ON A MEASURE OF NONCONVEXITY AND APPLICATIONS*

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1. Introduction. The measure of noncompactness which was introduced by *Kuratowski* [8] (in 1930) has now become an important tool in nonlinear analysis (although its value in that regard was not appreciated until much later). Following Kuratowski we introduce a measure of nonconvexity which has many properties in common with the measure of noncompactness and therefore we may now have "convex" where previously we had "compact" in the statements of some theorems.

For example, let E be a Banach space and consider the differential equation

$$x'=f(t,x)\ ,\quad x(t_0)=x_0\ , \tag{1.1}$$

where $f:R^+\times E\to E$. Let $x(t,t_0,x_0)$ denote the solution of (1.1) and

$$x(t,t_0,X)=\{x(t,t_0,x_0):x_0\in X\}\ , \tag{1.2}$$

where X is a subset of E. Let $\gamma(A)$ denote the measure of noncompactness of A. *Ambrosetti* [1] and *Szufla* [10] used the condition that f is α-Lipschitzian, i.e., there is an $L>0$ such that

$$\gamma(f(X))\leq L\gamma(X)\ , \tag{1.3}$$

to guarantee that

$$\gamma(x(t,t_0,X))\leq e^{L(t-t_0)}\gamma(X)\ ,\quad t\geq t_0\ . \tag{1.4}$$

More generally one introduces a function $g(t,u)$, which may be nonlinear, and replaces condition (1.3) by weaker condition [6]

$$\gamma(f(t,A))\leq g(t,\gamma(A))\ , \tag{1.5}$$

to guarantee that

$$\gamma(x(t,t_0,X))\leq r(t,t_0,\gamma(X)) \tag{1.6}$$

Here $r(t,t_0,x_0)$ is the maximal solution of the scalar differential equation

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$$u' = g(t, u), \quad u(t_0) = u_0 \geq 0. \tag{1.7}$$

Similarly, if α denotes the measure of nonconvexity then we may obtain the analogous estimate

$$\alpha(x(t, t_0, X)) \leq r(t, t_0, \alpha(X)), \quad t \geq t_0. \tag{1.8}$$

If the maximal solution of (1.7) is identically zero, when $u_0 = 0$, then in view of (1.8) ⟨(1.6)⟩, the closure of $x(t, t, X)$ is convex ⟨compact⟩ for $t > t_0$ if it is convex ⟨compact⟩ at $t = t_0$.

The concept of a comparison map, such as ψ, which bounds the measure of noncompactness has been used in fixed point theory in conection with the contraction mapping and Schauder principles [4], [5]. In Section 2 we make some general remarks concerning comparison maps, the measures of non-compactness and nonconvexity and fixed point theory. The role of the measure of nonconvexity in differential equations is discussed in Section 3.

2. A measure of nonconvexity. Let E be a Banach space (with norm $\|\cdot\|$) and A a subset in E. Denote by $\mathrm{co}(A)$ the convex hull of A. We say that A is α-measurable with measure $\alpha(A)$ if

$$\alpha(A) = \sup_{b \in \mathrm{co}(A)} \inf_{a \in A} \|b - a\| < \infty, \tag{2.1}$$

Alternatively, if $H(X, Y)$ denotes the Hausdorff distance between two subsets X and Y,

$$\alpha(A) = H(A, \mathrm{co}(A)). \tag{2.2}$$

Clearly, a bounded set is α-measurable.

From the definition the following properties of α can be derived in a straightforward manner.

$\alpha(A) = 0$ iff $\bar{A}$ (the closure of A) is convex; (2.3)

$\alpha(\lambda A) = |\lambda| \alpha(A)$ for $\lambda \in R^1$ (where $\lambda A = \{\lambda a | a \in A\}$; (2.4)

$\alpha(A + B) \leq \alpha(A) + \alpha(B)$; (2.5)

$|\alpha(A) - \alpha(B)| \leq \alpha(A - B)$; (2.6)

$\alpha(\bar{A}) = \alpha(A)$; (2.7)

$\alpha(A) \leq \mathrm{diam}(A)$ (the diameter of A); (2.8)

$|\alpha(A) - \alpha(B)| \leq 2H(A, B)$. (2.9)

Note that all these properties are shared by the measure of noncompactness γ. Recall $\gamma(A) = \inf\{d > 0 | A$ can be covered by a finite number of sets of diameter $\leq d\}$. α is not monotone in the sense that $\alpha(A) \leq \alpha(B)$ if $A \subset B$. If it did, then every closed set would be convex which is not true. Unfortunately $\alpha(A)$ measures only the nonconvexity of $\bar{A}$ and not A itself if A is not closed.

