ON SOME CLASSES OF OPERATORS ASSOCIATED WITH OPERATOR RADII OF HOLBROOK*

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The purpose of this paper is to establish some properties of ρ-oid operators and to extend the class of operators satisfying the growth condition-(G_1) by considering the growth condition upon the operator radius of the resolvent of an operator.

For an operator (a bounded linear transformation) T on a camplex Hilbert space H, let $\sigma(T)$, $\overline{W(T)}$, $r(T)$ and $|W(T)|$ denote respectively, the spectrum, the closure of the numerical range $W(T)$, the spectral radius and the numerical radius of T. If S is a set of complex numbers then we write ∂S and con(S) for the boundary and the convex hull of S.

Let $C_\rho(\rho>0)$ be the class of operators T on H for which there exists a Hilbert space K containing H as a subspace and a unitary operator U on K satisfying the following relation:

$$T^n x=\rho P U^n x \ (n=1,2,3,\cdots)\,, \quad x\in H\,.$$

The following theorem due to *B. Sz. Nagy* and *Foias* characterizes the class C_ρ.

Theorem A. [8, Theorem I 11.1]. *An operator T belongs to C_ρ if and only if*

$$(\rho-2)\|(I-zT)x\|^2+2Re\langle(I-zT)x, x\rangle\geq 0\,,$$

for all x in H, $|z|\leq 1$.

Recently, *Holbrook* [4] has introduced the following concept of operator radii $\omega_\rho(T)$ $(0<\rho<\infty)$:

$$\omega_\rho(T)=\inf\{u: u>0 \text{ and } u^{-1}T\in C_\rho\}\,.$$

In particular, $\omega_1(T)=\|T\|$ and $\omega_2(T)=|W(T)|$. Furthermore, he has obtained the following characterization of C_ρ in terms of operator radii.

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Theorem B. *$T\in C_\rho$ if and only if $\omega_\rho(T)\leq 1$.*

In Section 1, some properties of ρ-oid operators are obtained. Section 2 is devoted to the study of the classes of operators associated with operator radii which are more general than the class of operators satisfying the growth condition-(G_1).

1. ρ-oid operators.

According to Furuta [2] an operator T is called ρ-oid if $\omega_\rho(T^k)=(\omega_\rho(T))^k$ $(k=1, 2, 3, \cdots)$. Clearly 1-oid and 2-oid operators are normaloid and spectraloid. Also for $\rho\geq 1$, $\omega_\rho(T)=r(T)$ if and only if T is ρ-oid [2].

First we give an alternate but a simple proof of a result due to *Eckstein* [1, Lemma] and use it to obtain some properties of ρ-oid operators.

Theorem 1. *Let z be a non-zero complex number such that $|z|=\omega_\rho(T)$ $(0<\rho<\infty)$ and let $\{x_n\}$ be a bounded sequence of vectors. Then*

$$\|(T-z)x_n\|\to 0 \text{ implies } \|(T^*-z^*)x_n\|\to 0$$

Proof. Setting $T_0=z^{-1}T$, we see that $\omega_\rho(T_0)=1$. Therefore by Theorem B and Theorem A, we get $(\rho-2)\|(I-T_0)x\|^2+2\operatorname{Re}\langle(I-T_0)x, x\rangle\geq 0$, for all x in H, or $\operatorname{Re}\langle((\rho-2)(I-T_0)^*(I-T_0)+2(I-T_0))x, x\rangle\geq 0$.

This inequality shows that the operator $\operatorname{Re} S\geq 0$, where $S=(\rho-2)(I-T_0)(I-T_0)^*+2(I-T_0)$. Since $\|(I-T_0)x_n\|\to 0$, we have $\|Sx_n\|\to 0$. This, in turn gives $\|(\operatorname{Re} S)^{1/2}x_n\|=\langle(\operatorname{Re} S)x_n, x_n\rangle=\operatorname{Re}\langle Sx_n, x_n\rangle\to 0$; thus $\|(\operatorname{Re} S)x_n\|\to 0$. Since $\|Sx_n\|\to 0$, we conclude that $\|S^*x_n\|\to 0$. Consequently, $\|(I-T_0)^*x_n\|\to 0$, or $\|(T^*-z^*)x_n\|\to 0$. This proves the result.

