ELIMINABILITY OF DESCRIPTIVE DEFINITIONS IN MANY-VALUED LOGICS

By

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(Received October 22, 1975)

1. Introduction.

At various stages in the formal development of a mathematical theory in many-valued logics, new formation rules and postulates are added to a given formal system to obtain another system. In this paper, some theorems in many-valued logics demanded under such circumstances shall be proved generally and syntactically.

2. The formal system of many-valued logics.

We shall introduce the system M-LK formulated by M. Takahashi [1], which generally represents many-valued logics.

2.1. Truth values and primitive symbols.

Let M be an integer ≥ 2. We take the set T of all the truth values;

$$T=\{1, 2, \cdots, M\} .$$

M-LK contains as symbols, denumerably many free variables $a, b, \cdots$, denumerably many bound variables $x, y, \cdots$, function letters of rank k ($k=0, 1, \cdots$), predicate letters of rank k ($k=1, 2, \cdots$), propositional connectives of rank k ($k=0, 1, \cdots$), universal quantifiers $\forall x$, $\forall y, \cdots$, existential quantifiers $\exists x$, $\exists y, \cdots$, parentheses and comma. Specially, function letters of rank 0 denote individual constants and propositional connectives of rank 0 denote propositional constants. Each propositional connective of rank k is associated a truth value function from T^k into T.

To refine symbols of a system M-LK, we must choice and fix the following sets: $\mathfrak{V}$ is an infinite set of free variables, and $\mathfrak{B}$ is an infinite set of bound variables, and $\mathfrak{F}$ is a set of function letters, and $\mathfrak{P}$ is a set of predicate letters, and $\mathfrak{C}$ is a set of propositional connectives. We call the notation $\langle\mathfrak{V}, \mathfrak{B}, \mathfrak{F}, \mathfrak{P}, \mathfrak{C}\rangle$ the frame of a system M-LK.

We assume that the definitions of "term" and "formula" in a system are well-known; their formation rules are applied to the symbols of the system.

2.2. g-matrices and matrices.

When $\Gamma^{(1)}, \cdots, \Gamma^{(M)}$ are sets of formulae, we call the following expression a g-matrix;

$$\begin{pmatrix} \Gamma^{(1)} \\ \vdots \\ \Gamma^{(M)} \end{pmatrix}.$$

Next let

$$K=\begin{pmatrix} \Gamma^{(1)} \\ \vdots \\ \Gamma^{(M)} \end{pmatrix} \text{ and } L=\begin{pmatrix} \Pi^{(1)} \\ \vdots \\ \Pi^{(M)} \end{pmatrix}$$

be g-matrices. We define g-matrix $K \cup L$ to be

$$\begin{pmatrix} \Gamma^{(1)} \cup \Pi^{(1)} \\ \vdots \\ \Gamma^{(M)} \cup \Pi^{(M)} \end{pmatrix}.$$

Moreover, $\{\Gamma\}_{\mu_1,\dots,\mu_k}$ or $\{\Gamma\}_R$ denotes the g-matrix such that $\Gamma^{(\mu_i)}$ is Γ $(i=1, \cdots, k)$ and $\Gamma^{(\nu)}$ is the empty set $(\nu\neq\mu_1, \cdots, \mu_k)$ where Γ is a set of formulae and $R=\{\mu_1, \cdots, \mu_k\}$. Often we write $\hat{\mu}$ as $T-\{\mu\}$.

Specially we call the g-matrix a matrix when all $\Gamma^{(1)}, \cdots, \Gamma^{(M)}$ are finite sequences of formulae. We define $\Gamma \cup \Pi$ to be $\Gamma\Pi$ when Γ, Π are finite sequences of formulae.

2.3. Beginning matrices.

A matrix of the form $\{A\}_T$ or $\{F_0\}_{f0}$ is called a logical matrix where A is a formula and F_0 is a propositional connective of rank 0, a propositional constant, associated a truth value function f_0.

Let S be a set of closed formulae. A matrix of the form $\{A\}_1$ is called a mathematical matrix where $A \in S$. When S is not null, we call our system M-LK with S. But we simply call our system M-LK when there can be no misunderstanding. We often call S the set of axioms of M-LK (with S).

Logical matrices and mathematical matrices are called beginning matrices.

2.4. Inference rules.

We introduce inference rules and K, L, N stand for matrices and $A, B, A_1, A_2, \cdots$ stand for formulae.

