* if it satisfies

(3) $$[a, (a, a, b)]=0$$

in addition to the conditions above, and shows that if the characteristic is not 2, any generalized standard algebra of *Schafer* [15] is W-admissible with $\rho=-2$ and $\sigma=\tau=0$. Lie algebras are also weakly W-admissible with $\rho=\tau=0$. Therefore, weakly W-admissible algebras generalize alternative, Lie, standard, and hence Jordan algebras. *Shestakov* [16] recently defined another large class of algebras and called a noncommutative Jordan algebra R *admissible* if R satisfies

(4) $$([x, y], y, y)=0 .$$

In view of (1) and (3), a W-admissible algebra is admissible. However, a weakly W-admissible algebra is not in general admissible since, as *Thedy* [17, p. 178] points out, any Lie algebra R over a field is weakly W-admissible but (4) is equivalent to $2L_y^3=0$ in R where L_y is the left multiplication by y, and hence

R is not admissible unless R is a nilpotent Lie algebra (in the finite-dimensional case).

Let $w_i(x_1, \cdots, x_n) = w(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$, $i=1, 2, \cdots, n$ and $n \geq 2$. If $A_1, \cdots, A_n$ are ideals of R, one easily checks

$$w(A_1, A_2, \cdots, A_n) \subseteq \sum_{i=1}^{n} w_i(A_1, \cdots, A_n) .$$

In generalizing Shestakov's definition [16, Definition 11], we have

Definition 2.3. An algebra R over Φ is said to be p-ideal-admissible if there exists an element $p(x_1, \cdots, x_n, x_{n+1}) \in \Phi\langle X\rangle$ which is homogeneous in each of $x_1, \cdots, x_n$, such that for any ideals $A_1, \cdots, A_n$ of R the submodule $p(A_1, \cdots, A_n, R)$ is also an ideal of R and

$$w(A_1, \cdots, A_n) \subseteq p(A_1, \cdots, A_n, R) \subseteq \sum_{i=1}^{n} w_i(A_1, \cdots, A_n)$$

holds. Here $p(x_1, \cdots, x_n, x_{n+1})$ is not necessarily homogeneous in x_{n+1} and x_{n+1} may not occur in p at all. In this case $p(x_1, \cdots, x_n, x_{n+1})$ is called an *ideal-admissible polynomial* for R.

Shestakov [16] calls an admissible algebra R *σ-admissible* if there exists an ideal-admissible polynomial $\sigma(x_1, x_2, x_3, x_4)$ for R.

Example 2.4. *Block* [1] defines a class of noncommutative Jordan algebras satisfying

$$([x, y], z, z) = 0 , \tag{5}$$

$$([x, y], z, w) + (z, [x, y], w) = 0 . \tag{6}$$

It is shown in [16] that any noncommutative Jordan algebra satisfying (5) and (6) is σ-(ideal-)admissible with

$$\begin{aligned}\sigma(x_1, x_2, x_3, x_4) = w(x_1, x_2, x_3) + \sum_{u \in S_3} [x_4(x_{u(1)}(x_{u(2)}x_{u(3)})) \\ + ((x_{u(2)}x_{u(3)})x_{u(1)})x_4]\end{aligned}$$

where S_3 is the symmetric group on $\{1, 2, 3\}$. Thus, for ideals A, B of R, setting $A*B = \sigma(A, B, B, R)$ yields a strong left-additive *-operation in R such that

$$A^5 \subseteq A*A \subseteq A^2$$

(see [16, p. 265]).

Example 2.5. Let R be a weakly W-admissible algebra. A careful examination of the proof of *Thedy* [17, Lemma 7] reveals that $w(A_1, A_2, A_3)$ is an ideal of R for any ideals A_1, A_2, A_3 of R. Thus R is σ-ideal-admissible with

$$\sigma(x_1, x_2, x_3, x_4)=w(x_1, x_2, x_3)\ .$$

We note

$$w(x_1, x_2, x_3)=\sum_{u\in S_3}[x_{u(1)}(x_{u(2)}x_{u(3)})+(x_{u(2)}x_{u(3)})x_{u(1)}]\ .$$

Setting $A*B=w(A, B, B)$, we obtain a strong left-additive $*$-operation in R such that $A*A=A^3$. This also proves the existence of a $*$-operation in a weakly W-admissible ring satisfying Condition (*4) in [12]. We will denote $A*B$ by $\langle A, B\rangle$ in this case.

It is readily seen that, for ideals A, B in R, $\langle A, B\rangle$ is reduced to

$$\langle A, B\rangle=AB^2+B^2A+(AB)B+B(AB)+(BA)B+B(BA)\ . \tag{7}$$

In relation of Example 2.5 with Example 2.4, one should note that a weakly W-admissible algebra does not necessarily satisfy (5) or (6). Using the flexible law, $\langle A, B\rangle$ is further reduced to

$$\begin{aligned}\langle A, B\rangle&=AB^2+(AB)B+B(AB)+(BA)B+B(BA)\\&=B^2A+(AB)B+B(AB)+(BA)B+B(BA)\ .\end{aligned} \tag{8}$$

Let R be an alternative algebra and let $a\in A$ and $b, c\in B$. From $(a, b, c)=-(b, a, c)$ we have

$$\langle A, B\rangle=(AB)B+B(AB)+(BA)B+B(BA) \tag{9}$$

for ideals A, B in R. Let R now be a flexible algebra satisfying $S(a, b, c)=(a, b, c)+(b, c, a)+(c, a, b)=0$. For ideals A, B in R, let $a\in A$ and $b, c\in B$. Then by the flexible law $(b, c, a)+(a, c, b)=0$ we have $B(BA)\subseteq B^2A+(AB)B+AB^2$, and $S(b, a, c)=0$ implies $B(AB)\subseteq(BA)B+B^2A+B(BA)+AB^2+(AB)B\subseteq AB^2+B^2A+(AB)B+(BA)B$. Hence $B(BA)+B(AB)\subseteq AB^2+B^2A+(AB)B+(BA)B$ and similarly we get $(AB)B+(BA)B\subseteq B(AB)+B(BA)+AB^2+B^2A$. Therefore in this case $\langle A, B\rangle$ is reduced to

$$\begin{aligned}\langle A, B\rangle&=AB^2+B^2A+(AB)B+(BA)B\\&=AB^2+B^2A+B(AB)+B(BA)\ .\end{aligned} \tag{10}$$

In particular, (10) holds for any standard algebra. Since a Jordan algebra is a standard algebra, in a Jordan algebra $\langle A, B\rangle$ is further reduced to

$$\langle A, B\rangle=AB^2+(AB)B\ . \tag{11}$$

If R is a Lie algebra, in view of the Jacobi identity and anticommutativity, we have

$$\langle A, B\rangle = (AB)B\ . \tag{12}$$

We also note that, in any algebra R where AB is an ideal of R for ideals A, B, the operation $\langle A, B\rangle$ given by (7) yields a strong *-operation in R such that $\langle A, A\rangle = A^3$.

Example 2.6. Another class of ideal-admissible algebras is the class of $(-1,1)$-algebras. A right alternative algebra R is called a $(-1,1)$-algebra if the identity $S(a,b,c)=0$ holds in R. *Dorofeev* [4] proves that $AB+BA$ is an ideal for any ideals A, B in R. Thus R is p-ideal-admissible with $p(x_1, x_2)=w(x_1, x_2)$. Hence setting $A*B=AB+BA$ gives rise to a strong left-additive *-operation in R. In any algebra we will denote

$$A\circ B = AB+BA\ . \tag{13}$$

Example 2.7. Let R be a noncommutative Jordan algebra over Φ in the sence of *McCrimmon* [9]. An ideal-building operation in R is the well-known quadratic operation U defined by $yU_x = x(xy+yx)-x^2y$ for $x, y \in R$. *McCrimmon* [9, Lemma 5] proves that the submodule AU_B is an ideal of R for any ideals A, B in R. Thus U yields a strong left-additive *-operation in R.

3. Prime ideals and radicals.

We first recall some definitions and known results from [12]. Let * be a *-operation defined in an arbitrary algebra R and f be the function in Definition 1.4.

An ideal P of R is called *f*-prime* if $f(A)*f(B)\subseteq P$ for ideals A, B implies $f(A)\subseteq P$ or $f(B)\subseteq P$. An ideal P is called *f*-semiprime* if $f(A)*f(A)\subseteq P$ implies $f(A)\subseteq P$ for any ideal A in R. If $f(a)=(a)$ for all $a\in R$, an f^*-prime or f^*-semiprime ideal is simply called **-prime* or **-semiprime*. A nonempty subset M of R is called an *f*-system* if, for $A, B\in \mathscr{I}(R)$, $f(A)\cap M\neq\emptyset$ and $f(B)\cap M\neq\emptyset$ imply $f(A)*f(B)\cap M=\emptyset$. Given an ideal A of R, the *f*-prime radical*, $r_f^*(A)$, of A is defined to be the set of elements $x\in R$ such that any f^*-system containing x meets A, and shown to be the intersection of all f^*-prime ideals containing A. In particular $r_f^*(0)$ is called the *f*-prime radical* of R and denoted by $P_f^*(R)$. If $f(a)=(a)$ for all $a\in R$, these are called a *-system and the *-prime radical of A or of R, and denoted by $r^*(A)$ or $P^*(R)$, respectively. Call R *f*-semisimple* (or **-semisimple*) if $P_f^*(R)=0$ (or $P^*(R)=0$).