Corollary 1. (*Furuta* [8]). *If $T^2=T$, then T is a projection if either* (i) *T is a ρ-oid, or* (ii) *$\omega_\rho(T)\leq 1$.*

Proof. Since $T^2=T$, $r(T)^2=r(T)$. Therefore either $r(T)=0$ or $r(T)=1$. If $r(T)=0$, then $T=0$. If not, then $Tx\neq 0$ for some $x\in H$ and hence $1\in\sigma(T)$ as $T(Tx)=Tx$, which is a contradiction. So $T=0$, obviously, a projection. Assume then that $r(T)=1$. If (i) holds then $\omega_\rho(T)=1$. Since $T(Tx)=Tx$ for all x in H, an application of Theorem 1 gives $T^*(Tx)=Tx$ for all x in H or $T^*T=T$. This shows that T is a projection. Lastly assume (ii). Then $r(T)\leq\omega_\rho(T)\leq 1=r(T)$ or T is ρ-oid. Hence the result follows from (i).

The following corollary is easy to prove.

Corollary 2. *Every ρ-oid operator on a Hilbert space H with* $\dim H\leq 2$ *is normal.*

Remark 1. The above result is not valid if dim $H>2$. To see this, consider the operator $T=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ on a Hilbert space H with dim $H=3$. Clearly T is normaloid and hence ρ-oid, but it is not normal.

Let $L_\rho(\rho\geq 1)$ be the collection of ρ-oid operators. We know that $L_\rho\subseteq L_{\rho'}$ for $\rho'>\rho$ [2, Theorem 3]. To sharpen this result, we establish.

Theorem 2. *If dim $H>2$, the class $L_\rho(1\leq\rho<\infty)$ increases with ρ that is, $L_\rho\subseteq L_{\rho'}$ and $L_\rho\neq L_{\rho'}$ for $1\leq\rho<\rho'<\infty$.*

Proof. We construct for every ρ and ρ' with $1\leq\rho<\rho'<\infty$ an operator $T_{\rho'}$ in $L_{\rho'}$ such that $T_{\rho'}\notin L_\rho$.

Let M be the two-dimensional subspace of H. Write $H=M\oplus M^\perp$. Let $A=\begin{bmatrix} 0 & \rho' \\ 0 & 0 \end{bmatrix}$ be the nilpotent operator on M and B, the identity operator on $M^\perp$. Let $T_{\rho'}=A\oplus B$. Clearly $r(T_{\rho'})=1$. Since $\|\rho'^{-1}A\|=1$ and $(\rho'^{-1}A)^2=0$, we have by [4, Theorem 4.5] $\omega_{\rho'}(\rho'^{-1}A)=\rho'^{-1}$ or $\omega_{\rho'}(A)=1$. Moreover, as B is the identity operator on $M^\perp$ and $\rho'>1$, we have by [4, Theorem 4.3], $\omega_{\rho'}(B)=1$. An application of Theorem 4.1[5] gives $\omega_{\rho'}(T_{\rho'})=\max\{\omega_{\rho'}(A),\ \omega_{\rho'}(B)\}=\max\{1,1\}=1$. Thus $\omega_{\rho'}(T_{\rho'})=1=r(T_{\rho'})$, showing $T_{\rho'}\in L_{\rho'}$.

Next we claim that $T_{\rho'}\notin L_\rho$. Since $r(T_{\rho'})=1$, it will suffice to show that $\omega_\rho(T_{\rho'})>1$. Now as argued before, we obtain $\omega_\rho(A)=\rho'/\rho$ and so, as $\rho'>\rho$, $\omega_\rho(A)>1$. Moreover, $\omega_\rho(B)=1$ as $\rho\geq 1$. Therefore, again using [5, Theorem 4.1], we get $\omega_\rho(T_{\rho'})>1$. This shows $\omega_\rho(T_{\rho'})>1=r(T_{\rho'})$.