Contraction: $\dfrac{K\cup\{A\}_\mu\cup\{A\}_\mu}{K\cup\{A\}_\mu}$ $(\mu\in T)$.

Weakening: $\dfrac{K}{K\cup\{A\}_\mu}$ $(\mu\in T)$.

Exchange: $\dfrac{K\cup\{A\}_\mu\cup L\cup\{B\}_\mu\cup N}{K\cup\{B\}_\mu\cup L\cup\{A\}_\mu\cup N}$ $(\mu\in T)$.

Cut: $\dfrac{K\cup\{A\}_\mu \qquad L\cup\{A\}_\nu}{K\cup L}$ $(\mu, \nu\in T, \mu\neq\nu)$.

Inferences on propositional connectives: Let F be a propositional connective of rank k and f be its truth value function and let $\mu=f(\mu_1, \cdots, \mu_k)$,

$$\frac{K\cup\{A_1\}_{\mu_1},\ K\cup\{A_2\}_{\mu_2},\ \cdots,\ K\cup\{A_k\}_{\mu_k}}{K\cup\{F(A_1, \cdots, A_k)\}_\mu}$$

Inferences on quantifiers:

$$\frac{K\cup\{A(a)\}_{1, 2, \cdots, \mu},\ K\cup\{A(t)\}_\mu}{K\cup\{\forall x A(x)\}_\mu} \quad (\mu\in T)$$

$$\frac{K\cup\{A(a)\}_{\mu,\mu+1,\cdots,M},\ K\cup\{A(t)\}_\mu}{K\cup\{\exists x A(x)\}_\mu} \quad (\mu\in T)\ ,$$

where the free variable a does not occur in the conclusion and $A(t)$ means the formula obtained from the formula $A(a)$ by substituting the term t for a in both inferences. Moreover, a is called the eigen-variable of these inferences.

A matrix is called provable, if it is obtained from beginning matrices by a finite number of applications of inference rules given above.

2.5. Some propositions of M-LK'.

To prove our theorems we need some preliminaries. The following propositions hold and their proofs should be referred to § 4 in [1].

2.5.1. *Let $R_j\subseteq T(j=1, 2, \cdots, r)$. If for each j*

$$K\cup\{A\}_{R_j}$$

is provable, then $K\cup\{A\}_R$

is also provable where R denotes $\bigcap_{j=1}^{r} R_j$.

2.5.2. *For each $\mu\in T$,*

$$\{\exists x A(x)\}_{1,\cdots,\mu-1}\cup\{A(t)\}_{\mu,\cdots,M}\ ,$$

$$\{\forall x A(x)\}_{\mu+1,\cdots,M}\cup\{A(t)\}_{1,\cdots,\mu}\ ,$$

are provable.

2.5.3. *If a matrix is provable, then it is provable without cut.*

2.5.4. *Let G be a generalized propositional connective defined by the propositional connectives in M-LK with its truth value function g of rank n, and g be composed by the truth value functions in M-LK. For any* $\mu_1, \cdots, \mu_n \in T$, *if*

$$K \cup \{A_1\}_{\mu_1}, \ldots, K \cup \{A_n\}_{\mu_n},$$

are provable, then $K \cup \{G(A_1, \cdots, A_n)\}_{g(\mu_1, \ldots, \mu_n)}$ *is provable.*

3. Functionally completeness.

A functionally complete system denotes that any function from T^k into T ($k=1, 2, \cdots$) can be represented by the truth value functions in the system (see [2]). We hereafter treat of the functionally complete systems and suppose that a system M-LK can be regarded as the system containing all functions from T^k into T ($k=1, 2, \cdots$) as the truth value functions. As the following connectives are often used in this paper, we are concerned about some properties of them (refer to § 3 in [3]).

3.1. The connectives $\vee^k$ of rank k ($k \geq 2$).

The truth value function of this connective corresponds an element of T^k to the minimum of all factors of it.

We write $\vee$ when there can be no misunderstanding.

3.1.1. *If a matrix* $K \cup \{A_i\}_\mu$ *is provable for some* i ($1 \leq i \leq k$), *then the matrix* $K \cup \{\vee^k(A_1, \cdots, A_k)\}_{1, \ldots, \mu}$ *is provable.*

3.1.2. *If a matrix* $K \cup \{A_i\}_{\mu, \ldots, M}$ *is provable for each* i ($1 \leq i \leq k$), *then the matrix* $K \cup \{\vee^k(A_1, \ldots, A_k)\}_{\mu, \ldots, M}$ *is provable.*

3.2. The connective $\wedge^k$ of rank k ($k \geq 2$)'

The truth value function of this connective corresponds an element of T^k to the maximum of all factors of it.