Definition 3.1. An ideal P is called prime if $AB\subseteq P$ implies $A\subseteq P$ or $B\subseteq P$ for any ideals A, B of R. An ideal P of R is called σ-prime if $\langle A, B\rangle\subseteq P$ implies

$A \subseteq P$ or $B \subseteq P$. If P is prime in the sense of the operation $A \circ B$ given by (13), P is called $\circ$-prime. Finally, if R is a Jordan algebra and P is prime in the sense of *Tsai* [19], P is called Q-prime. The prime radicals of an ideal A or of R will be denoted by $r(A), r_\sigma(A), r_0(A)$ or by $P(R), P_\sigma(R), P_0(R)$, respectively. A prime algebra is also defined in the usual sense for each of the operations above.

For the basic properties of prime ideals and radicals in terms of *-operation, consult *Myung* [12]. We first prove

Lemma 3.2. *Let R be an algebra such that, for any ideals A, B in R, AB is an ideal of R. Then an ideal P of R is prime if and only if P is σ-prime if and only if P is $\circ$-prime.*

Proof. Let A, B be any ideals of R. We first observe $(BA)^2 \subseteq AB$ since $(BA)^2 = (BA)(BA) \subseteq AB$. Let P be σ-prime and let $AB \subseteq P$. Then $(BA)^2 \subseteq AB \subseteq P$ and hence from (7) $\langle BA, BA \rangle = (BA)^3 \subseteq P$, so $BA \subseteq P$ since P is σ-prime. Therefore we have $\langle A, B \rangle = AB^2 + B^2A + B(AB) + B(BA) + (AB)B + (BA)B \subseteq P$. This implies $A \subseteq P$ or $B \subseteq P$ and P is prime. Suppose P is prime and let $\langle A, B \rangle \subseteq P$. Then $(AB)B \subseteq P$ and hence $A \subseteq P$ or $B \subseteq P$; that is, P is σ-prime. Now, let P be $\circ$-prime and let $AB \subseteq P$. Then $(BA)^2 \subseteq AB \subseteq P$ and so $(BA) \circ (BA) \subseteq P$, which implies $BA \subseteq P$. Thus $A \circ B = AB + BA \subseteq P$ and so $A \subseteq P$ or $B \subseteq P$. Clearly, if P is prime, it is $\circ$-prime.

Since an alternative or Lie algebra R satisfies the condition in Lemma 3.2, we have

Corollary 3.3. *An ideal P in an alternative or Lie algebra is prime if and only if P is σ-prime if and only if P is $\circ$-prime.*

Therefore, the prime ideal theory for alternative algebras is a special case of that for $(-1, 1)$-algebras in terms of the operation $A \circ B$, or for weakly W-admissible algebras in terms of $\langle A, B \rangle$. For a Jordan algebra we obtain.

Lemma 3.4. *Let P be an ideal in a (linear) Jordan algebra J where $2y = x$ has a unique solution for all $x, y \in J$. Then P is Q-prime if and only if P is σ-prime.*

Proof. Suppose P is a σ-prime ideal of J and A, B are ideals of J such that $AU_B \subseteq P$ where U is the quadratic operation. Recall $AU_A = A^3$. Setting $C = A \cap B$, we get $C^3 = \langle C, C \rangle = CU_C \subseteq AU_B \subseteq P$. Hence $C \subseteq P$ since P is σ-prime. But then $\langle A, B \rangle \subseteq A \cap B = C \subseteq P$, so $A \subseteq P$ or $B \subseteq P$. Thus P is Q-prime. Now, suppose P is a Q-prime ideal of J and $\langle A, B \rangle \subseteq P$ for ideals A, B. Then clearly $AU_B \subseteq \langle A, B \rangle \subseteq P$ by (11) and hence $A \subseteq P$ or $B \subseteq P$, showing that P is σ-prime.

In the past five years there has been a great deal of study to classify some large classes of prime rings. Recently, *Thedy* [18] introduced a class of nonassociative rings R defined by the identities

$$[x, (y, x, y)]=0 , \tag{14}$$

$$(x, [y, z], x)=0 , \quad (x, [y, z], w)=([y, z], w, x)=(w, x, [y, z]) . \tag{15}$$

The identity (15) is to say that the commutators $[R, R]$ are completely alternative. *Thedy* [18] proves that any prime ring of characteristic $\neq 3$ satisfying (14) and (15) is alternative or commutative. Rings satisfying (14) and (15) generalize generalized standard rings. *Kleinfeld, Kleinfeld* and *Kosier* [6] prove also the same result for generalized accessible rings which generalize generalized standard rings. It should be noted that any generalized accessible ring satisfies (14) and (15) (see [6, Lemma 2]). However, a weakly W-admissible algebra does not necessarily satisfy (14) or (15). The 11-dimensional nilpotent Lie algebra of [17, p. 178] reveals that $(e_1, e_1, [e_2, e_3])=(e_1, e_1, 2e_6)=-4e_{11}$ but $(e_1, [e_2, e_3], e_1)=0$. Thedy brought in a letter our attention to this example. In a weakly W-admissible algebra R there is no direct comparision between the primeness and the σ-primeness of R. In fact, if R is an algebra where $AB=0$ implies $BA=0$ for ideals A, B of R, then the σ-primeness implies the primeness since if $AB=BA=0$, $\langle A, B\rangle=0$. On the other hand, if R is an algebra such that AB is an ideal of R for any ideals A, B then by Lemma 3.2 the primeness implies the σ-primenss. The known results about prime rings however suggest the following.

Conjecture. *Any σ-prime (or prime) weakly W-admissible algebra is alternative or Jordan.*

As a particular case of this conjecture we can prove.

Theorem 3.5. *Any σ-prime or prime generalized standard algebra of characteristic $\neq 2, 3$ is alternative or Jordan.*

Proof. Note the result is well known for the primeness. Suppose R is σ-prime. Much of the proof is done by *Thedy* [18]. If $A^2=0$ for an ideal A of R, $\langle A, A\rangle=A^3=0$ and so $A=0$. Hence R is semiprime. It is shown in [18, Theorem 1] that R is a subring of the algebra direct sum $A\oplus C$ of an alternative algebra A and a commutative algebra C such that any alternator of the form (x, y, y) or (y, y, x) lies in C and any commutator $[x, y]$ lies in A. Hence the ideal I generated by the alternators of R and the ideal J generated by the commutators of R are contained in C and A, respectively. Thus $IJ=JI=0$ and, as remarked above, $\langle I, J\rangle=0$. Therefore $I=0$ or $J=0$ and R is alternative or Jordan.

For the proof one could also use an argument in [6] where two ideals similar to I and J are constructed. We now return to an arbitrary algebra equipped with a *-operation. The following definition will be used throughout this paper.

Definition 3.6. Let $*$ be a strong *-operation defined in an algebra R. Given an ideal A of R, define $D_*^n(A)$ and $d_*^n(A)$ recursively as

$$D_*^0(A)=A\ , \quad D_*^{n+1}(A)=D_*^n(A)*D_*^n(A)\ ,$$
$$d_*^0(A)=A\ , \quad d_*^{n+1}(A)=A*d_*^n(A)\ .$$

An ideal A is said to be *-solvable if $D_*^n(A)=0$ for some n, and similarly, A is called *-nilpotent if $d_*^n(A)=0$ for some n.

One can easily check by induction that

$$D_*^n(D_*^m(A))=D_*^{n+m}(A)\ ,$$
$$D_*^n(A)\subseteq d_*^n(A)\ ,$$
$$d_*^n(d_*^m(A))\subseteq d_*^{n+m}(A)\ .$$

Thus if an ideal A is *-nilpotent, it is *-solvable. This definition seems natural in our situation because of its similarity with that for Lie algebras and with "one-sided" nilpotence in nonassociative algebras; for example, "right" or "left" nilpotence (see *Shestakov* [16]). If R is an alternative algebra and $A*B=A\circ B$ for ideals A, B in R, then *Hentzel* and *Slater* [5] show that the *-nilpotence in R coincides with the nilpotence in the ordinary sense. More precisely, they prove.

Lemma 3.7. *Let R be an alternative algebra and let $A*B=A\circ B$ for ideals A, B in R. Then $d_*^n(A)=A^n$ for any ideal A and positive integer n.*

Theorem 3.8. *Let $*$ be a strong *-operation in R. Then R is *-semisimple if and only if R contains no nonzero *-solvable ideals.*

Proof. Note that R is *-semisimple if and only if the ideal (0) is *-semiprime [12, Theorem 2.2]. If A is a nonzero *-solvable ideal of R, there exists an n such that $D_*^{n+1}(A)=0$ and $D_*^n(A)\neq 0$. Since $D_*^n(A)*D_*^n(A)=D_*^{n+1}(A)=0$, we have that (0) is not *-semiprime. The converse is immediate.

