GENERAL FUNCTIONAL DIFFERENTIAL EQUATIONS AND THEIR ASYMPTOTIC OSCILLATORY BEHAVIOR

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1. Introduction.

A great deal of literature exists on the oscillation and nonoscillation of

(1) $$y^{(n)}(t)+a(t)y(t)=0\ , \quad (n\geq 1 \text{ an interger}),$$

and

(2) $$y^{(n)}(t)+a(t)y(g(t))=0\ ,$$

when $a(t)>0$ on some positive half line. For this, see [8, 9, 10] and the refcrences cited in them. However nothing much seems to be known about equations of the type

(3) $$y^{(n)}(t)+a(t)y(t)=f(t)\ ,$$

where $a(t)$ and $f(t)$ are continuous on R (real line) and change sign arbitrarily to the right of the origin. Asymptotic results in relation to boundedness and approach to zero of the solutions of

(4) $$y''(t)+a(t)y(t)=f(t)$$

for nonnegative $a(t)$ can be found in [4, 5, 6]. *Hammett* [6] recently studied the equation

(5) $$(p(t)x'(t))'+q(t)g(x(t))=f(t)$$

and proved that if $p(t)$ and $q(t)$ are positive, continuous and bounded away from zero; and, if $f(t)$ was integrable on the positive half real line, then all nonoscillatory solutions of equation (5) approach zero asymptotically. Hammett's approach is based upon a theorem of *Bhatia* [1] which does not apply to delay equations of the type

(6) $$y''(t)+a(t)y(t-\tau(t))=0\ .$$

In fact *Travis* [13] showed that the equation

(7) $$y''(t)+\frac{\sin t}{2-\sin t}y(t-\pi)=0$$

has the nonoscillatory solution $2+\sin t$ even though

$$\int^{\infty}(\sin t)/(2-\sin t)dt=\infty .$$

But according to Bhatia's theorem all solutions of the equation

(8) $$y''(t)+\frac{\sin t}{2-\sin t}y(t)=0$$

are oscillatory.

This unusually different behavior between equations (6) and (7) motivates us to study the general nth order equation

(9) $$y^{(n)}(t)+a(t)y(g(t))=f(t)$$

and extend Hammett's type study. In what follows, conditions have been found to ensure that bounded nonoscillatory and oscillatory solutions approach zero asymptotically. Examples have been given to show the applications of the theorems proved.

In regard to equations (3) and (9), the following assumptions hold for the rest of this paper:

(A) $a(t), g(t), f(t)$: $R\to R$ and continuous,

(B) $g(t)\leq t$, $g(t)\to\infty$ as $t\to\infty$ and has a bounded derivative on R which is nonnegative on R.

The results of this paper remain true if condition (B) on $g(t)$ is replaced by:

g is absolutely continuous and

$$\lim_{t\to\infty}\int_t^{t+1}|g'(s)|ds<\infty .$$

We call a function on $C[\alpha,\infty]$ oscillatory if it has arbitrarily large zeros. Otherwise we call it nonoscillatory. The term "solution" will henceforth apply to continuously extendable solutions on some positive half line.

An interesting aspect of equations (3) and (9) being considered here is that $a(t)$ and $f(t)$ have been allowed to change signs arbitrarily often.

2. Main results.

Lemma (2.1). *Suppose*

(i) *there exists a $k>0$ such that*

$$\liminf_{t\to\infty}\int_t^{t+k}a^+(t)dt\geq\varepsilon>0 ,$$

(ii) $$\int^{\infty} a^{-}(t)dt < \infty$$

(iii) $$\int^{\infty} |f(t)|dt < \infty .$$

Let $y(t)$ be a bounded nonoscillatory solution of equation (9). *Then $y^{(i)}(t) \to 0$ as $t \to \infty$, $i = 1, 2, \cdots, n-1$.*

Proof. We can assume, without any loss, that $y(t) > 0$ eventually. The case when $y(t) < 0$ can be treated similarly. Let T_1 be sufficiently large so that both $y(t)$ and $y(g(t))$ are positive to the right of T_1. Integrating equation (9) between T_1 and t we have

(10) $$y^{(n-1)}(t) - y^{(n-1)}(T_1) + \int_{T_1}^{t} a^{+}(s)y(g(s))ds - \int_{T_1}^{t} a^{-}(s)y(g(s))ds = \int_{T_1}^{t} f(s)ds$$

from which

(11) $$\begin{aligned} y^{(n-1)}(t) - y^{(n-1)}(T_1) &+ \int_{T_1}^{t} a^{+}(s)y(g(s))ds \\ &\leq \int_{T_1}^{t} a^{-}(s)y(g(s))ds + \int_{T_1}^{t} |f(s)|ds \leq M\int_{T_1}^{t} a^{-}(s)ds + \int_{T_1}^{t} |f(s)|ds \end{aligned}$$

where $y(g(t)) \leq M$ in $[T_1, \infty]$. Now as $t \to \infty$, the right hand side of (11) is finite. We will show that