We write $\wedge$ when there can be no misunderstanding.

3.2.1. *If a matrix* $K \cup \{A_i\}_\mu$ *is provable for some* i ($1 \leq i \leq k$), *then the matrix* $K \cup \{\wedge_k(A_1, \cdots, A_1)\}_{\mu, \ldots, M}$ *is provable.*

3.2.2. *If a matrix* $K \cup \{A_i\}_{1, \ldots, \mu}$ *is provable for each* i ($1 \leq i \leq k$), *then the matrix* $K \cup \{\wedge^k(A_1, \cdots, A_k)\}_{1, \ldots, \mu}$ *is provable.*

3.3. The connective $G^{R_1,R_2}_{\mu,\nu}$ of rank n.

The truth value function of this connective corresponds an element of R_1 to μ and an element of R_2 to ν where $R_1 \cup R_2 = T^n$ and $R_1 \cap R_2 = \phi$. Instead of $G^{\{\mu\},\hat{\mu}}_{1,M}$, we write G_μ.

3.3.1. *A matrix* $K \cup \{A\}_\mu$ *is provable, if and only if a matrix* $K \cup \{G_\mu(A)\}_1$ *is provable.*

3.4. The connective $\equiv$ of rank 2.

A formula $A \equiv B$ denotes $\bigwedge_{\mu \in T} (\bigvee_{\nu \in \hat{\mu}} G_\nu(A) \vee G_\mu(B))$.

3.4.1. *If two matrices* $\{A \equiv B\}_1$ *and* K *are provable, then the matrix* K' *obtained from* K *by replacing some occurrences of* A *in* K *by* B *is also provable.*

4. The extended system MLE.

Hereafter, we assume that the predicate letter $=$ of rank 2 is contained in $\mathfrak{P}$ in the frame of our system M-LK. Suppose not, we can add the new predicate letter $=$ of rank 2 to $\mathfrak{P}$.

In this section, we shall introduce two formal systems of many-valued logics with equality by two methods. The one, which is called **MLE**, is obtained from M-LK by adding the inference rules of equality. Another, which is called **MLE′**, is obtained from M-LK by joining the set Γ_e of equality axioms to the set S of M-LK which composes the mathematical matrices (refer to 2.3) Next we shall show the relations among **MLE** and **MLE′** and M-LK.

First of all, we introduce **MLE** obtained from M-LK by adding the following inference rules of equality:

$$\frac{K \cup \{t=t\}_1}{K} \qquad \frac{K \cup \{P(s)\}_\mu \qquad K \cup \{P(t)\}_{\hat{\mu}}}{K \cup \{s=t\}_M} \qquad (\mu \in T)$$

where $P(a)$ expresses a prime formula and s, t express terms.

Then we can prove the cut-elimination theorem for the system **MLE** and the following proposition.

4.1. *The following matrices are provable in* **MLE** *where* $P(a)$ *denotes a prime formula and* $f(a)$, s, t *denote terms;*

1) $\{t=t\}_1$, 2) $\{s=t\}_M \cup \{s=t\}_1$,

3) $\{t=s\}_M \cup \{s=t\}_1$, 4) $\{s=t\}_M \cup \{t=r\}_M \cup \{s=r\}_1$,

5) $\{s=t\}_M \cup \{f(s)=f(t)\}_1$, 6) $\{s=t\}_M \cup \{P(s)\}_{\hat{\mu}} \cup \{P(t)\}_\mu \quad (\mu \in T)$.

Let Γ_e be the set of the following closed formulae;

1) $\forall x(x=x)$, $\forall x\forall y(G_M(x=y)\vee y=x)$, $\forall x\forall y\forall z(G_M(x=y)\vee G_M(y=z)\vee x=z)$,

2) $\forall x_1\cdots\forall x_n\forall y_1\cdots\forall y_n(\bigvee_{1\le k\le n}(G_M(x_k=y_k))\vee(f(x_1,\cdots,x_n)=f(y_1,\cdots,y_n)))$,

3) $\forall x_1\cdots\forall x_n\forall y_1\cdots\forall y_n(\bigvee_{1\le k\le n}(G_M(x_k=y_k))\vee(P(x_1,\cdots,x_n)\equiv P(y_1,\cdots,y_n)))$,

where $f(*_1,\cdots,*_n)$ and $P(*_1,\cdots,*_n)$ express a function letter of rank n and a predicate letter of rank n respectively ($n=1,2,\cdots$).