As a consequence of (2.9) and a similar inequality for γ, $|\gamma(A)-\gamma(B)|\leq H(A,B)$, the measures α and γ are continuous with respect to the Hausdorff metric, that is,

Proposition 2.1. *Let A_n be a sequence of subsets of E such that A_n approaches a subset A_∞ in the Hausdorff metric. Then* (i) *if A_n are α-measurable,*

$$\lim_{n\to\infty}\alpha(A_n)=\alpha(A_\infty)\ . \tag{2.10}$$

(ii) *if A_n are bounded*

$$\lim_{n\to\infty}\gamma(A_n)=\gamma(A_\infty)\ . \tag{2.11}$$

Proposition 2.2 (*Kuratowski*). *Let (X,ρ) be a complete metric space and let $A_0\supset A_1\supset\cdots$ be a decreasing sequence of nonempty, closed subsets of E. Assume $\gamma(A_n)\to 0$. Then if we write $A_\infty=\bigcap_{n\geq 0}A_n$, A_∞ is a nonempty compact set and A_n approaches A_∞ in the Hausdorff metric.*

Proposition 2.3. *Let $A_0\supset A_1\supset\cdots$ be a decreasing sequence of closed bounded subsets of E. Let $A_\infty=\bigcap_{n\geq 0}A_n$. Then A_∞ is nonempty, convex and compact and A_n converges to A_∞ in the Hausdorff metric iff $\alpha(A_n)\to 0$ and $\gamma(A_n)\to 0$.*

Proof. Suppose $\gamma(A_n)\to 0$. It follows from Proposition 2.2 that A_n converges to the nonempty compact set A_∞ in the Hausdorff metric. If, in addition, $\alpha(A_n)\to 0$ then in view of (2.10), $\alpha(A_\infty)=0$. Since A is also closed, A_∞ is convex by (2.3).

Suppose $A_n\to A_\infty$ in the Hausdorff metric and $\alpha(A_\infty)=\gamma(A_\infty)=0$. Then by (2.10) and (2.11), $\alpha(A_n)\to 0$ and $\gamma(A_n)\to 0$.

Proposition 2.4. *Let $A_0\supset A_1\supset\cdots$ be a decreasing sequence of closed, bounded subsets of E such that $\alpha(A_n)\to 0$ and $\gamma(A_n)\to 0$. Suppose T is a continuous map of $A_0\to A_0$ such that*

$$Tx\in A_n \text{ if } x\in A_n,\ n=0,1,\cdots. \tag{2.12}$$

Then there exists an $x\in A_\infty=\bigcap_{n\geq 0}A_n$ such that

$$Tx=x\ , \tag{2.13}$$

Proof. The result is a corollary of the Schauder principle ([7], p. 67) since, from Proposition 2.3, A_∞ is nonempty, convex and compact and T maps A_∞ into itself.

Closely associated with the notion of measure of noncompactness is the concept of *k-set-contraction* (also due to *Kuratowski* [8]). Let (X_1, d_1) and (X_2, d_2) be metric spaces and suppose $T:X_1 \to X_2$ is a continuous map. We say T is a k-set-contraction if given any bounded set A in X_1, $T(A)$ is bounded and $\gamma_2(T(A)) \leq k\gamma_1(A)$ where γ_i denotes the measure of noncompactness in X_i, $i=1, 2$.

Proposition 2.5. (*Darbo* [3].) *Let C be a closed, bounded, convex set and $T:C \to C$ a k-set-contraction, $k<1$. Then T has a fixed point, i.e., a point x satisfying* (2.13).