(12) $$\int_{T_1}^{\infty} a^{+}(s)y(g(s))ds < \infty .$$

To see this if

$$\int_{T_1}^{\infty} a^{+}(s)y(g(s))ds = \infty$$

is true, then $y^{(n-1)}(t) \to -\infty$ as $t \to \infty$ on the left side of (11). But this will force $y(t)$ to be negative which is a contradiction. Hence (12) holds. From inequality (12) we get

(13) $$\liminf_{t\to\infty} y(t) = \liminf_{t\to\infty} y(g(t)) = 0 .$$

Now $y^{(n-1)}(t)$ must be oscillatory because otherwise from (13) we must have $\lim_{t\to\infty} y(t) = 0$ and also $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 1, 2, \cdots, n-2$. Taking T_1 as a large zero of $y^{(n-1)}(t)$, from (11) and (12) we see that $y^{(n-1)}(t) \to 0$ as $t \to \infty$. Now we invoke an extension of Kolmogorov's theorem for bounded derivative—see *Shoenberg* [11].

There exist constants M, N and C such that

(14) $$|y(t)|\leq M \quad \text{on } (0, \infty)$$

(15) $$|y^{(n-1)}(t)|\leq N \quad \text{on } (0, \infty) .$$

These imply

(16) $$|y^{(i)}(t)|\leq CM^{1-i/(n-1)}N^{i/(n-1)} ,$$

where

$$i=1, 2, \cdots, n-2 .$$

Hence if $y^{(n-1)}(t)\to 0$ as $t\to\infty$, so does $y^{(i)}(t)$, $i=1, 2, \cdots, n-2$. This completes the proof of the lemma.

Theorem (2.1). *Suppose the conditions of Lemma* (2.1) *hold. Then bounded nonoscillatory solutions of equation* (9) *approach zero as* $t\to\infty$.

Proof. From the proof of lemma (2.1), we have conclusion (13) namely

(13) $$\liminf_{t\to\infty} y(g(t))=0$$

where $y(t)$ is nonoscillatory, bounded and (without any loss) non-negative solution of equation (9).
Suppose

(17) $$\limsup_{t\to\infty} y(g(t))>r>0 .$$

In view of (12) and (17), there exists a sequence $\{\beta_n\}$, $n\geq 0$ with the following properties—See *Hammett* [6].

(D) $\lim_{n\to\infty}\beta_n=\infty$, $\beta_n\geq T_1$ for all n, T_1 is the same as in lemma (2.1).

(E) For each $n\geq 1$, $y(g(\beta_n))>r$ and there exists a β'_n such that $\beta_{n-1}<\beta'_n<\beta_n$ and $y(g(\beta'_n))<r/2$.

Let α_n be the largest number less than β_n such that $y(g(\alpha_n))=r/2$ and δ_n be the smallest number greater than β_n such that $y(g(\delta_n))=r/2$ for $n\geq 1$. Now in the interval (α_n, β_n), there exists a number ξ_n such that

(18) $$y'(g(\xi_n))g'(\xi_n)=\frac{y(g(\beta_n))-y(g(\alpha_n))}{\beta_n-\alpha_n} > \frac{r}{2(\beta_n-\alpha_n)}$$

by mean value theorem. But by lemma (2.1), $y'(g(\xi_n))\to 0$ as $n\to\infty$ since $g(\xi_n)\to\infty$ as $n\to\infty$. Also $g'(\xi_n)$ remains bounded. Hence it follows from (18)

(19) $$\lim_{n\to\infty}(\delta_n-\alpha_n)=\infty\ .$$

Also because of the way α_n and δ_n were chosen

$$y(g(t))\geq r/2>0$$

on $[\alpha_n, \delta_n]$. Now from (12) in the proof of lemma (2.1), we have

$$\infty>\int_{T_1}^{\infty}a^+(t)y(g(t))dt\geq\sum_{n=1}^{\infty}\int_{\alpha_n}^{\delta_n}a^+(s)y(g(s))ds$$
$$\geq r/2\sum_{n=1}^{\infty}\int_{\alpha_n}^{\delta_n}a^+(s)ds=\infty\ ,$$

a contradiction, since from conditions of lemma (2.1), the last integral diverges as a result of

$$\lim_{n\to\infty}(\delta_n-\alpha_n)=\infty\ .$$

The proof is now complete.

Example 1. To justify this theorem consider the equation

(20) $$y'''(t)+e^{-\pi}y(t-\pi)=2e^{-t}\cos t+e^{-t}\sin t\ .$$

Here all the conditions of theorem (2.1) are satisfied. In fact it has

$$y(t)=e^{-t}(2+\sin t)$$

as a bounded nonoscillatory solution which approaches zero as $t\to\infty$.

Example 2. The equation

(21) $$y''(t)+e^{-\pi}y(t-\pi)=4e^{-t}-2e^{-t}\cos t-e^{-t}\sin t$$

also has

$$y(t)=e^{-t}(2+\sin t)$$

as the solution that satisfies our theorem.