If R is an s-algebra, we note that there exists a strong *-operation in R such that $A*A=A^s$ for any ideal A in R. The same proof as in Theorem 3.8 then shows

Theorem 3.9. *Let $*$ be a strong $*$-operation in an s-algebra R such that $A*A=A^s$. Then R is $*$-semisimple if and only if R contains no nonzero nilpotent ideals.*

Corollary 3.10. *The $*$-prime radical $P^*(R)$ contains all $*$-solvable (so $*$-nilpotent) ideals of R. If R is an s-algebra such that $A*A=A^s$ then $P^*(R)$ contains all nilpotent ideals in R.*

Proof. Let A be a $*$-solvable ideal in R and let $\bar{R}=R/P^*(R)$. Then clearly $\bar{A}$ is $*$-solvable in $\bar{R}$ and since $\bar{R}$ is $*$-semisimple, by Theorem 3.8 $\bar{A}=(\bar{0})$, thus $A\subseteq P^*(R)$. If R is an s-algebra, then the result is immediate from Theorem 3.9.

The following lemma will be frequently used.

Lemma 3.11. *Let h be a homomorphism of R onto $\bar{R}$. For ideals A in R, the mapping $A\to h(A)$ induces a 1—1 correspondence between the set of f^*-prime ideals in R containing the kernel of h and the set of f^*-prime ideals in $\bar{R}$.*

Proof. It is enough to show that $h(P)=\bar{P}$ is f^*-prime in $\bar{R}$ for any f^*-prime ideal P in R containing the kernel of h and that, for any f^*-prime ideal $\bar{P}$ in $\bar{R}$, $h^{-1}(\bar{P})$ is f^*-prime in R. The first part of this is proved in [12, Lemmas 2.3 and 2.5]. Let $\bar{P}$ be an f^*-prime ideal in $\bar{R}$ and let $P=h^{-1}(\bar{P})$. Suppose $f(A)*f(B)\subseteq P$ for ideals A, B in R. Then by (*3) and (f3), we have $f(\bar{A})*f(\bar{B})\subseteq\bar{P}$ and hence $f(\bar{A})\subseteq\bar{P}$ or $f(\bar{B})\subseteq\bar{P}$, which implies $f(A)\subseteq P$ or $f(B)\subseteq P$ since P contains the kernel of h. Hence $h^{-1}(\bar{P})$ is f^*-prime.

Theorem 3.12. *Let $*$ be a $*$-operation in R (not necessarily strong) and A be an ideal of R. If the quotient algebra R/A is f^*-semisimple, then $P^*_f(R)\subseteq A$.*

Proof. Let $\{P_i\}$ be the collection of f^*-prime ideals in R containing A. By Lemma 3.11, $\{P_i/A\}$ is the collection of f^*-prime ideals in R/A. Hence $\bigcap_i(P_i/A)=0$ and so $\bigcap_i P_i\subseteq A$. Since $P^*_f(R)$ is the intersection of all f^*-prime ideals in R, we have $P^*_f(R)\subseteq A$.

Corollary 3.13. *If R is an s-algebra and $*$ is strong such that $A*A=A^s$ for every ideal A, then $P^*(R)$ is the smallest ideal L of R such that R/L contains no nonzero nilpotent ideals.*

Proof. If L is an ideal such that R/L contains no nonzero nilpotent ideal, then by Theorem 3.9 R/L is $*$-semisimple. Hence by Theorem 3.12 $L\supseteq P^*(R)$.

Corollary 3.14. *Let $*$ be strong. Then the $*$-prime radical $P^*(R)$ is the* *intersection of all ideals S of R such that R/S has no nonzero *-solvable ideals.*

Proof. Let $K=\bigcap_i S_i$ be the intersection of all ideals S_i such that R/S_i has no nonzero *-solvable ideals. Let $P^*(R)=\bigcap_\alpha P_\alpha$ be the intersection of all *-prime ideals in R. Since R/P_α is *-semisimple, by Theorem 3.8 $K\subseteq P^*(R)$. On the other hand, if R/S_i contains no nonzero *-solvable ideals, then by Theorem 3.8 again, R/S_i is *-semisimple. Therefore by Theorem 3.12 $S_i\supseteq P^*(R)$ and $K\supseteq P^*(R)$, so $K=P^*(R)$.

We close this section with an analogous result of *Brown* and *McCoy* [2, p. 250]. If S is a subalgebra of R such that any ideal of S is also an ideal of R, then, in view of (*2), any *-operation in R induces a *-operation in S.

Theorem 3.15. *Let S be a subalgebra of an algebra R equipped with a *-operation such that any ideal of S is also an ideal of R. Then we have $P^*(S)=P^*(R)\cap S$.*

Proof. Let P be a *-prime ideal in R and let $A*B\subseteq P\cap S$ for ideals A, B in S. Since A, B are ideals in R and P is *-prime in R, we have $A\subseteq P\cap S$ or $B\subseteq P\cap S$. Hence $P^*\cap S$ is *-prime in S. Let $P^*(R)=\bigcap_i P_i$, the intersection of all *-prime ideals in R. Then $P^*(R)\cap S=(\bigcap_i P_i)\cap S=\bigcap_i (P_i\cap S)\supseteq P^*(S)$ since $P_i\cap S$ is *-prime in S. Conversely, let $a\in P^*(R)\cap S$ and M be any *-system in S containing a. Let A, B be ideals in R such that $A\cap M\neq\varnothing$ and $B\cap M\neq\varnothing$, so $(A\cap S)\cap M\neq\varnothing$ and $(B\cap S)\cap M\neq\varnothing$. This implies $\varnothing\neq(A\cap S)*(B\cap S)\cap M\subseteq A*B\cap M$. Hence M is a *-system in R, and since $a\in P^*(R)$, M contains 0. Thus $a\in P^*(S)$ and $P^*(R)\cap S\subseteq P^*(S)$.

Corollary 3.16. *If R is a direct sum of ideals A_i, $i\in I$, then $P^*(A_i)=P^*(R)\cap A_i$ for all $i\in I$ and hence*

$$\sum_i \oplus P^*(A_i)\subseteq P^*(R)\ .$$

Corollary 3.17. *If K is the algebra obtained by adjoining an identity to R, then $P^*(R)=R\cap P^*(K)$.*

4. The Baer lower radical.

In this section we introduce the Baer lower radical in an arbitrary algebra R in terms of a *-operation and the function f. For this we need the following additional condition:

(f4) $f(a+b)\subseteq f(a)+f(b)$ for all $a, b\in R$.

For a fixed subset S of R and any element $a \in R$, the ideal (S, a) generated by a and S in R is still the basic example of the function f satisfying (f1)-(f4). For any family $\{A_\alpha | \alpha \in I\}$ of ideals in R, if (f4) holds, one easily checks

$$\sum_{\alpha \in I} f(A_\alpha) = f(\sum_{\alpha \in I} A_\alpha) .$$

Thus the function f satisfying (f4) will be called *additive*. For any homomorphic image $\bar{R}$ of R, if f is additive, we further observe

$$\sum_{\alpha \in I} f(\bar{A}_\alpha) = f(\sum_{\alpha \in I} \bar{A}_\alpha) = \overline{f(\sum_{\alpha \in I} A_\alpha)} .$$

Using this, if $\{N_\alpha/A | \alpha \in I\}$ is a family of ideals in $\bar{R} = R/A$, then we can easily check

$$\sum_{\alpha \in I} f(N_\alpha/A) = f(\sum_{\alpha \in I} N_\alpha)/A . \tag{16}$$

Throughout this section we assume that * is strong and f is additive. To introduce the Baer lower radical in R, we proceed as in *Divinsky* [3]. Recall $\mathscr{I}_f(R) = \{f(A) | A \in \mathscr{I}(R)\}$.

Let N_0' be the sum of all *-nilpotent ideals in $\mathscr{I}_f(R)$. Then we have $N_0' = f(N_0)$ for some ideal N_0 in R since f is additive. Let N_1' be the ideal in R such that $N_1'/f(N_0)$ is the sum of all the *-nilpotent ideals in $\mathscr{I}_f(R/f(N_0))$. Then by (16) we have $N_1' = f(N_1)$ for some ideal N_1 of R. For every ordinal α, which is not a limit ordinal, we define N_α' to be the ideal in R such that $N_\alpha'/f(N_{\alpha-1})$ is the sum of all the *-nilpotent ideals in $\mathscr{I}_f(R/f(N_{\alpha-1}))$. We then have $N' = f(N_\alpha)$ for some ideal N_α in R. If α is a limit ordinal, we define

$$N_\alpha' = \sum_{\beta < \alpha} N_\beta' = \sum_{\beta < \alpha} f(N_\beta) = f(\sum_{\beta < \alpha} N_\beta) .$$

Then we have an increasing sequence of ideals in $\mathscr{I}_f(R)$

$$f(N_0) \subseteq f(N_1) \subseteq \cdots \subseteq f(N_\alpha) \subseteq \cdots .$$

Let τ be the smallest ordinal such that

$$f(N_\tau) = f(N_{\tau+1}) = \cdots .$$

This process leads to

Definition 4.1. The ideal $f(N_\tau)$ in $\mathscr{I}_f(R)$ is called the f*-Baer lower radical of R and is denoted by $B_f^*(R)$. If $f(a) = (a)$ for all $a \in R$, we call this the *-Baer lower radical of R and denote it by $B^*(R)$.