The above generalization of the Schauder principle was further extended [5] by introducing a comparison function ϕ which has the following properties: (i) ϕ maps a conical segment of regular cone in a partially ordered space into itself; (ii) ϕ is monotone; (iii) ϕ is upper semi-continuous from the right; (iv) $\phi(x)=x$ iff $x=\theta$ (the zero of the space). Then Darbo's condition $\gamma(T(A)) \leq k\gamma(A)$, $k<1$, is replaced by the weaker condition

$$\gamma(T(A)) \leq \phi(\gamma(A)) . \tag{2.14}$$

In [5] vector-valued measure of noncompactness was considered. Our present discussion is limited to the cone of nonnegative numbers in R. In this context it is more appropriate (see Proposition 2.7, below) to use a comparison function which was introduced by *Boyd and Wong* [2] in their discussion of the contraction mapping principle.

Definition 2.1. A function $\phi: [0, \infty) \to [0, \infty)$ is a comparison function if (i) $\phi(t)<t$ for $t>0$, (ii) $\phi(0)=0$, ϕ is upper semi-continuous from the right.

Proposition 2.6. *Let ϕ be a comparison map and let $S_0, S_1 \cdots$ be a sequence of nonnegative real numbers such that $S_n \leq \phi(S_{n-1})$, $n=1, 2, \cdots$. Then the sequence S_n converges to zero.*

Proof. Since $S_n \leq \phi(S_{n-1}) \leq S_{n-1}$, the sequence S_n converges monotonically. Suppose $S_\infty = \lim S_n > 0$. Then $\phi S_\infty < S_\infty \leq S_n$, $n=1, 2, \cdots$. But this contradicts the upper semi-continuity from the right.

Proposition 2.7. *Let ϕ: $[0, a) \to [0, a]$ be nonincreasing, upper semicontinuous from the right, and $\phi(t)=t$ iff $t=0$. Then ϕ has an extension to $[0, \infty)$ which is a comparison function.*

Proof. Since the interval $[0, a]$ is a segment of the regular cone (of nonnegative real numbers) it follows from Theorem 3.1 in [4] that if $t \leq \psi(t)$ then $t \leq t_0$ where t_0 is the maximal solution of $\psi(t)=t$. By assumption $t_0=0$. Thus $t \leq \psi(t)$ iff $t=0$. If we define $\phi(t)=\psi(a)$, $t \geq a$ then ϕ is a comparison function.

Definition 2.2. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces and suppose $T: X_1 \to X_2$ is a continuous map. We say that T is a ψ-set-contraction with respect to convexity ⟨compactness⟩ if given any α_1-measurable ⟨bounded⟩ set A in X_1, $T(A)$ is α_2-measurable ⟨bounded⟩ and

$$\alpha_2(T(A)) \leq \psi(\alpha_1(A)) \tag{2.15}$$

$$\langle \gamma_2(T(A)) \leq \psi(\gamma_1(A)) \rangle \tag{2.16}$$

where $\alpha_i \langle \gamma_i \rangle$ denotes the measure of nonconvexity ⟨noncompactness⟩ in X_i, $i=1, 2$. We say that T is a ψ-contraction if $\|Tx-Ty\|_2 \leq \psi(\|x-y\|_1)$ for every $x, y \in X_1$. The following result is a generalization of a similar result due to Darbo [3] in regard to relating the notion of k-contraction, i.e., a ψ-contraction with $\psi(t)=kt$, to the notion of k-set-contraction.

Proposition 2.8. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces. Let T be a ψ-contraction, then* (i) *T is a ψ-set-contraction with respect to compactness;* (ii) *$H(TA, TB) \leq \psi(H(A, B))$ whenever $H(A, B) < \infty$;* (iii) *if for every α-measurable set A, $co(TA) \subset \overline{T(\overline{co}(A))}$ (where $\overline{co}(X)$ denotes the convex closure of X), then T is ψ-set-contraction with respect to convexity.*