3. On oscillation.

Theorem. (3.1). *Suppose*

$$\int^{\infty}|a(t)|dt<\infty$$

$$\int^{\infty}|f(t)|dt<\infty\ .$$

Let $y(t)$ be a bounded oscillatory solution of equation (9). *Then*

$$y^{(i)}(t) \to 0 \quad as \quad t \to \infty$$

for $i=1, 2, 3, \cdots, n-1$.

Proof. From equation (9) for $t \geq T_1$ we have

$$|y^{(n-1)}(t) - y^{(n-1)}(T_1)| \leq \int_{T_1}^{t} |a(s)||y(g(s))|ds + \int_{T_1}^{t} |f(s)|ds \tag{21}$$

Since $y^{(n-1)}(t)$ must be oscillatory, let $y^{(2n-1)}(T_1)=0$. Here again we see that

$$y^{(n-1)}(t) \to 0 \quad \text{as} \quad t \to \infty ,$$

since the right hand side of (21) can be made as small as we please by choosing T_1 a sufficiently large zero of $y^{(n-1)}(t)$.

From here the proof is the same as that of lemma (2.1) from inequality (14) and (15) onward. The proof is now complete.

In order to justify this theorem consider the following example.

Example 3. The equation

$$y^{(iv)}(t) + e^{-t-\pi} y(t-\pi) = -4e^{-t} \sin t - e^{-2t} \sin t \tag{22}$$

has $y(t)=e^{-t} \sin t$ as the bounded oscillatory solution such that $y'(t)$, $y''(t)$ and $y'''(t)$ all tend to zero as $t \to \infty$. Here $y(t) \to 0$ as $t \to \infty$ also.

However, the next example tells a different story.

Example 4. Consider the equation

$$y'''(t) + e^{-t} y(t) = \frac{3 \sin(\sqrt{t})}{t^2} - \frac{\cos(\sqrt{t})}{t^{3/2}} + \frac{3 \cos(\sqrt{t})}{t^{5/2}} + 8e^{-t} \sin(\sqrt{t}) . \tag{23}$$

which has

$$y(t) = 8 \sin(\sqrt{t})$$

as a bounded oscillatory solution. All the conditions of theorem (3.1) are satisfied. Although $y'(t) \to 0$ and $y''(t) \to 0$ as the theorem claims, yet $y(t)$ itself does not approach any limit.

This example suggests the following theorem.

Theorem (3.2). *Suppose conditions of theorem* (3.1) *hold. Let* $y(t)$ *be a bounded oscillatory solution of equation* (9) *with finitely spaced zeros i.e. to say that if* $\{t_n\}_{n=1}^{\infty}$ *is a sequence of consecutive zeros of* $y(t)$ *then*

$$\limsup_{n \to \infty} (t_{n+1} - t_n) < \infty .$$

Then $y(t)\to 0$ *as* $t\to\infty$.

Proof. From the proof of theorem (3.1), $y'(t)\to 0$ as $t\to\infty$. Now

$$\liminf_{t\to\infty} |y(g(t))|=0 \tag{24}$$

since $y(t)$ is oscillatory. Now suppose

$$\limsup_{t\to\infty} |y(g(t))|>r>0\ . \tag{25}$$

In a manner of the proof of theorem (2.1) we have a sequence $\{j_n\}$ such that

(F_1) $\lim_{n\to\infty} j_n=\infty$, $j_n\geq T_1$ for all n where T_1 is a conveniently large number.

(F_2) For each $n\geq 1$, $|y(g(j_n))|>r$.

(G) For each $n\geq 1$, there exists a number m_n such that $j_{n-1}<m_n<j_n$ and

$$|y(g(m_n))|<r/2\ .$$

Let p_n be the largest number less than j_n such that

$$|y(g(p_n))|=r/2$$

and q_n be the smallest number greater than j_n such that

$$|y(g(q_n))|=r/2$$

for $n\geq 1$. Now in the interval (p_n, j_n), there exists a number r_n such that

$$y'(g(r_n))g'(r_n)=\frac{y(g(j_n))-y(g(p_n))}{j_n-p_n}$$

from where

$$|y''(g(r_n))|g'(r_n)>\frac{||y(g(j_n))|-|y(g(p_n))||}{j_n-p_n} \tag{26}$$

$$>\frac{r}{2(q_n-p_n)}\ .$$

Since $y'(g(r_n))\to 0$ as $r_n\to\infty$ with n and $g'(r_n)$ is bounded we obtain from (26)

$$\lim_{n\to\infty} (q_n-p_n)=\infty\ . \tag{27}$$

Now because of the choice of p_n and q_n, the interval $[p_n, q_n]$ cannot include any zero of $y(g(t))$. Hence p_n and q_n are between two consecutive zeros of $y(g(t))$. But this contradicts the hypothesis of the theorem which essentially states that consecutive zeros of $y(g(t))$ are finitely spaced. This contradiction proves the theorem.

Remark. Coming back to example (4) we find that its solution

$$y=8\sin(\sqrt{t})$$

has zeros at

$$t_n=n^2\pi^2$$

$$t_{n+1}-t_n=[(n+1)^2-n^2]\pi^2\to\infty \quad \text{as } n\to\infty .$$

This justifies our last theorem.

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