As usual, $\{\Gamma\}_{\hat{1}}\cup K$ is said to be provable in a system if there is a finite subset Γ' of Γ such that the matrix $\{\Gamma'\}_{\hat{1}}\cup K$ is provable in the system where Γ is the set of formulae.

4.2. *If A is a formula contained in Γ_e, then the matrix $\{A\}_1$ is provable in* **MLE**.

4.3. *A matrix K is provable in* **MLE** *if and only if $\{\Gamma_e\}_{\hat{1}}\cup K$ is provable in M-LK.*

Proof. Suppose that K is provable in **MLE**. By the cut-elimination theorem for **MLE**, there exists a cut-free proof of K. We prove our assertion by induction on the number of inference rules used in the proof. Therefore we consider only the following case;

$$\frac{L\cup\{P(s)\}_\mu \qquad L\cup\{P(t)\}_{\hat{\mu}}}{L\cup\{s=t\}_M} \quad \text{(for any } \mu) .$$

Then two g-matrices $\{\Gamma_e\}_{\hat{1}}\cup L\cup\{P(s)\}_\mu$ and $\{\Gamma_e\}_{\hat{1}}\cup L\cup\{P(t)\}_{\hat{\mu}}$ are provable in M-LK by the induction hypotheses. On the other hand, $\{\Gamma_e\}_{\hat{1}}\cup\{s=t\}_M\cup\{P(s)\equiv P(t)\}_1$ is provable in M-LK. Hence $\{\Gamma_e\}_{\hat{1}}\cup\{s=t\}_M\cup L\cup\{P(t)\}_\mu$ is provable in M-LK by applying the Proposition 3.4.1 to $\{\Gamma_e\}_{\hat{1}}\cup L\cup\{P(s)\}_\mu$. Therefore $\{\Gamma_e\}_{\hat{1}}\cup\{s=t\}_M\cup L$ is provable in M-LK by applying the Proposition 2.5.1 to two g-matrices $\{\Gamma_e\}_{\hat{1}}\cup L\cup\{P(t)\}_{\hat{\mu}}$ and $\{\Gamma_e\}_{\hat{1}}\cup\{s=t\}_M\cup L\cup\{P(t)\}_\mu$.

4.4. *Each matrix K, which occurs in a cut-free proof of* **MLE** *and which contains no predicate letter $=$ except in prime formulae, satisfies at least one of the following conditions;*

1) *a matrix of the form $\{s=t\}_M$, where s and t are distinct terms, is contained in K,*

2) *a matrix of the form $\{t=t\}_1$ is contained in K,*

3) *the matrix obtained by deleting all formulae containing the predicate letter $=$ from K is provable in M-LK.*

Proof. This proposition is proved by the induction on the number of inference rules used in the proof. As a formula containing $=$ is not sub-formula of a logical inference rule by the premises, we consider only the following case;

$$\frac{L\cup\{P(s)\}_{\mu} \qquad L\cup\{P(t)\}_{\hat{\mu}}}{L\cup\{s=t\}_{M}} \quad (\text{for any } \mu\in T)\ .$$

The lower matrix satisfies the condition 1) when s and t are distinct terms. Therefore we may consider the following case;

$$\frac{L\cup\{P(t)\}_{\mu} \qquad L\cup\{P(t)\}_{\hat{\mu}}}{L\cup\{t=t\}_{M}} \quad (\text{for any } \mu\in T)\ .$$

In this case, we can easily prove our assertion by considering not only the conditions satisfied by the upper matrices but the value of μ and the form of $P(t)$. For example, the lower matrix satisfies condition 3) when the right upper matrix satisfies the condition 3) and the form of $P(t)$ is $u=v$ and μ is M.

5. Some propositions of MLE (MLE′)

First of all, we need the following preliminaries to state some propositions of **MLE** (**MLE′**).

Now let $B(a_1, \cdots, a_n)$ be a formula and let $x_1, \cdots, x_n$ be distinct bound variables not occurring in $B(a_1, \cdots, a_n)$. We introduce the expression $\lambda x_1 \cdots x_n B(x_1, \cdots, x_n)$, which is called a variety of rank n. If $t_1, \cdots, t_n$ are terms and V denotes a variety $\lambda x_1 \cdots x_n B(x_1, \cdots, x_n)$, then $V(t_1, \cdots, t_n)$ means the formula $B(t_1, \cdots, t_n)$.