We note that $R/B_f^*(R)$ contains no nonzero *-nilpotent ideals in $\mathscr{I}_f(R/B_f^*(R))$. For, if $\mathscr{I}_f(R/B_f^*(R))$ contains a nonzero *-nilpotent ideal $f(Q)/B_f^*(R)$ then $f(Q) \supsetneqq$

$B_f^*(R)$ but, by definition of $B_f^*(R)$, $f(Q)\subseteq B_f^*(R)$, a contradiction.

We now prove the well-known characterization of $B_f^*(R)$.

Theorem 4.2. *The f^*-Baer lower radical $B_f^*(R)$ is the intersection of all ideals $f(Q_i)$ in $\mathscr{I}_f(R)$ such that $R/f(Q_i)$ has no nonzero *-nilpotent ideals in $\mathscr{I}_f(R/f(Q_i))$.*

Proof. Let $W=\bigcap_i f(Q_i)$ and let $B_f^*=B_f^*(R)$. Clearly $W\subseteq B_f^*$ since R/B_f^* contains no nonzero *-nilpotent ideals in $\mathscr{I}_f(R/B_f^*)$. Conversely, take any ideal $f(Q_i)\in\mathscr{I}_f(R)$ such that $\mathscr{I}_f(R/f(Q_i))$ contains no nonzero *-nilpotent ideals. Then $f(N_0)\subseteq f(Q_i)$. To use transfinite induction, assume that $f(N_\alpha)\subseteq f(Q_i)$ for every $\alpha<\beta$. If β is a limit ordinal, then by the construction we have

$$f(N_\beta)=\sum_{\alpha<\beta} f(N_\alpha)\subseteq f(Q_i)\ .$$

If β is not a limit ordinal, then $\beta-1$ exists and $f(N_{\beta-1})\subseteq f(Q_i)$. Now, suppose $f(N_\beta)$ is not contained in $f(Q_i)$. Then there exists a *-nilpotent ideal $f(C)/f(N_{\beta-1})$ not contained in $f(Q_i)/f(N_{\beta-1})$. We then consider

$$\frac{f(C)+f(Q_i)}{f(Q_i)}\cong\frac{f(C)}{f(C)+f(Q_i)}\ .$$

For some n, $d_*^n(f(C))\subseteq f(N_{\beta-1})\subseteq f(Q_i)$ by (*3) and so $d_*^n(f(C))\subseteq f(C)\cap f(Q_i)$ by (*2'). Hence $f(C)/f(C)\cap f(Q_i)$ is *-nilpotent and thus $[f(C)+f(Q_i)]/f(Q_i)$ is a nonzero *-nilpotent ideal in $R/f(Q_i)$ since $f(C)\not\subseteq f(Q_i)$. This is a contradiction. Therefore $f(N_\beta)\subseteq f(Q_i)$ and $W=B_f^*$.

Definition 4.3. Let R be an s-algebra. For an ideal A of R, let A^G and $P_s(R)$ respectively denote the prime radical of A and of R in the sense of *Zwier* [21]. Also, $B(R)$ denotes the Baer lower radical of R in the sense of ordinary nilpotence.

The same proof as for Theorem 4.2 shows

Theorem 4.4. *The Baer lower radical $B(R)$ for an s-algebra R is the intersection of all ideals Q_i such that R/Q_i has no nonzero nilpotent ideals.*

If $f(a)=(a)$ for all $a\in R$, in relation between $P^*(R)$ and $B^*(R)$ we have

Theorem 4.5. *Let R be an arbitrary algebra equipped with a strong *-operation. Then $B^*(R)\subseteq P^*(R)$.*

Proof. The proof is immediate from Corollary 3.14 and Theorem 4.2 since $f(A)=A$ for all ideals A in R.

Theorem 4.6. *Let R be an s-algebra equipped with a strong *-operation such that $A*A=A^s$ for every ideal A of R. Then $P^*(R)=B(R)$.*

Proof. Let $P^*(R)=\bigcap_i P_i$, the intersection of all *-prime ideals in R. Then since R/P_i is *-semisimple, by Theorem 3.9 R/P_i has no nonzero nilpotent ideals. Hence by Theorem 4.4 we have $P^*(R)\supseteq B(R)$. Conversely, if R/P_i has no nonzero nilpotent ideals, by Theorem 3.9 again R/P_i is *-semisimple and hence by Theorem 3.12 we get $P^*(R)\subseteq Q_i$ and $P^*(R)\subseteq B(R)$. This proves the theorem.

Since *Rich* [14] proves $P_s(R)=B(R)$, we have

Corollary 4.7. *Let R be the same s-algebra as in Theorem 4.6. Then we have*

$$P^*(R)=P_s(R)=B(R)\ .$$

Corollary 4.8. *If R is the same algebra as in Theorem 4.6, then for any ideal A of R we have $r^*(A)=A^G$.*

Proof. Let $\bar{R}=R/A$ and let $r^*(A)=\bigcap_i P_i$, the intersection of *-prime ideals P_i in R containing A. Then $\overline{r^*(A)}=\overline{\bigcap_i P_i}=\bigcap_i \bar{P}_i=r^*(\bar{0})=P^*(R)$ by Lemma 3.11, and so $P^*(\bar{R})=r^*(A)/A$. Similarly we show $P_s(\bar{R})=A^G/A$ (see *Zwier* [21]). Hence by Corollary 4.7 $r^*(A)/A=A^G/A$ and $r^*(A)=A^G$.

This shows that in an s-algebra R any strong *-operation such that $A*A=A^s$ for any ideal A leads to the same prime radical as in *Zwier* [21]. We have shown that any s-algebra possesses such a *-operation. In particular, if J is a Jordan algebra, then for any ideal A of J, we have

$$A^Q=r_\sigma(A)=A^G$$

where A^Q is the prime radical of A in the sense of the quadratic operation. If R is an alternative or Lie algebra then

$$r(A)=r_\sigma(A)=r_0(A)=A^G\ .$$

(See Section 3 for notations.)

5. Primary ideals.

In the rest of the paper we will assume that any *-operation is strong and left-additive, and $f(a)=(a)$ for all $a\in R$. Recall an arbitrary algebra always possesses a strong left-additive *-operation. Our definition of primary ideal is essentially the *-operation-analog of that for quadratic Jordan algebras given by *Tsai* and *Foster* [20] and thus that in the classical theory for the associative case. We begin with.

Definition 5.1. Let R be an arbitrary algebra equipped with a strong left-additive $*$-operation. An ideal Q of R is called left $*$-primary if $A*B\subseteq Q$ for ideals A, B of R implies that $A\subseteq Q$ or $B\subseteq r^*(Q)$.

Henceforth we will say "$*$-primary" rather than "left" $*$-primary. If R is an algebra where $A*B=\langle A, B\rangle$ given in (7) is an ideal for any ideals A, B of R, then a $*$-primary ideal is called σ-primary. Similarly, if $A*B=A\circ B$, AU_B, or AB in a $(-1, 1)$-algebra, Jordan algebra, or alternative algebra, then we will say "$\circ$-primary", "Q-primary", or "primary". We note that every $*$-prime ideal of R is $*$-primary.

We first give some "internal" characterization of $*$-primary ideals by using the same definition as in [20].

Theorem 5.2. *An ideal Q of R is $*$-primary if and only if $(\bar{0})$ is $*$-primary in R/Q.*

Proof. If we let $\bar{R}=R/Q$, by Lemma 3.11 we have $\overline{r^*(Q)}=r^*(\bar{0})$. Suppose Q is $*$-primary and $\bar{A}*\bar{B}=\bar{0}$ for ideals A, B in R such that $Q\subseteq A$ and $Q\subseteq B$. Then by (*3) $A*B\subseteq Q$ and so $A\subseteq Q$ or $B\subseteq r^*(Q)$. Hence $\bar{A}=\bar{0}$ or $\bar{B}\subseteq\overline{r^*(Q)}=r^*(\bar{0})$ and $(\bar{0})$ is $*$-primary in $\bar{R}$. If $(\bar{0})$ is $*$-primary and $A*B\subseteq Q$, $\bar{A}*\bar{B}=\bar{0}$ and so $\bar{A}=\bar{0}$ or $\bar{B}\subseteq r^*(\bar{0})=\overline{r^*(Q)}$; $B\subseteq r^*(Q)$. Hence Q is $*$-primary in R.

Following [20], we give

Definition 5.3. Given an ideal A of R and a $*$-system M. Then we call N a $*$-M-system if $R\supseteq N\supseteq M$ and $(m)*(n)\cap N\neq\emptyset$ for all $m\in M$ and all $n\in N$. If $A\cap M=\emptyset$, then the $*$-lower M-component A^*_M of A is the set $\{x\in R|(x)*(m)\subseteq A$ for some $m\in M\}$ and the $*$-upper M-component A^M_* of A is the set $\{x\in R|$ every $*$-M-system containing x meets $A\}$. An element $a\in R$ is $*$-prime to A if $(x)*(a)\subseteq A$ implies that $x\in A$. An ideal B is $*$-prime to A if B contains an element that is $*$-prime to A. Finally, when $A\cap M=\emptyset$, we say that A is related to M if every element of M is $*$-prime to A.

For an ideal A of R, we always have $A\subseteq A^*_M$ since $(a)*(m)\subseteq A*(m)\subseteq A$. Every $*$-system M is a $*$-M-system. Let $\{N_i\}$ be a collection of $*$-M-systems. Tnen $N=\bigcup_i N_i$ is also a $*$-M-system. Hence for a $*$-system M not meeting A, there exists a unique maximal $*$-M-system not meeting A.