Proof. (i) Let A be a bounded set in X_1 and suppose $\gamma_1(A)=d$. Then given $\varepsilon>0$, we can write $A=\bigcup_{j=1}^{m} S_j$, $\operatorname{diam}(S_j) \leq d+\varepsilon$. Thus $T(A)=\bigcup_{j=1}^{m} T(S_j)$ and since T is a ψ-contraction, $\operatorname{diam}(T(S_j)) \leq \psi(d+\varepsilon)$. Let ε_i be a sequence of positive numbers converging to zero such that $\psi(d+\varepsilon_i)$ converges and let $b=\lim \psi(d+\varepsilon_i)$. Then by upper semi-continuity from the right, $b \leq \psi(d)$. Hence $\gamma_2(TA) \leq \psi(d)$. (ii) Let A and B be sets such that $H(A, B)=d<\infty$. Let $b \in B$. Then $\inf\{\|Tb-Ta\|_2, a \in A\} \leq \inf\{\psi(\|b-a\|_1), a \in A\} \leq \psi d$ by the upper semi-continuity from the right of the function ψ. Similarly $\inf\{\|Ta-Tb\|_2, b \in B\} \leq \psi d$. Thus $H(TA, TB) \leq d$. (iii) Let A be an α-measurable set in X_1. Then from (ii), $\alpha(TA)=H(TA, \overline{co}(TA))=H(TA, \overline{\overline{co}(TA)}) \leq H(TA, \overline{T(\overline{co}(A))})=H(TA, T(\overline{co}(A))) \leq \psi(H(A, \overline{co}A))=\psi\alpha(A)$.

Theorem 2.9. *Let A be a closed subset of a Banach space and T a map from A onto itself. If T is set contractive with respect to convexity*

⟨compactness⟩ then A is convex ⟨compact⟩. In particular, the set of fixed points of a set contractive with respect to convexity ⟨compactness⟩ map of a closed subset of a Banach space $\mathscr{B}$ into $\mathscr{B}$ is convex ⟨compact⟩.

Proof. Set $m=\alpha(T(A))=\alpha(A)$ $(m=\gamma(T(A))=\gamma(A))$. Then $m\leq\psi(m)$. If $m>0$ then $\psi(m)<m$. But this is impossible. Clearly $m=0$.

Theorem 2.10. *Let C be a closed, bounded set and $T:C\rightarrow C$ a ψ_1-set-contraction with respect to convexity and a ψ_2-set-contraction with respect to compactness. The set of fixed points of T is nonempty, convex, and compact.*

Proof. Let $C_0=C$, and $C_{n+1}=\overline{T(C_n)}$. Then $C_{n+1}\subset C_n$. Let $s_n=\gamma(C_n)$, $t_n=\alpha(C_n)$, then it follows from Proposition 2.6 that $s_n\rightarrow 0$ and $t_n\rightarrow 0$. By Proposition 2.4, the set $F(T)$ of fixed points of T is nonempty and, by Theorem 2.9, it is also convex and compact.

3. Convexity of solutions of differential equations.

Let E be a real Banach space and let $\|\cdot\|$ denote the norm in E. We let $B=\{x\in E \mid \|x\|\leq b\}$ denote the ball of radius b and let $R_0=[t_0, t_0+a]\times B$ where $t_0\geq 0$, $a>t_0$.

Consider the differential equation

$$x'=f(t, x)\ , \quad x(t_0)=x_0\ , \tag{3.1}$$

where $f\in C[R_0, E]$. There are several known results which guarantee the existence of solutions to (3.1). We mention in particular those given in [6]. One of the conditions given there is:

(I) f is uniformly continuous in R_0.

Another is a compactness condition which is similar to the convexity condition II stated below.

For any subset $A\subset B$ and for small $h>0$ set

$$A_h(f)=\{y \mid y=x+hf(t, x): x\in A\}\ .$$

We introduce a (comparison) scalar differential equation

$$u'=g(t, u)\ , \quad u(t_0)=0 \tag{3.2}$$

where $g\in C[[t_0, t_0+a]\times R^+, R]$. Assume that $u\equiv 0$ is the unique solution of (3.2). Then the convexity condition on f is

(II) $$\liminf_{h\rightarrow 0^+}\{h^{-1}[\alpha(A_h(f))-\alpha(A)]\}\leq g(t, \alpha(A))$$

for any subset $A \subset B$.

We also require the following condition on a set $A \subset B$:

(III) The set of solutions $x(t, x_0)$, $x_0 \in A$ of (3.1) exists and is equicontinuous. Condition III is satisfied under the conditions given in [6] when A is pre-compact.