2. Subclasses of convexoid operators.

An operator T is defined to be convexoid if $\operatorname{con}(\sigma(T))=\overline{W(T)}$. If $z\notin\sigma(T)$, then $R(T,z)=(T-z)^{-1}$ exists and is called the resolvent of T. An operator T is said to satisfy the growth condition-(G_1) if $\|R(T,z)\|=1/d(z,\sigma(T))$, for all $z\notin\sigma(T)$, where $d(z,\sigma(T))$ denotes the distance of z from $\sigma(T)$. Clearly, T satisfies the growth condition-(G_1) if and only if $R(T,z)$ is normaloid for all $z\notin\sigma(T)$. For the various properties of such operators, we refer to [6], [10], [11], [12], [13], [14] and [15]. Since $\omega_\rho(T)\leq\|T\|$ for $\rho\geq 1$, it is natural to introduce and study the following generalization of this class.

Let $\rho\geq 1$. Then an operator T is defined to be of class M_ρ if $\omega_\rho(R(T,z))=1/d(z,\sigma(T))$, $z\notin\sigma(T)$. Obviously $T\in M_\rho$ if and only if $R(T,z)$ is ρ-oid for all $z\notin\sigma(T)$. Also for $\rho'>\rho$, $M_{\rho'}\supseteq M_\rho$ and so, in particular M_1 which is nothing but the class of operators satisfying the growth condition-(G_1) is contained in M_ρ

for all $\rho\geq 1$.

Orland [9] established the following remarkable characterization of convexoid operators: An operator T is convexoid if and only if $\|R(T,z)\|\leq 1/d(z,\mathrm{con}(\sigma(T))$ for all $z\notin \mathrm{con}\,(\sigma(T))$. From this it follows immediately that the operators of class M_1 are convexoid. However, since this characterization fails to say whether operators of class M_ρ are convexoid for $\rho\geq 1$, we present the following general criterion of convexoid operators.

Theorem 3. *Let $\rho\geq 1$. If X is a closed convex subset of the complex plane, then $X\supseteq\overline{W(T)}$ if and only if $\omega_\rho(R(T,z))\leq 1/d(z,X)$ for all $z\notin X$. In particular, T is convexoid if and only if $\omega_\rho(R(T,z))\leq 1/d(z,\mathrm{con}\,(\sigma(T)))$ for all $z\notin\mathrm{con}\,(\sigma(T))$.*

Proof. If $X\supseteq\overline{W(T)}$, then as $\|R(T,z)\|\leq 1/d(z,\overline{W(T)})$ for all $z\notin\overline{W(T)}$, we have $\omega_\rho(R(T,z))\leq\|R(T,z)\|\leq 1/d(z,\overline{W(T)})\leq 1/d(z,X)$, for all $z\notin X$.

Conversely, suppose that the resolvent of T satisfies the indicated growth condition. To prove $X\supseteq\overline{W(T)}$, it will suffice to show that every half-plane M containing X also contains $\overline{W(T)}$. By the suitable rotation and translation, we assume $M=\{z:\mathrm{Re}\,z\geq 0\}$. Since $M\supseteq X$, we have for $t>0$,

$$\omega_\rho(R(tT,-1))=t^{-1}\omega_\rho(R(T,-t^{-1}))\leq 1\ .$$

Therefore, by Theorem B, $R(tT,-1)\in C_\rho$. Now applying Theorem A, we get

$$(\rho-2)\|(I-R(tT,-1)x\|^2+2\,\mathrm{Re}\,\langle(I-R(tT,-1))x,x\rangle\geq 0\ ,$$

or

$$(\rho-2)\|tTR(tT,-1)x\|^2+2\,\mathrm{Re}\,\langle tTR(tT,-1)x,x\rangle\geq 0\ ,$$

for all x in H.

Dividing this inequality by t and taking $t\to 0$, we get $\mathrm{Re}\,\langle Tx,x\rangle\geq 0$ for all $x\in H$ and hence $\overline{W(T)}\subseteq X$.

The second assertion follows directly from the first one by taking $X=\mathrm{con}(\sigma(T))$.

Remark 2. The above theorem is a more general form of Theorem 4(2) proved in [7].

As an immediate consequence of Theorem 3, one has

Corollary 3. $\overline{W(T)}=\bigcap_k\{\mathrm{con}\,(X_k)$: *$\omega_\rho(R(T,z))\leq 1/d(z,\mathrm{con}\,(X_k))$ for all $z\notin\mathrm{con}\,(X_k)$, where X_k is a bounded closed set in the complex plane*$\}$.