Lemma 5.4. *Let A be an ideal of R.*

(a) *If M is a $*$-system in R not meeting A, then A^*_M is an ideal in R.*

(b) *A is related to a *-system M not meeting A if and only if $c(A)$ is a *-M-system.*

(c) *If M is a *-system and N is an arbitrary *-M-system not meeting A, then A is contained in an ideal $\tilde{A}$ such that $\tilde{A}$ is maximal relative to $\tilde{A}\cap N=\emptyset$. Moreover, $\tilde{A}$ is related to M.*

(d) *Let M be a *-system not meeting A. Then B is a maximal element in the set $M=\{C$ is an ideal in $R|A\subseteq C$ and C is related to $M\}$ if and only if $c(B)$ is a *-M-system not meeting A.*

Proof. The proof is essentially the same as in [20] and we prove only (a) and (b).

(a) Let $x, z\in A^*_M$. Then $(x)*(c)$ and $(y)*(d)$ are contained in A for some $c, d\in M$. Since M is a *-system, there exists an element $e\in(c)*(d)\cap M$. Since * is left-additive, $(x-y)*(e)\subseteq(x)*(e)+(y)*(e)\subseteq(x)*(c)+(y)*(d)$ since $e\in(c)*(d)\subseteq(c)\cap(d)$. Thus $(e)\subseteq(c)$ and $(x)*(e)\subseteq(x)*(c)$. This implies $(x-y)*(e)\subseteq A$, showing A^*_M is a submodule of R. Let $(a)*(m)\subseteq A$ for some $m\in M$. For any $x\in R$, we have $(xa)*(m)\subseteq(a)*(m)\subseteq A$. Hence A^*_M is an ideal of R.

(b) Suppose A is related to a *-system M not meeting A. For every elements $a\in c(A)$, $m\in M$, since m is *-prime to A, we have $(a)*(m)\cap c(A)\neq\emptyset$. Hence $c(A)$ is a *-M-system. Conversely, if $c(A)$ is a *-M-system then $A\cap M=\emptyset$. For every element m in M, if x is not in A, $(x)*(m)\neq\emptyset$ since $c(A)$ is a *-M-system. Hence $(x)*(m)$ is not contained in A, and so A is related to M.

Using Lemma 5.4, the following theorem can be proved exactly the same as in [20].

Theorem 5.5. *Let A be an ideal of R and M be a *-system not meeting A. Then*

(a) *A^M_* is the intersection of all ideals L such that $A\subseteq L$ and L is related to M. Hence A^M_* is an ideal of R.*

(b) *$c(A^M_*)$ is the uniquely determined maximal *-M-system not meeting A.*

(c) *$A\subseteq A^*_M\subseteq A^M_*$.*

Using Theorem 5.5, we can prove the following characterization of *-primary ideals (also see [20])

Theorem 5.6. *Let Q be an ideal of R. Then the following are equivalent.*

(a) *Q is *-primary.*

(b) *If P is a *-prime ideal of R such that $P\neq R$ and $Q\subseteq P$, then $Q=Q^{c(P)}_*$.*

(c) *If P is a *-prime ideal of R such that $P\neq R$ and $Q\subseteq P$, then $Q=Q^*_{c(P)}$.*

(d) *all elements not in* $r^*(Q)$ *are* $*$*-prime to* Q.

The following lemma is immediate from the definition of the $*$-prime radical of ideals.

Lemma 5.7. *If* A, B *are ideals in* R, *then*

(a) *if* $A \subseteq B$, $r^*(A) \subseteq r^*(B)$,

(b) $r^*(r^*(A)) = r^*(A)$.

(c) $r^*(A \cap B) = r^*(A) \cap r^*(B)$.

Lemma 5.8. *If* Q *and* Q' *are* $*$*-primary ideals in* R *such that* $r^*(Q) = r^*(Q')$, *then* $Q \cap Q'$ *is* $*$*-primary and* $r^*(Q \cap Q') = r^*(Q) = r^*(Q')$.

Proof. By Lemma 5.7, $r^*(Q \cap Q') = r^*(Q) \cap r^*(Q') = r^*(Q) = r^*(Q')$. Suppose $A*B \subseteq Q \cap Q'$ for ideals A, B in R. If $A \not\subseteq Q \cap Q'$; say $A \not\subseteq Q$, then $B \subseteq r^*(Q)$ since $A*B \subseteq Q$. Hence $B \subseteq r^*(Q) = r^*(Q \cap Q')$ and so $Q \cap Q'$ is $*$-primary.

Definition 5.9. We say that an ideal A of R has an irredundant representation by the ideals $B_1, \cdots, B_k$ of R if $A = B_1 \cap B_2 \cap \cdots \cap B_k$ and no one of the B_i's contains the intersection of the other ones. An irredundant representation $B_1 \cap B_2 \cap \cdots \cap B_k$ is called normal if $r^*(B_i) \neq r^*(B_j)$ for $i \neq j$. If each B_i is $*$-primary, the representation is called a normal $*$-primary representation.

Theorem 5.10. *Let* $A = Q_1 \cap \cdots \cap Q_k$ *be an irredundant representation by* $*$*-primary ideals* $Q_1, \cdots, Q_k$. *Then* A *is* $*$*-primary if and only if* $r^*(Q_i) = r^*(Q_j)$ *for* $i, j = 1, \cdots, k$.

Proof. If $r^*(Q_i) = r^*(Q_j)$ for $i, j = 1, \cdots, k$, then by repeated application of Lemma 5.8, we have that A is $*$-primary. Conversely, suppose A is $*$-primary. Let $B = Q_2 \cap \cdots \cap Q_k$. Since $B*Q \subseteq Q_1 \cap B = A$ and $B \not\subseteq A$, we have $Q_1 \subseteq r^*(A)$. By Lemma 5.7, $r^*(Q_1) \subseteq r^*(A) = r^*(Q_1) \cap \cdots \cap r^*(Q_k)$ and so $r^*(Q_1) \subseteq r^*(Q_j)$ for $j = 1, \cdots, k$. Repeating this implies that $r^*(Q_i) \subseteq r^*(Q_j)$ for $i, j = 1, \cdots, k$ and the proof is complete.

As a corollary of Theorem 5.10 we have

Corollary 5.11. *If an ideal* A *in* R *can be represented as a finite intersection of* $*$*-primary ideals, then* A *has a normal* $*$*-primary representation.*

6. The Lasker-Noether Theorem.

Following [11], we give

Definition 6.1. Let A be an ideal of R. An ideal P is called a minimal

$*$-prime ideal belonging to A if $A\subseteq P$ and $c(P)$ is a maximal $*$-system in R.

Lemma 6.2. *Let A be an ideal of R and M be a $*$-system not meeting A. Then there exists a minimal $*$-prime ideal P belonging to A such that $P\cap M=\emptyset$.*

Proof. By Zorn's lemma, we first find a maximal $*$-system M' such that $M'\supseteq M$ and $M'\cap A=\emptyset$. Next, by Zorn's lemma again, we find a maximal ideal P such that $P\supseteq A$ but $P\cap M'=\emptyset$. To show P is $*$-prime, let B, C be ideals in R such that $B\cap c(P)\neq\emptyset$ and $C\cap c(P)\neq\emptyset$. Let $x\in B\cap c(P)$ and $y\in C\cap c(P)$. Then by the maximality of M', $[(x)+P]\cap M'\neq\emptyset$ and $[(y)+P]\cap M'\neq\emptyset$, and since M' is a $*$-system, $[(x)+P]*[(y)+P]\cap M'\subseteq[(x)*(y)+P]\cap M'\neq\emptyset$ (also see [12, Lemma 1.1]). From this we get $(x)*(y)\cap(P+M')\neq\emptyset$ and so $(x)*(y)\cap c(P)\neq\emptyset$ since $P+M'\subseteq c(P)$. Thus $c(P)$ is a $*$-system and P is $*$-prime. But then since M' is a maximal $*$-system with respect to $M'\cap A=\emptyset$, we must have $c(P)=M'$. Hence P is a minimal $*$-prime ideal belonging to A.

Corollary 6.3. *Let A be an ideal in R and P be a $*$-prime ideal of R containing A. Then there exists a minimal $*$-prime ideal P' belonging to A such that $P'\subseteq P$.*

Proof. Since $c(P)$ is a $*$-system such that $c(P)\cap A=\emptyset$, by Lemma 6.2, there exists a minimal $*$-prime ideal P' belonging to A such that $P'\cap c(P)=\emptyset$, so $P'\subseteq P$.

The following corollary is immediate from Corollary 6.3.

Corollary 6.4. *For any ideal A of R, the $*$-prime radical $r^*(A)$ is the intersection of all minimal $*$-prime ideals belonging to A.*

Definition 6.5. An arbitrary algebra R is called noetherian if R satisfies the maximal condition on ideals.

The proof of the following is similar to [20].

Theorem 6.6. *Suppose R is noetherian and A is an ideal of R. Then there exists only a finite number of minimal $*$-prime ideals belonging to A.*

Corollary 6.7. *Suppose R is noetherian and A is an ideal of R. Let $P_1, \cdots, P_m$ be the minimal $*$-prime ideals belonging to A such that $P_1\neq R$. Then the following are equivalent.*

(a) *A is $*$-primary;*

(b) $A=A_*^{c(P_i)}$, $i=1, 2, \cdots, m$;

(c) $A=A^*_{c(P_i)}$, $i=1, 2, \cdots, m$.

