ON OPERATOR COMMUTATORS

By

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Throughout this note H will denote a separable (complex) Hilbert space and D_T, $W_e(T)$, sp(T) and $\sigma_e(\hat{T})$ will stand for the domain, the essential numerical range, the spectrum and the left essential spectrum of an operator T on H. X_H will denote the collection of self-adjoint commutators on H i.e. of operators of the form $AB-BA$, where A and B are bounded operators on H and B self-adjoint [1]. For other notations and terminologies we refer to [4].

Putnam [6] has proved that if A and B are bounded operators on H, B seminormal, then their commutator

$$C=AB-BA$$

satisfies

$$\operatorname*{Inf}_{||x||=1} |(Cx, x)|=0 .$$

In [2] and [3] we obtained generalizations of Putnam's result to operators of the form A_1B-BA_2, where A_1 and A_2 are operators on H satisfying certain nice conditions. One of the objects of the present note is to prove the following:

Theorem 1. *Let A_1 and A_2 be bounded operators and B an unbounded self-adjoint operator on H satisfying the following conditions*:

(i) *A_1-A_2 is a compact operator,*

(ii) *$A_2(D_B)\subset D_B$.*

(iii) *If $(B+D-\alpha)^{-1}$ exists for some α and for some bounded D then* $0\notin\sigma_l[(B+D-\alpha)^{-1}]^\wedge$.

If C is any operator such that

$$A_1B-BA_2\subset C ,$$

then

$$\operatorname*{Inf}_{\substack{||x||=1\\ x\in D_C}} |(Cx, x)|=0 .$$

Proof. *Case I*: Let sp(B) be a proper subset of the real line. Choose $\alpha\notin$

sp(B). Then $(B-\alpha)^{-1}$ is a self-adjoint bounded operator on H. Now if $x\in D_B$, we have

$$(A_1B-BA_2)x=Cx\ ,$$

$$\text{i.e. } A_1(B-\alpha)x-(B-\alpha)A_2x+\alpha(A_1-A_2)x=Cx\ ,$$

and hence

$$(B-\alpha)^{-1}A_1(B-\alpha)x-A_2x+\alpha(B-\alpha)^{-1}(A_2-A_1)x=(B-\alpha)^{-1}Cx\ ,$$

Let y be a unit vector. Put $x=(B-\alpha)^{-1}y$. Then $x\in D_B$. Hence we get

$$(B-\alpha)^{-1}A_1(B-\alpha)(B-\alpha)^{-1}y-A_2(B-\alpha)^{-1}y+\alpha(B-\alpha)^{-1}(A_2-A_1)(B-\alpha)^{-1}y$$
$$=(B-\alpha)^{-1}C(B-\alpha)^{-1}y\ ,$$

which implies

$$(B-\alpha)^{-1}A_1y-A_1(B-\alpha)^{-1}y+(A_1-A_2)(B-\alpha)^{-1}y+\alpha(B-\alpha)^{-1}(A_2-A_1)(B-\alpha)^{-1}y$$
$$=(B-\alpha)^{-1}C(B-\alpha)^{-1}y;$$

or,

$$((B-\alpha)^{-1}A_1-A_1(B-\alpha)^{-1})y+((A_1-A_2)(B-\alpha)^{-1}+\alpha(B-\alpha)^{-1}(A_2-A_1)(B-\alpha)^{-1})y$$
$$=(B-\alpha)^{-1}C(B-\alpha)^{-1}y\ .$$

Now $(B-\alpha)^{-1}A_1-A_1(B-\alpha)^{-1}\in X_H$. Also $(A_1-A_2)(B-\alpha)^{-1}+\alpha(B-\alpha)^{-1}(A_2-A_1)(B-\alpha)^{-1}$ is a compact operator by hypothesis. Hence $(B-\alpha)^{-1}C(B-\alpha)^{-1}\in X_H$ [1, Cor 6.9]. Therefore by Theorem 6.13 of *Anderson* [1], there exists an infinite orthonormal sequence $\langle e_n\rangle$ of vectors such that

$$((B-\alpha)^{-1}C(B-\alpha)^{-1}e_n,e_n)=(C(B-\alpha)^{-1}e_n,\ (B-\alpha)^{-1}e_n)\to 0 \quad \text{as } n\to\infty\ .$$

Suppose that

$$\operatorname*{Inf}_{\substack{\|x\|=1\\ x\in D_C}} |(Cx,\ x)|=K\neq 0\ .$$

Then

$$K\|x\|^2\leq|(Cx,x)|\ ,\quad \forall x\in D_C\ .$$

Now as

$$(B-\alpha)^{-1}e_n\in D_B\subset D_C\ ,$$
$$K\|(B-\alpha)^{-1}e_n\|^2\leq|(C(B-\alpha)^{-1}e_n,\ (B-\alpha)^{-1}e_n)|\to 0 \quad \text{as } n\to\infty\ .$$