Corollary 4. *If $R(T,z)$ is convexoid for all $z\notin\sigma(T)$, then T is convexoid.*

Remark 3. It follows from this corollary that each condition in Column A of Corollary 1 established in [13] can be omitted without affecting the conclusion.

In fact, Professor *J.G. Stampfli* provided us with an independent proof of Corollary 4 which led us to consider the more general situation when $R(T,z)$ is ρ-oid for all $z\notin\sigma(T)$. As his proof is interesting in itself, we are quoting here with his kind permission:

Assume $\|T\|=1$. It suffices to show that any support line to the set $\text{con}(\sigma(T))$ is also support line for $\overline{W(T)}$. Let $Z_x=\{x+iy;\ -\infty<y<\infty\}$. We may assume $\text{Re con}(\sigma(T))\leq 0$ and hence Z_0 is a support line for $\text{con}(\sigma(T))$. Then Z_s is a support line for $\text{con}(\sigma(T+s))$ when $s>0$. If $z\in\sigma(T+s)$, then $|z|=\sqrt{s^2+1}=\delta$. Thus $\text{Re}(R(T,-s))\geq s\sigma^{-2}=\alpha$. Since Z_α is a support line for $\text{con}(\sigma(R(T,-s))$, it is a support line for $\overline{W(R(T,-s))}$ and hence $\text{Re}\,z\geq\alpha$ is a spectral set for $R(T,-s)$. Thus the disc $\{z: |z-\alpha^{-1}/2|\leq\alpha^{-1}/2\}$ is a spectral set for $T+s$ and hence $\text{Re}\,\overline{W(T+s)}\leq\alpha^{-1}=s+s^{-1}$. Thus $\text{Re}\,\overline{W(T)}\leq s^{-1}$, and since s is arbitrary, we conclude that $\text{Re}\,\overline{W(T)}\leq 0$. We have just checked the support line Z_0. The argument for other support lines is identical and hence we have shown that $\overline{W(T)}\subseteq\text{con}(\sigma(T))$.

It is well-known that a semi-bare point of the spectrum of an operator of class M_1 turns out to be a normal approximate eigenvalue of that operator (see [16]) (A complex number z is called a normal approximate eigenvalue of T if $\{\{x_n\}: x_n\in H,\ \|x_n\|=1,\ \|(T-z)x_n\|\to 0\}=\{\{x_n\}: x_n\in H,\ \|x_n\|=1,\ \|(T^*-z^*)x_n\|\to 0\}$). In Theorem 4, we extend this result for $T\in M_\rho$.

Theorem 4. *Let $T\in M_\rho$. If z is a semi-bare point of $\sigma(T)$, then it is a normal approximate eigenvalue of T.*

Proof. Since $T-\lambda I\in M_\rho$ whenever $T\in M_\rho$, we can assume $z=0$. Let $z_0\neq 0$ be a complex number such that $\{\alpha: |\alpha-z_0|\leq|z_0|\}\cap\sigma(T)=\{0\}$. Then $d(z_0,\sigma(T))=|z_0|$ and $\omega_\rho(R(T,z_0))=1/|z_0|$. If $S=-z_0R(T,z_0)$, then $\omega_\rho(S)=1$. Since $0\in\partial\sigma(T)$, 0 is an approximate eigenvalue of T. If $\{x_n\}$ is a sequence of unit vectors such that $\|Tx_n\|\to 0$, then $\|Sx_n-x_n\|\to 0$. Therefore, as $\omega_\rho(S)=1$, we conclude by Theorem 1 that $\|S^*x_n-x_n\|\to 0$, or $\|T^*x_n\|\to 0$ as $\|S^*x_n-x_n\|=\|-z^*R(T^*,z^*)x_n-x_n\|=\|R(T^*,z^*)T^*x_n\|$.