Proof. The proof for (a)$\Rightarrow$(b)$\Rightarrow$(c) is the same as in Theorem 5.6. To show (c)$\Rightarrow$(a), suppose A is not *-primary. Then by Theorem 5.6 there exists a *-prime ideal $P\supseteq A$ such that $A\neq A^*_{c(P)}$ (note always $A\subseteq A^*_{c(P)}$). By Corollary 6.3 there exists a minimal *-prime ideal P_i such that $P_i\supseteq A$ and $P_i\subseteq P$. But then, since $c(P)\subseteq c(P_i)$, it follows from Theorem 5.5 that $A^*_{c(P_i)}\supseteq A^*_{c(P)}$ and hence $A\neq A^*_{c(P_i)}$.

Theorem 6.8. *Suppose R is noetherian and A is an ideal of R. Then there exists a nonnegative integer $k=k(A)$ such that $D^k_*(r^*(A))\subseteq A$.*

Proof. If A is *-prime, the result is obvious. We proceed as in [20]. Suppose the theorem is false and A is a maximal element in the set of ideals for which the result does not hold. Since A is not *-prime, there exist ideals B, C in R properly containing A such that $B*C\subseteq A$. Hence we have $D^p_*(r^*(B))\subseteq B$ and $D^q_*(r^*(C))\subseteq C$ for some p, q. Setting $k=\max(p, q)$ implies that $D^k_*(r^*(A))\subseteq B\cap C$ since $r^*(A)\subseteq r^*(B)\cap r^*(C)$. Hence $D^{k+1}_*(r^*(A))\subseteq(B\cap C)*(B\cap C)\subseteq B*C\subseteq A$, a contradiction.

Recall that an ideal A is called *-solvable if $D^m_*(A)=0$ for some m and that the *-prime radical $P^*(R)$ of R contains all *-solvable ideals in R (Corollary 3.10). If R is noetherian, we have the following stronger result.

Corollary 6.9. *If R is noetherian, the *-prime radical $P^*(R)$ is the unique maximal *-solvable ideal of R.*

Using this, we prove the following characterization of *-primary ideals.

Corollary 6.10. *Suppose R is noetherian. An ideal Q of R is *-primary if and only if $A*B\subseteq Q$ and $A\not\subseteq Q$ for ideals A, B of R imply that there exists a positive integer n such that $D^n_*(B)\subseteq Q$.*

Proof. Suppose Q is *-primary and $A*B\subseteq Q$ and $A\not\subseteq Q$ for ideals A, B. Then $B\subseteq r^*(Q)$ and by Theorem 6.8 we have $D^n_*(B)\subseteq Q$ for some n. Conversely, suppose Q is not *-primary. Then there exist ideals A, B in R such that $A*B\subseteq Q$ but $A\not\subseteq Q$ and $B\not\subseteq r^*(Q)$. Setting $\bar{R}=R/Q$ implies $\bar{A}\neq\bar{0}$ and $\bar{B}\not\subseteq\overline{r^*(Q)}=r^*(\bar{0})=P^*(\bar{R})$ by Lemma 3.11. Suppose $D^n_*(B)\subseteq Q$ for some $n>0$. Then $D^n_*(\bar{B})=\bar{Q}=\bar{0}$; that is, $\bar{B}$ is *-solvable, and hence $\bar{B}\subseteq P^*(\bar{R})$. This implies $B\subseteq r^*(Q)$, a contradiction. Hence $D^n_*(B)\not\subseteq Q$ for all n.

Definition 6.11. (a) An ideal A of R is said to be meet irreducible if $A=B\cap C$ for ideals B, C always implies $A=B$ or $A=C$.

(b) An arbitrary algebra R equipped with a strong left-additive *-operation is said to satisfy the right *-Artin-Rees property on ideals if, for every ideals A, B in R, there exists a nonnegative integer $n=n(A, B)$ such that $A\cap D_*^n(B)\subseteq A*B$.

This definition is a *-operation-analog of the Artin-Rees property for associative algebras or quadratic Jordan algebras [20]. The proof of the following lemma is well known and easy.

Lemma 6.12. (a) *An ideal A of R is meet irreducible if and only if* (0) *is meet irreducible in R/A.*

(b) *If R is noetherian then every ideal of R is an intersection of a finite number of ideals in R.*

Lemma 6.13. *If R satisfies the right *-Artin-Rees property, then every meet irreducible ideal of R is *-primary.*

Proof. Let S be a meet irreducible ideal of R. By Theorem 5.2 and Lemma 6.12 we may assume $S=(0)$. Let A, B be ideals in R such that $A*B=0$. By tne right *-Artin-Rees property there exists an n such that $A\cap D_*^n(B)=0$ and hence $A=0$ or $D_*^n(B)=0$. If $A\neq 0$, $D_*^n(B)=(0)\subseteq r^*(0)$ $(=P^*(R))$. By Corollary 6.9, since $r^*(0)$ is *-solvable, $D_*^m(D_*^n(B))=D_*^{m+n}(B)=0$. Thus B is *-solvable and is contained in $r^*(0)$; that is, (0) is *-primary.

We now state the main theorem.

Theorem 6.14 (Lasker-Noether Theorem) *Suppose R is noetherian. Then a necessary and sufficient condition that every ideal of R has a normal *-primary representation is that R satisfies the right *-Artin-Rees property on ideals.*

The proof is based on an argument using Theorem 6.8, Corollary 5.11, and Lemmas 6.12, 6.13, and so the same as in [20].

7. Tertiary ideals and decompositions.

Throughout we let R be an arbitrary algebra equipped with a strong left-additive *-operation. In this section we introduce a definition of tertiary ideal and radical for R by using a *-operation. Our definition is similar to that given by *Kurata* [7] and that more recently given by *Tsai* and *Foster* [20] for quadratic Jordan algebras. Also, the present one is a best generalization of *-primary decomposition since we can show that any ideal in a noetherian algebra possesses a normal *-tertiary decomposition without any additional condition. We begin with

Definition 7.1. Let A, B be ideals of R. Then the set $[B: A]^*=\{x\in R\mid (x)*A\subseteq B\}$

is called the (left) *-quotient of A in B.

A notion similar to this was defined in *Myung* [13]. From the definition one can easily show that $[B: A]^*$ is an ideal in R containing B.

Lemma 7.2. *Let A be an ideal of R. Then the following sets are equal to each other.*

(a) $T_1=\{a\in R|[A:(a)]^*\cap(c)\subseteq A$ *implies* $c\in A\}$;

(b) $T_2=\{a\in R|$ *for any* $b\in c(A)$, *there exists* $c\in(b)\cap c(A)$ *such that* $(c)*(a)\subseteq A\}$;

(c) $T_3=\{a\in R|[A:(a)]^*\cap B\subseteq A$ *for any ideal B in R implies* $B\subseteq A\}$.

Proof. To show $T_2\subseteq T_1$, let $a\in T_1$. Then there is a $b\in c(A)$ such that $[A:(a)]^*\cap(b)\subseteq A$. Now, choose any $c\in(b)\cap c(A)$ and suppose $(c)*(a)\subseteq A$. Then $c\in[A:(a)]^*$ and so $c\in[A:(a)]^*\cap(b)\subseteq A$, a contradiction. Hence $(c)*(a)\nsubseteq A$ and so $a\notin T_2$, or $T_2\subseteq T_1$. To see $T_1\subseteq T_2$, let $a\notin T_2$. Then there is a $b\in c(A)$ such that, for all $c\in(b)\cap c(A)$, we have $(c)*(a)\nsubseteq A$. It is enough to show $[A:(a)]^*\cap(b)\subseteq A$. For this, let $x\notin A$. If $x\in(b)\cap c(A)$, then $(x)*(a)\nsubseteq A$ and so $x\notin[A:(a)]^*$. Thus $[A:(a)]^*\cap(b)\subseteq A$ and $a\notin T_1$. That $T_1=T_3$ is immediate.

Definition 7.3. Let A be an ideal of R. Any one of the sets in Lemma 7.2 is called the *-tertiary radical of A and is denoted by $t^*(A)$. An ideal T is called *-tertiary if $A*B\subseteq T$ for ideals A, B in R implies $A\subseteq T$ or $B\subseteq t^*(T)$.

For any ideal A, we note from the definition that $A\subseteq t^*(A)$ and if $A\neq R$ then $A\subsetneqq[A:(b)]^*$ for all $b\in t^*(A)$.

Lemma 7.4. *Every meet irreducible ideal A in R is *-tertiary.*

Proof. Suppose A is not *-tertiary. Then there exist ideals B, C such that $B\nsubseteq A$ and $C\nsubseteq t^*(A)$ but $B*C\subseteq A$. Choose an element $b\in C$ with $b\notin t^*(A)$. By Lemma 7.2(a) there exists $c\notin A$ such that

$$[A:(b)]^*\cap(c)\subseteq A\ .$$

Hence $A\subseteq[A:(b)]^*\cap[(c)+A]$. Now, let $x\in[A:(b)]^*\cap[(c)+A]$. Then $x=y=c_0+a$ for $c_0\in(c)$, $y\in[A:(b)]^*$, $a\in A$ and so $c_0=y-a\in[A:(b)]^*$. Thus $c_0\in A$ and $x=c_0+a\in A$, which implies $A=[A:(b)]^*\cap[(c)+A]$. Choosing a $u\in B$ with $u\notin A$ gives $u\in[A:(b)]^*$ since $(u)*(b)\subseteq B*C\subseteq A$. Hence $A\subsetneqq[A:(b)]^*$ and since $c\notin A$, we have that A is not meet irreducible.