Theorem 3.1. *Let $A \subset B$ have convex closure and let condition I, II, and III be satisfied for* (3.1). *Then the set*

$$x(t, t_0, A) = \{x(t, t_0, x_0) | x_0 \in A\}$$

has convex closure for $t \in [t_0, t_0 + a]$.

Proof. Set $m(t) = \alpha(x(t, A))$ where α is the measure of nonconvexity and $x(t, A) = x(t, t_0, A)$. Our claim is then $m(t) = 0$. Now $m(t+h) - m(t) = \alpha(x(t+h, A)) - \alpha(x(t, A)) = [\alpha(x(t+h), A) - \alpha(A_h(f))] + [\alpha(A_h(f)) - \alpha(x(t, A))]$. If we know that

$$\liminf_{h \to 0+} h^{-1}[\alpha(x(t+h, A)) - \alpha(A_h(f))] \leq 0 \tag{3.3}$$

then it follows from condition II that $D_+ m(t) \leq g(t, m(t))$ where D_+ denotes a Dini derivative. It follows further from the theory of differential inequalities [9] that $m(t) \equiv 0$. Thus it remains to verify (3.3).

By properties (2.4), (2.6) and (2.9)

$$\begin{aligned} h^{-1}[\alpha(x(t+h, A)) - \alpha(A_h(f))] &\leq \alpha[h^{-1}(x(t+h, A) - A_h(f))] \\ &\leq 2 \sup_{x_0 \in A} |h^{-1}[x(t+h, x_0) - x(t, x_0)] - f(t, x(t, x_0))| \,. \end{aligned}$$

Hence it suffices to show that

$$h^{-1}(x(t+h, x_0) - x(t, x_0)) \to f(t, x(t, x_0))$$

uniformly in x_0. Now

$$\|h^{-1}(x(t+h, x_0) - x(t, x_0)) - f(t, x(t, x_0))\|$$

$$\leq h^{-1} \int_t^{t+h} \|f(t+s, x(t+s, x_0)) - f(t, x(t, x_0))\| ds \,.$$

By the uniform continuity of f and by the equicontinuity of $x(t, x_0)$ this last expression can be made arbitrarily small, independent of x_0, by taking h sufficiently small. This concludes the argument.

Remark. Suppose further that A is compact and that the semi-group map $x_0 \to x(t, x_0)$ is continuous for each $t \in [t_0, t+a]$. Then $x(t, A)$ is closed and hence, by Theorem 3.1, also convex. In particular, if $y_0, z_0 \in A$, set $y = x(\bar{t}, y_0)$, $z = x(\bar{t}, z_0)$ for some $\bar{t} \in [t_0, t_0 + a]$. Then we know that if w lies on the line segment connecting y and z there exists w_0 on the line segment connecting y_0 to z_0 such that $x(t, w_0)=w$, i.e., w is an attainable target.

PEFERENCES

[1] A. Ambrosetti, *Un teorema di esistenza par le equazioni differenziali negli spazi di Banach*, Rend. Sem. Mat. Univ. Padua 39(1967), 349-360.

[2] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. 20(1969), 458-464.

[3] G. Darbo, *Punti uniti in transformazioni a codominio non compatto*, Rend. Sem. Mat. Univ. Padua, 24(1955), 84-92.

[4] J. Eisenfeld and V. Lakshmikantham, *Comparison principle and nonlinear contractions in abstract spaces*, J. Math. Anal. Appl. 49(1975), 504-511.

[5] J. Eisenfeld and V. Lakshmikantham, *Fixed point theorems through abstract cones*, J. Math. Anal. Appl., to appear.

[6] J. Eisenfeld and V. Lakshmikantham, *On the existence of solutions of differential equations in a Banach space*, University of Texas at Arlington Technial Report No. 8.

[7] M. A. Krasnoselskii, *"Positive Solutions of Operator Equations,"* Noordhoff, Groningen, 1964.

[8] C. Kuratowski, *Sur les espaces complets*, Fund. Math., 15 (1930), 301-309.

[9] V. Lakshmikantham and S. Leela, *"Differential and Integral Inequalities; Theory and Applications,"* Vol. I, Academic Press, New York, 1969.

[10] S. Szufla, *Some remarks on ordinary differential equations in Banach space*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys. 16 (1968), 795-800.

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