Let h be a function letter of rank p, and H be a predicate letter of rank $p+1$. We define the operation "$*$" (relative to h and H) for terms and formulae as follows:

1) a^* is $\lambda x(x=a)$ where a is a free variable.

2) $h(t_1, \cdots, t_p)^*$ is $\lambda y \forall x_1 \cdots \forall x_p(\bigvee_{1\leq j\leq p} G_M(t_j(x_j)) \vee H(x_1, \cdots, x_p, y))$.

3) $f(t_1, \cdots, t_n)^*$ is $\lambda y \forall x_1 \cdots \forall x_n(\bigvee_{1\leq j\leq n} G_M(t_j(x_j)) \vee y=f(x_1, \cdots, x_n))$ where f is any function letter of rank n other than h.

4) $P(t_1, \cdots, t_n)^*$ is $\forall x_1 \cdots \forall x_n(\bigvee_{1\leq j\leq n} G_M(t_j(x_j)) \vee P(x_1, \cdots, x_n))$ where P is any prodicate letter of rank n.

5) F_0^* where F_0 is F_0 is any propositional connective of rank 0.

6) $F(A_1, \cdots, A_n)^*$ is $F(A_1^*, \cdots, A_n^*)$ where F is any propositional connective of rank n $(n=1, 2, \cdots)$.

7) $(\forall x B(x))^*$ and $(\exists x B(x))^*$ are $\forall x B^*(x)$ and $\exists x B^*(x)$ respectively where

$B^*(a)$ means $B(a)^*$.

Specially, we omit the useless symbols in order to make the expressions formulae when rank p or rank n is 0 in 2) or 3) of above-mentioned definition.

K^* often denotes the g-matrix obtained from a g-matrix K by operating $*$ on all formulae in K.

Furthermore, we define the expression E^0 as the result of substitution of $\bar{h}$ for H throughout an expression E where $\bar{h}$ is an abbreviation of the expression $\lambda x_1 \cdots x_p y(y=h(x_1, \cdots, x_p))$. We simply write $E^{\#}$ in place of $(E^*)^0$.

It is clear from the definitions that the following propositions hold.

5.1. *Let E be a term or a formula or a g-matrix.*

1) *A free variable occurs in E^* iff it occurs in E.*

2) *A function letter occurs in E^* iff it differs from h and occurs in E.*

3) *For any predicate letter other than H, $=$ it occurs in E^* iff it occurs in E.*

4) *H occurs in E^* iff either h or H occurs in E.*

5) *If none of h, H occurs in E, then E^* coincides with $E^{\#}$.*

$\exists! x A(x)$ denotes $\exists x A(x) \wedge \forall y \forall z (G_M(A(y)) \vee G_M(A(z)) \vee y=z)$, and H' denotes $\forall x_1 \cdots \forall x_p (\exists! x H(x_1, \cdots, x_p, x))$ and H'' denotes $\forall x_1 \cdots \forall x_p H(x_1, \cdots, x_p, h(x_1, \cdots, x_p))$.

Hereafter we show some propositions of **MLE** (**MLE′**) which can be transformed to the propositions of M-LK by the proposition 4.3. They are proved by induction or by applying the propositions in § 2 and § 3.

5.2. *Let t be a term and let A be a formula not containing H. Then the following matrices are provable in* **MLE** *and there exists the proof of each of them such that it has no expression containing H:*

1) $\{t^{\#}(a) \equiv t=a\}_1$. 2) $\{A^{\#} \equiv A\}_1$.

5.3. *Let a term t and a formula A not to contain h. Then the following matrices are provable in* **MLE** *and there exists the proof of each of them such that it has no expression containing h:*

1) $\{H'\}_{\hat{1}} \cup \{t^*(a) \equiv t=a\}_1$. 2) $\{H'\}_{\hat{1}} \cup \{A^* \equiv A\}_1$.

5.4. *Let t, $T(a)$ be terms and let $A, B, A(a)$ be formulae. Then the following matrices are provable in* **MLE**:

1) $\{H\}_{\hat{1}} \cup \{(H')^*\}_1$. 2) $\{H'\}_{\hat{1}} \cup \{(H'')^*\}_1$.