For another characterization of *-tertiary radical we put

Definition 7.5. Let R be an arbitrary algebra. An ideal A in R is called essential in R if for any nonzero ideal B of R we have $A\cap B\neq 0$.

An example of an algebra R where every nonzero ideal is essential in R is a subdirectly irreducible algebra. By definition, R is called subdirectly irreducible if the intersection of all nonzero ideals in R is not zero.

Lemma 7.6. *Let A be an ideal of R and let $\bar{R}=R/A$. Then*

$$t^*(A)=\{b\in R \mid [\bar{0}:(\bar{b})]^* \text{ is essential in } \bar{R}\}\ .$$

Proof. For any ideals A, B, C in R, by the same argument as in Lemma 7.4 we see that $[A:B]^*\cap C\subseteq A$ if and only if $[A:B]^*\cap(C+A)=A$. From this and Lemma 7.2 it follows that

$$\begin{aligned} t^*(A)&=\{b\in R \mid B\not\subseteq A \text{ for any ideal } B \text{ implies } [A:(b)]^*\cap B\not\subseteq A]\\ &=\{b\in R \mid B\not\subseteq A \text{ implies } [A:(b)]^*\cap(B+A)\supsetneqq A\}\\ &=\{b\in R \mid \bar{B}\neq\bar{0} \text{ implies } [\bar{0}:(\bar{b})]^*\cap\bar{B}\neq\bar{0}\}\\ &=\{b\in R \mid [\bar{0}:(\bar{b})]^* \text{ is essential in } \bar{R}\}. \end{aligned}$$

Theorem 7.7. *Let A be an ideal of R. If $*$ is both left- and right-additive, then $t^*(A)$ is an ideal in R.*

Proof. Let $r\in R$, $b\in t^*(A)$ and let $\bar{R}=R/A$. Since $(rb)\subseteq(b)$, $[\bar{0}:(\bar{b})]^*\subseteq[\bar{0}:(\overline{rb})]^*$. Hence by Lemma 7.6 $rb\in t^*(A)$ and similarly $\alpha b\in t^*(A)$ for $\alpha\in\Phi$. Let $b, c\in t^*(A)$. Since $(b-c)\subseteq(b)+(c)$ and $*$ is right-additive, we have

$$[\bar{0}:(\bar{b}-\bar{c})]^*\supseteq[\bar{0}:(\bar{b})]\cap[\bar{0}:(\bar{c})]^*\ .$$

If K is any nonzero ideal in $\bar{R}$, then since $[\bar{0}:(\bar{b})]^*$ and $[\bar{0}:(\bar{c})]^*$ are essential in $\bar{R}$, we get

$$K\cap[\bar{0}:(\bar{b}-\bar{c})]^*\supseteq\{K\cap[\bar{0}:(\bar{b})]^*\}\cap[\bar{0}:(\bar{c})]^*$$

and so $[\bar{0}:(\bar{b}-\bar{c})]^*$ is essential in $\bar{R}$ too. Thus by Lemma 7.6 $b-c\in t^*(A)$ and $t^*(A)$ is an ideal of R.

If R is an algebra where the set of all ideals in R forms a chain by the inclusion, then any $*$-operation in R is left-and right-additive. A nontrivial example of this algebra is given in [2, p. 247].

Lemma 7.8. *Let $A_1, \cdots, A_n$ be ideals in R. Then*

$$t^*(A_1\cap A_2\cap\cdots\cap A_n)\subseteq t^*(A_1)\cap t^*(A_2)\cap\cdots\cap t^*(A_n)\ .$$

Proof. By induction it is enough to show the result for $n=2$. For $b\in t^*(A_1\cap A_2)$, let $c\in R$ be such that $[A_1\cap A_2:(b)]^*\cap(c)\subseteq A_1\cap A_2$. Then we have $[A_1\cap A_2:(b)]^*\cap(c)=[A_1:(b)]^*\cap[A_2:(b)]^*\cap(c)\subseteq A_2$. Since $b\in t^*(A_2)$, from this and Lemma 7.2(c) it follows that $[A_1:(b)]^*\cap(c)\subseteq A_2\subseteq[A_2:(b)]^*$. Hence $[A_1\cap A_2:(b)]^*\cap(c)=[A_2:(b)]^*\cap([A_1:(b)]^*\cap(c))=[A_1:(b)]^*\cap(c)\subseteq A_1$ and so $c\in A_1$ since $b\in t^*(A_1)$. Similarly we show $c\in A_2$ and so $c\in A_1\cap A_2$; that is, $b\in t^*(A_1\cap A_2)$.

Furthermore, for irredundant representations we have

Lemma 7.9. *Suppose an ideal A of R has an irredundant representation by *-tertiary ideals $T_1, \cdots, T_n$. Then $t^*(A)=t^*(T_1)\cap\cdots\cap t^*(T_n)$.*

This follows from Lemma 7.8 and the same argument as in [20, Lemma 7.1]. As an immediate consequence of Lemma 7.9 we have

Corollary 7.10. *Let $T_1, \cdots, T_n$ be *-tertiary ideals of R such that $t^*(T_1)=\cdots=t^*(T_n)$. Then $T=T_1\cap\cdots\cap T_n$ is also *-tertiary and $t^*(T)=t^*(T_1)$.*

Definition 7.11. An irredundant representation by *-tertiary ideals $T_1, \cdots, T_n$ is called a normal *-tertiary representation if $t^*(T_i)\neq t^*(T_j)$ for all $i\neq j$.

From Corollary 7.10 we obtain

Corollary 7.12. *Any ideal represented by a finite number of *-tertiary ideals has a normal *-tertiary representation.*

We are now prepared to state the main result

Theorem 7.13. *Suppose R is noetherian. Then every ideal A has a normal *-tertiary representation, and if $A=T_1\cap\cdots\cap T_m=S_1\cap\cdots\cap S_n$ are two normal *-tertiary representations for A then $m=n$ and $t^*(T_i)=t^*(S_i)$, $i=1, 2, \cdots, m$, for a suitable ordering of the components.*

The proof is essentially based on Lemmas 6.11(b), 7.4 and Corollary 7.12, and hence standard (see, for example, [20] or [7]). In a noetherian algebra, the right *-Artin-Rees property yields the following relation between *-tertiary and *-primary ideals.

Theorem 7.14. *Suppose R is noetherian. Then the following are equivalent.*

(a) *R satisfies the right *-Artin-Rees property;*

(b) *Every *-tertiary ideal of R is *-primary.*

For the proof, see [20] or [7].

Finally we prove an analogous result of the Krull Intersection Theorem in commutative, associative rings.

Theorem 7.15. *Suppose R is noetherian and satisfies the right *-Artin-Rees property. For any ideal A of R, let $B=\bigcap_{n>0} D_*^n(A)$. Then we have $B*A=B$. In particular, if $A\subseteq P^*(R)$, then $\bigcap_{n>0} D_*^n(A)=0$.*

Proof. We first note that $B\cap D_*^n(A)\subseteq B*A$ for some n. Since $B\subseteq D_*^n(A)$, $B=B\cap D_*^n(A)\subseteq B*A\subseteq B$ and so $B*A=B$. If $A\subseteq P^*(R)$, then by Theorem 6.8

$D_*^n(A) \subseteq D_*^n(P^*(R))=0$ for some n and hence $B=0$.

8. Applications

To apply the present theory to other systems which generalize "binary" algebras, let G be a unital module over a commutative associative ring Φ with identity. Let Ω be a set of finitary operations defined on G. Here, as usual, an n-ary operation ω on G is a function of $G \times \cdots \times G$ (n times) into G. We denote by $\omega(a_1 a_2 \cdots a_n)$ the image of $(a_1, a_2, \cdots, a_n)$ by $\omega \in \Omega$. Thus a Φ-module G together with Ω is regarded as a universal algebra where the module operations in G yield a set of unary operations and a binary operation. Following *Kurosh* [8, p. 99], we will call such a G a *Φ-module with a system Ω of multi-operators*, or simply call it an *(Ω, Φ)-group* if G satisfies the condition

$$\omega(00 \cdots 0)=0 \quad \text{for all} \quad \omega \in \Omega \,. \tag{17}$$

An (Ω, Φ)-group G and (Ω', Φ)-group G' are said to be of *the same type* if Ω and Ω' are of the same type; that is, there exists a one-one correspondence $\omega \leftrightarrow \omega'$ between Ω and Ω' such that whenever ω is an n-ary operation on G, so is ω' on G'. If G and G' are of the same type, a mapping h from G into G' is called a *homomorphism* if h is a module homomorphism of G into G' and in addition satisfies

$$h(\omega(a_1 \cdots a_n)) = \omega'(h(a_1) \cdots h(a_n))$$

for all $a_i \in G$ and all $\omega \in \Omega$.