Corollary 5. *Let $\dim H<\infty$. Then the classes M_ρ coincide with the class of normal operators.*

As a particular case, the preceding theorem asserts that an isolated point in $\sigma(T)$ is a normal approximate eigenvalue, whenever $T\in M_\rho$. However, one cannot say from this result whether this isolated point is actually a normal eigenvalue of T. For operators of class M_1, the result is well-known (see [12, Theorem C]). Here, we show this property being retained by operators of class M_ρ, when $1\leq\rho\leq 2$.

Theorem 5. *Let $T\in M_\rho(1\leq\rho\leq 2)$. If z_0 is an isolated point in $\sigma(T)$, then it is a normal eigenvalue of T, that is, $N(T-zI)=N(T^*-z^*I)(N(\cdot)=$null space).*

Proof. Assume $z_0=0$. Choose $r>0$ sufficiently small so that 0 is the only point of $\sigma(T)$ contained in the disc $\{z: |z|\leq r\}$ and $d(\{z: |z|=r\}, \sigma(T))=r$. Define

$$P=-(2\pi i)^{-1}\int_{|z|=r} R(T,z)dz$$

Then $P^2=P$. Since, for $\rho\leq 2$, the function $\omega_\rho(\cdot)$ is a norm on the space of operators [5, Theorem 3.2] and $T\in M_\rho(1\leq\rho\leq 2)$, we have

$$\omega_\rho(P)\leq(2\pi)^{-1}\int_{|z|=r}\omega_\rho(R(T,z))dz\leq(2\pi)^{-1}2\pi r r^{-1}=1\,.$$

Then Corollary 1 (ii) assures us that P is a projection. If x is in the range of P, then $\|Tx\|\leq r$. Since r is arbitrary, $Tx=0$. Moreover, as 0 is a semi-bare point of $\sigma(T)$, the desired conclusion follows from Theorem 4.

Corollary 6. *Operators of class $M_\rho(1\leq\rho\leq 2)$ with finite spectra are normal.*

Our next theorem shows the existence of operators in M_ρ with some interesting properties not possessed by all members of M_ρ.

Theorem 6. *There exists a non-singular operator T in M_ρ such that*

(i) *$T^2\notin M_\rho$ for any $\rho\geq 1$,*

and

(ii) *$T^{-1}\notin M_\rho$ for any $\rho\geq 1$.*

Proof. Let $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and N be a normal operator with $\sigma(N)=W(A)$. If $T=A\oplus N$, then as shown in [6, Theorem 1.3], $T\in M_1$, $0\notin\operatorname{con}(\sigma(T)^2)=\operatorname{con}(\sigma(T^2))$ and $0\in W(A^2)\subseteq W(T^2)$. Thus $\overline{W(T^2)}\neq\operatorname{con}\sigma(T^2)$ or T^2 is not convexoid. Consequently, $T\in M_\rho$ for all $\rho\geq 1$ but $T^2\notin M_\rho$ for any $\rho\leq 1$. This proves (i).

To prove (ii) it suffices to exhibit that T^{-1} is not convexoid. Since $W(A)=\{z: |1-z|\leq 1/2\}$, $\{W(A)\}^{-1}=\{z: |4/3-z|\leq 2/3\}$. Now $\sigma(T^{-1})=\sigma(A^{-1})\cup\sigma(N^{-1})=\{1\}\cup$

$\sigma(N^{-1})=\{1\}\cup\sigma(N)^{-1}=\{1\}\cup\{W(A)\}^{-1}$. But $1\in\{W(A)\}^{-1}$. Therefore $\sigma(T^{-1})=\{W(A)\}^{-1}$. Obviously $\text{con}\,\sigma(T^{-1})=\text{con}\,\{W(A)\}^{-1}=\{W(A)\}^{-1}$. Next we see that as $1/2\in W(A^{-1})$, $1/2\in \text{con}\,\{W(A^{-1})\cup\{W(A)\}^{-1}\}=\text{con}\,\{W(A^{-1})\cup W(N^{-1})\}=W(T^{-1})$ and hence $1/2\in \overline{W(T^{-1})}$. However, as $1/2\notin\{W(A)\}^{-1}=\text{con}\,\sigma(T^{-1})$, we arrive at the conclusion that $\text{con}\,\sigma(T^{-1})\neq\overline{W(T^{-1})}$. This proves the desired assertion.

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