3) $\{H'\}_{\hat{1}} \cup \{B^*\}_1$ (*where* $B \in \Gamma_e$). 4) $\{H'\}_{\hat{1}} \cup \{\exists! x t^*(x)\}_1$.

5) $\{H'\}_{\hat{1}} \cup \{t^*(a)\}_M \cup \{T(t)^*(b) \equiv T(a)^*(b)\}_1$.

6) $\{H'\}_{\hat{1}} \cup \{t^*(a)\}_M \cup \{A(t)^* \equiv A(a)^*\}_1$.

7) $\{H', H''\}_{\hat{1}} \cup \{H(a, b) \equiv b = h(a)\}_1$.

8) $\{H', H''\}_{\hat{1}} \cup \{t^*(a) \equiv t = a\}_1$. 9) $\{H', H''\}_{\hat{1}} \cup \{A^* \equiv A\}_1$.

5.5. *If a matrix K is provable in M-LK, then the matrix* $\{H'\}_{\hat{1}} \cup K^*$ *is provable in* **MLE.**

6. Eliminability of descriptive definitions.

In this section, we shall generally discuss the eliminability of descriptive definitions in many-valued logics (refer to [4]).

6.1. Elimination relations.

At various stages in the formal development of a mathematical theory in many-valued logics, we add new formation rules introducing new formal symbols or notations and the new postulates providing for their use deductively to a given formal system S_1 in order to obtain another system S_2. The notions of "a beginning matrix", "an inference rule" and "provability" etc. are closely related to the symbols of a system: If a new symbol is added to a system, new beginning matrices and inference rules can be added to the system. Then the set of formulae (provable matrices) of S_1 becomes the subset of those of S_2.

Under such circumstances, we say that the new notations or symbols (with their postulates) are *eliminable* (from S_2 in S_1) if there is an effective process $+$ by which the formula E^+ (the matrix K^+) of S_1 can be found for any formula E (any matrix K) of S_2:

(I) If E is a formula of S_1, then E^+ is E.

(II) A matrix $\{E^+ \equiv E\}_1$ is provable in S_2.

(III) If a matrix K is provable in S_2, then K^+ is provable in S_1.

We call (I)-(III) *the elimination relations*. When the elimination relations hold, then furthermore:

(IV) A matrix K is provable in S_2 iff the matrix K^+ is provable in S_1.

(V) For any matrix L of S_1, L is provable in S_2 iff L is provable in S_1.

Often, we may consider S_1, S_2 as the systems of M-LK without the mathematical axioms because of the following proposition.

6.1.1. *A matrix K is provable in M-LK with S if and only if the g-matrix* $\{S\}_{\hat{1}} \cup K$ *is provable in M-LK without S.*

6.2. Eliminability of explicit definitions.

Let S_1 be a system of M-LK and let S_2 be the system of M-LK obtained from S_1 by adding the new predicate letter P of rank n and the new postulate (the mathematical matrix) of the form $\{\forall x_1 \cdots \forall x_n (P(x_1, \cdots, x_n) \equiv A(x_1, \cdots, x_n))\}_1$

where $\forall x_1 \cdots \forall x_n A(x_1, \cdots, x_n)$ be a closed formula of S_1. Then we can obtain the formula B^+ (the matrix K^+) of S_1 from a formula B (a matrix K) of S_2 by substituting $\lambda x_1 \cdots x_n A(x_1, \cdots, x_n)$ to P.

Example. Let S_1 be a system of **MLE** with a function letter h of rank p and let S_2 be the system of **MLE** from S_1 by adding the new predicate letter H of rank $p+1$ and the new postulate (the mathematical matrix) of the form $\{\forall x_1 \cdots \forall x_p \forall x(H(x_1, \cdots, x_p) \equiv h(x_1, \cdots, x_p)=x)\}_1$. Then the predicate letter H is eliminable from S_2 in S_1 by the effective process o in §5.

6.3. Eliminability of descriptive definitions.

Let S_3 be a system of **MLE** with the provable matrix $\{H'\}_1$ of the from $\{\forall x_1 \cdots \forall x_p \exists ! x H(x_1, \cdots, x_p, x)\}_1$ and let S_4 be the system of M-LK from S_3 by adding a new function letter h of rank p and the new postulate (the mathematical matrix) $\{H''\}_1$ of the form

$$\{\forall x_1, \cdots, \forall x_p H(x_1, \cdots, x_p h(x_1, \cdots, x_p))\}_1 .$$

We define $+$ for a formula A of S_3 or S_4:

1) If A does not contain h, then A^+ is A.