In order to introduce a notion of ideal in an (Ω, Φ)-group G, we adopt the definition given by *Kurosh* [8, p. 100].

Definition 8.1. Let G be an (Ω, Φ)-group. A nonempty subset A of G is called an ideal of G if

(a) A is a Φ-submodule of G,

(b) for any n-ary operation $\omega \in \Omega$, any element $a \in A$,

any elements $x_1, \cdots, x_n \in G$, and for all $i=1, 2, \cdots, n$, the following holds:

$$\omega(x_1 \cdots x_{i-1}(a+x_i)x_{i+1} \cdots x_n) \in \omega(x_1 x_2 \cdots x_n)+A \,. \tag{18}$$

If $\{A_i | i \in I\}$ is a family of ideals in G, by (18) an easy induction shows that the submodule $\sum_{i \in I} A_i$ is also an ideal of G. Let A be an ideal of G. If $\omega \in \Omega$ is an n-ary operation and we set

$$\omega((x_1+A) \cdots (x_n+A)) = \omega(x_1 \cdots x_n)+A$$

for $x_i \in G$, then ω is well defined on the quotient module G/A and so Ω acts on

G/A. Thus G/A is made into an (Ω, Φ)-group, called the (Ω, Φ)-*quotient group* of G. For the basic results regarding (Ω, Φ)-quotient groups, isomorphism theorems, and etc., the reader is referred to *Kurosh* [8, Chapter III]. Let G be an (Ω, Φ)-group. If every element ω in Ω is multi-linear on the Φ-module G, then (18) is reduced to

$$\omega(x_1 \cdots x_{i-1} a x_{i+1} \cdots x_n) \in A\ , \quad i=1, 2, \cdots, n \tag{19}$$

for all $\omega \in \Omega$, all $x_1, \cdots, x_n \in G$, and all $a \in A$.

Definition 1.1 can now be carried out to any (Ω, Φ)-group G to introduce a $*$-operation in G. All the previous results in terms of a $*$-operation remain to hold for any (Ω, Φ)-group equipped with a (strong left-additive) $*$-operation. Since Ω can arbitrarily act on G, the concept of ideal in G is usually much more complicated than that in a binary algebra. Thus it is not a surprise that the constructions of a strong left-additive $*$-operation in an (Ω, Φ)-group is somewhat less natural than those in a binary algebra. Consider an (Ω, Φ)-group such that an arbitrary n-ary operation $\omega \in \Omega$ has at least one argument with respect to which ω is Φ-linear. In this case, let $n(\omega)$ be the least index among such arguments. We further assume that

$$\omega(0 \cdots 0 x_{n(\omega)} 0 \cdots 0)=0$$

for all $\omega \in \Omega$ and $x_{n(\omega)} \in G$. It should be noted under this situation that Condition (17) is superfluous. For any ideals A, B in G, let $A*B$ be the ideal in G generated by the elements $\omega(x_1 \cdots x_n)$ with $x_i \in B$ for $i \neq n(\omega)$ and $x_i \in A$ for $i = n(\omega)$ where ω runs over all the n-ary operations in Ω. Then $*$ is clearly a strong left-additive $*$-operation in G. It is not difficult to form this kind of (Ω, Φ)-groups. For this, let G be a unital Φ-module and Λ be a set of mappings λ from $G \times \cdots \times G$ (n times) into $\mathrm{Hom}_\Phi(G, G)$ such that

$$\lambda(0, 0, \cdots, 0)=0\ ,$$

where n runs over a set of positive integers. Let Λ_0 be the set of finitary operations λ_0 on G such that

$$\lambda_0(x_1, x_2, \cdots, x_{n+1})=x_1\lambda(x_2, \cdots, x_{n+1})$$

for $\lambda \in \Lambda$ and for $x_1, x_2, \cdots, x_{n+1} \in G$. Then G is regarded as a (Λ_0, Φ)-group satisfying the condition above. We define the ideals in the module G with Λ to be those in G regarded as a (Λ_0, Φ)-group. Thus, for any ideals A, B in G, if we set $A*B$ to be the ideal in G generated by the elements $\lambda_0(ab_1b_2 \cdots b_n)$, $\lambda_0 \in \Lambda_0$, $a \in A$, $b_i \in B$, we obtain a strong left-additive $*$-operation in the (Λ_0, Φ)-group G.

This setting of a Φ-module with multi-operators is more natural because of its similarity with some well-known algebraic systems. Let $(J, U, 1)$ be a unital quadratic Jordan algebra over Φ, where J is a unital Φ-module and U is a quadratic mapping from J into $\mathrm{Hom}_\Phi(J, J)$ which satisfies the axioms of *McCrimmon* (see, for example, [20]). Here, a Φ-submodule A of J is called an inner (outer) ideal of J if, for all $x \in J$ and $a \in A$, $xU_a \in A (aU_x \in A)$, and A is called an ideal of J if A is both inner and outer. Any unital quadratic Jordan algebra over Φ is also regarded as an (Ω, Φ)-group in the above manner, where Ω consists of the binary operation $\omega(xy)=xU_y$. Thus we have two notions of ideal for J. We however prove

Theorem 8.2. *Let $(J, U, 1)$ be a unital quadratic Jordan over Φ. Then the ideals of J in the Jordan sense coincide with those of J in the sense of Definition* 8.1 *regarded J as an (Ω, Φ)-group.*

Proof. We first recall that if A is an outer ideal of J then $xU_{a,y} \in A$ for all $x, y \in J$ and $a \in A$, where $U_{a,y}=U_{a+y}-U_a-U_y$ (see [20, Proposition 1.1]). Suppose now that A is an ideal of J in the Jordan sense. For $a \in A$ and $x \in J$, we have $(x+a)U_y-xU_y=aU_y \in A$ and $xU_{a+y}-xU_y=xU_{a,y}+xU_a \in A$ since A is inner and outer. Hence A is an ideal of J in the sense of Definition 8.1. Conversely, let A be an ideal of J in the sense of Definition 8.1. Then, for $a \in A$ and $x, y \in J$, we have $(x+a)U_y-xU_y=aU_y \in A$ and so A is an outer ideal of J. Using this, we get $xU_a=xU_{a+y}-xU_y-xU_{a,y} \in A$ since A is outer. Hence A is inner too and is an ideal of J in the Jordan sense.

If J is a unital quadratic Jordan algebra then the submodule AU_B for any ideals A, B of J is shown to be an ideal of J. Therefore, the present theory applied to an (Ω, Φ)-group yields, as a special case, the primary ideal theory of *Tsai* and *Foster* [20] for quadratic Jordan algebras.

We close this section with a special type of Φ-module with multi-operators. For a fixed positive integer $\nu \geq 2$, a *ν-ary system* M is defined to be a unital Φ-module with a Φ-multi-linear mapping $(a_1, a_2, \cdots, a_\nu) \to \langle a_1, a_2, \cdots, a_\nu \rangle$ from $M \times \cdots \times M$ (ν times) into M. The best known ν-ary systems are the triple systems. An ideal in a ν-ary system is defined by Definition 8.1 or equivalently by (19). This definition of ideal is what has been used for general triple systems (see *Meyberg* [10]). A ν-ary system can be easily formed from a nonassociative algebra by ν times iteration of the binary product.

Let M be a ν-ary system. Then for a positive integer n we set

$$\nu(n)=\nu+(n-1)(\nu-1) .$$

If $A_1, A_2, \cdots, A_{\nu(n)}$ are submodules of M, denote by $\langle A_1A_2\cdots A_{\nu(n)}\rangle$ the submodule generated by all products of any $\nu(n)$ elements $a_1, a_2, \cdots, a_{\nu(n)}$ with $a_i \in A_i$ in all possible $\langle\ \rangle$-association. In particular, if $A_i=A$ for all $i=1\cdots, \nu(n)$, we denote $\langle A_1A_2\cdots A_{\nu(n)}\rangle=\langle A^{\nu(n)}\rangle$.

Definition 8.3. Let M be a ν-ary system. Then M is called a $\nu(s)$-system if there exists a positive integer s such that $\langle A^{\nu(s)}\rangle$ is always an ideal of M for any ideal A of M. As in *Zwier* [21], if M is a $\nu(s)$-system, then an ideal P of M is called $\nu(s)$-prime if $\langle A_1A_2\cdots A_{\nu(s)}\rangle\subseteq P$ for ideals A_i in M implies $A_i\subseteq P$ for some $i=1, 2, \cdots, \nu(s)$. An ideal A of M is called nilpotent if there exists a positive integer m such that $\langle A^{\nu(m)}\rangle=0$.

For a $\nu(s)$-system M, one can also develop the same prime ideal theory as for s-algebras given by *Zwier* [21]. If M is now an arbitrary ν-ary system, one can construct a strong left-additive *-operation in M as for binary algebras (Section 2). If M is a $\nu(s)$-system and A, B are ideals of M, we set $A*B$ to be the ideal in M generated by the submodule $\langle AB_2\cdots B_{\nu(s)}\rangle$ where $B_i=B$ for $i=2, \cdots, \nu(s)$. Then we obtain a strong left-additive *-operation in M such that $A*A=\langle A^{\nu(s)}\rangle$ for any ideal A of M. As in Section 4, those two notion of prime ideal for a $\nu(s)$-system lead to the same prime radical.

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