2) If A contains h, then A^+ is A^*.

Theorem 1. *The new function letter h with its postulate $\{H''\}_1$ is eliminable from S_4 in S_3 by the effective process $+$.*

Proof. We can prove by the Propositions 5.3, 5.4 and 5.5.

6.4. Replaceability of undefined functions by predicates.

We show the application of our arguments in 6.2 and 6.3.

For the above-mentioned systems S_2, S_4, we suppose that the frame (and the original set of mathematical matrices) of S_2 is same to one of S_4. The new postulate of the form $\{\forall x_1 \cdots \forall x_p \forall x(H(x_1, \cdots, x_p, x) \equiv h(x_1, \cdots, x_p)=x)\}_1$ of S_2 can be replaced in S_4 by the pair of the postulates $\{H'\}_1$ and $\{H''\}_1$ without changing the provability relationship. So we can equate S_2 with S_4.

Futhermore we consider the matrix $\{H'\}_1$ as the postulate of the predicate letter H in S_3.

Then the following proposition and Theorem 2 hold by passing through S_2 (or S_4) where o and $+$ are ones mentioned in 6.2 and 6.3 respectively.

6.4.1. *Let A (B) be a formula of S_1 (S_3) and let K (L) be a matrix of S_1 (S_3).*

(1) *$\{A \equiv A^{+o}\}_1$ is provable in S_1.*

(2) $\{B \equiv B^{0+}\}_1$ *is provable in* S_3.

(3) *If* K *is provable in* S_1, *then* K^+ *is provable in* S_3.

(4) *If* L *is provable in* S_3, *then* L^0 *is provable in* S_1.

Theorem 2. *Let* K (L) *be a matrix of* S_1 (S_3).

(1) K *is provable in* S_1 *if and only if* K^+ *is provable in* S_3.

(2) L *is provable in* S_3 *if and only if* L^0 *is provable in* S_1.

Note: In § 4-§ 6, we considered the predicate letter $=$ of rank 2, whose values are $1, M$, as equality symbol. Futhermore we may consider the predicate letter $|=|$ of rank 2, whose values are μ and ν where $\mu \neq \nu$, as generalized equality symbol. But we can reduce a formal system with $|=|$ to the suitable system with $=$ by applying the method in 6.2: $\forall x \forall y(x=y \equiv G_1^{\{\mu\}}{}^{\hat{}}{}_{M}^{\mu}(x|=|y))$ and $\forall x \forall y(x|=|y \equiv G_{\mu,\nu}^{\{1\}}{}^{\hat{1}}(x=y))$ are explicit definitions of $=$ and $|=|$, respectively. Hence we can consider our arguments in § 4-§ 6 as the general ones on equality with two values.

7. The applications.

In this section, we shall extend the interpolation theorem and Beth's theorem in many-valued logics with functionally completeness by applying the Theorem 2.

The notations should be referred to [3].

7.1. The extended interpolation theorem.

We prove an extension of the interpolation theorem which takes function letters into account (refer to [5]).

THE EXTENDED INTERPOLATION THEOREM. *Let* (K_1/K_2) *be a partition of a matrix* K *and let* $\mu, \nu \in T$. *If the matrix* K *is provable in a system* S_0 *of* M-*LK, then there exists the interpolation formula* C *for the ordered pairs* (K_1/K_2) *and* (μ, ν) *satisfying the following conditions*:

(1) *Both matrices* $K_1 \cup \{C\}_\mu$ *and* $K_2 \cup \{C\}_\nu$ *are provable in* S_0.

(2) *Each predicate letter in* C *occurs in* K_1 *and in* K_2.

(3) *Each free variable in* C *occurs in* K_1 *and in* K_2.

(4) *Each function letter in* C *occurs in* K_1 *and in* K_2.

Proof. We may consider systems of M-LK without the mathematical axioms by 6.1.1 in order to refine the postulates used in the proofs. In [3], we proved the original theorem which has the conditions (1)-(3). So there exists the interpolation formula C_0 in S_0 satisfying the conditions (1)-(3). Futhermore we may specially suppose that the original formula C_0 has only one function letter h of rank p as the one which does not satisfy the condition (4).

We define the systems S_1, S_3 in order to apply the Theorem 2; S_1 is the system with equality obtained from S_0 by adding equality and S_3 is the system with equality obtained from S_1 by omitting h and supplying instead the new predicate letter H of rank $p+1$ with the postulate H'.

Under these notations, we prove this theorem by considering the following three cases.

Case 1. The function letter h occurs neither in K_1 nor in K_2.

"a semiterm" is the result from a term by substituting bound variables for some occurrences of free variables. We call such an expression as $h(t_1, \cdots, t_p)$ h-semiterm where $t_1, \cdots, t_p$ are semiterms.

Consider the proofs of $K_1 \cup \{C_0\}_\mu$ and $K_2 \cup \{C_0\}_\nu$. We substitute the new free variable a for the h-semiterm, which there is no h-semiterm h' containing as the sub-semiterm of h', in each proof. The result are proofs.

So we see that $K_1 \cup \{\forall x C_1\}_\mu$ and $K_2 \cup \{\forall x C_1\}_\nu$ are provable in S_0 where C_1 is obtained from C_0 by the above-mentioned way of substitutions and $\forall x C_1$ is obtained from C_1 by binding the new free variable a.

Hence we see that $\forall x C_1$ satisfies the conditions (1)-(4).

Case 2. The function letter h occurs in K_1 but not in K_2.

By the condition (1) and the Theorem 2, $\{\Gamma_e, H'\}_1^\wedge \cup K_2^+ \cup \{C_0^+\}_\nu$ is provable in S_3; $\{\Gamma_e, H'\}_1^\wedge \cup K_2 \cup \{C_0^*\}_\nu$ is provable in S_3 because of the definition of $+$.

In the system S_3, there exists the interpolation formula C_1 by applying the original theorem for $(\{\Gamma_e, H''\}_1^\wedge \cup \{C_0^*\}_\nu / K_2)$ and (μ, ν). Hence the formula C_1 in S_3 satisfies the conditions (1)′-(3)′:

(1)′ $\{\Gamma_e, H'\}_1^\wedge \cup \{C^*\}_\nu \cup \{C_1\}_\mu$ and $K_2 \cup \{C_1\}_\nu$ are provable in S_3.

(2)′ Each predicate letter in C_1 occurs in $\{\Gamma_e, H'\}_1^\wedge \cup \{C_0^*\}_\nu$ and in K_2.

(3)′ Each free variable in C_1 occurs in $\{\Gamma_e, H'\}_1^\wedge \cup \{C_0^*\}_\nu$ and in K_2.

So each predicate letter (free variable) in C_1 occurs in C_0 because any predicate letter in Γe occurs in C_0^* and the predicate letters $=$, H don't occurs in K_2. Hence C_1 is the formula in S_1 and S_0. The provable matrices $\{C_0\}_\nu \cup \{C_1\}_\mu$ and $K_2 \cup \{C_1\}_\nu$ in S_0 are obtained from (1)′ by the Theorem 2 and 4.4. Futhermore $K_1 \cup \{C_1\}_\mu$ is provable in S_0 by applying cut to $\{C\}_\nu \cup \{C_1\}_\mu$ and $K_1 \cup \{C_0\}_\mu$.

We can regard C_1 as the required interpolation formula.

Case 3. The function letter h occurs in K_2 but not in K_1. We can take the required formula by the similar way in Case 2.

7.2. The extended Beth's theorem.

We proved the extended interpolation theorem in 7.1. Hence we can similarly prove the following theorem by the way used in § 5 in [3].

The extended Beth's theorem. *Let the following conditions be satisfied:*

(1) *P, Q are predicate letters of rank n.*

(2) *Q does not occur in a matrix $K(P)$.*

(3) *All formulae in $K(P)$ are closed formulae.*

(4) $K(P) \cup K(Q) \cup \{\forall x_1 \cdots \forall x_n (P(x_1, \cdots, x_n) \equiv Q(x_1, \cdots, x_n))\}_1$ *is provable where* $K(Q)$ *is the matrix obtained from* $K(P)$ *by substituting Q to P.*

Then there exists the formula $C(a_1, \cdots, a_n)$ *satisfying the following conditions:*

a) $K(P) \cup \{\forall x_1 \cdots \forall x_n (P(x_1, \cdots, x_n) \equiv C(x_1, \cdots, x_n))\}_1$ *is provable.*

b) *Each predicate (function) letter in* $C(a_1, \cdots, a_n)$ *occurs in* $K(P)$.

c) *The predicate letters P, Q are not contained in* $C(a_1, \cdots, a_n)$.

d) $\forall x_1 \cdots \forall x_n C(x_1, \cdots, x_n)$ *is the closed formula.*

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