Therefore $(B-\alpha)^{-1}e_n\to 0$, which implies $0\in\sigma_l[(B-\alpha)^{-1}]^\wedge$, a contradiction. Hence

$$\operatorname*{Inf}_{\substack{\|x\|=1\\ x\in D_C}} |(Cx,\ x)|=0\ .$$

Case II Let sp(B) be the whole real line. Consider the spectral resolution of B:

$$B=\int \lambda dE_\lambda ,$$

each λ being a real number. Let $\langle a_n \rangle$ be a sequence of positive real numbers converging to zero. Let $L_n=[0, a_n]$ and let $\langle \phi_n \rangle$ be a sequence of functions defined as follows:

$$\phi_n(t)=t \qquad \forall t \in L_n ,$$

$$\phi_n(t)=0 \qquad t \notin L_n .$$

Let

$$F_n=\int \phi_n(\lambda)\, dE_\lambda ,$$

and $B_n=B-F_n$. Since each F_n is bounded and $D_{B_n}=D_B$ for $n=1, 2\cdots$, therefore

$$A_2(D_{B_n})\subset D_{B_n} .$$

Also sp(B_n) is a proper subset of the real line and

$$A_1B_n-B_nA_2=A_1(B-F_n)-(B-F_n)A_2=A_1B-BA_2+(F_nA_2-A_1F_n)\subset C_n ,$$

where $C_n=C+F_nA_2-A_1F_n$. Thus A_1, B_n, A_2 and C_n satisfy the conditions of Case I. Hence

$$\operatorname*{Inf}_{\substack{\|x\|=1 \\ x\in D_{C_n}}} |(C_nx, x)|=0 , \qquad n=1, 2\cdots .$$

Now $\|F_n\|\leq a_n\to 0$ as $n\to\infty$. Hence $F_nA_2-A_1F_n\to 0$. Therefore $\|C_n-C\|\to 0$. Hence $(C_nx, x)\to(Cx, x)$ for all x. As $D_{C_n}=D_C$, $n=1, 2\cdots$, we obtain

$$\operatorname*{Inf}_{\substack{\|x\|=1 \\ x\in D_C}} |(Cx, x)|=0 .$$

Our next result is a generalization of Theorem 4 of *M. David* [4].

Theorem 2. *Let B be an unbounded self-adjoint operator. Let A_1 and A_2 be bounded operators such that A_1-A_2 is compact. Let C be any operator such that $(A_1B-BA_2)x=Cx$ on a linear subset Ω of H. If*

$$\operatorname*{Inf}_{\substack{\|x\|=1 \\ x\in D_C}} |(Cx, x)|=\alpha\neq 0 ,$$

then $B(\Omega)$ is not dense in H.

Proof. We shall essentially follow the arguments used by M. David [4]. We assume that $B(\Omega)$ is dense in H. Then $\forall x \in \Omega$

$$\alpha\|x\|^2 \leq |(Cx, x)| = |(A_1B - BA_2x, x)| = |(A_1Bx, x) - (BA_2x, x)|$$
$$\leq \|A_1\| \, \|Bx\| \, \|x\| + \|Bx\| \, \|A_2\| \, \|x\| .$$

Hence

$$\|x\| \leq \frac{\|A_1\| + \|A_2\|}{\alpha} \|Bx\| . \tag{2.1}$$

Let $y \in H$. Since $B(\Omega)$ is dense in H, there exists a sequence $\langle x_n \rangle$ in Ω such that $Bx_n \to y$. Now

$$\|x_n - x_m\| \leq \frac{\|A_1\| + \|A_2\|}{\alpha} \|Bx_n - Bx_m\| \to 0 \text{ as } n, m \to \infty .$$

Therefore there exists an element $x \in H$ such that $x_n \to x$. Since B is closed, $x \in D_B$ and $Bx = y$. Thus $x \in \bar{\Omega} \cap D_B$ and $B: \bar{\Omega} \cap D_B \to H$ is an onto map. Using (2.1) we get that B is one-one. Hence B is invertible and has a bounded inverse B^{-1} which is also self adjoint. Let $y \in B(\Omega)$, then there exists $x \in \Omega$ such that $B^{-1}y = x$. Hence

$$CB^{-1}y = A_1BB^{-1}y - BA_2B^{-1}y .$$

Therefore

$$B^{-1}CB^{-1}y = B^{-1}A_1y - A_2B^{-1}y \qquad \forall y \in B(\Omega) .$$

Let $0 \neq \mu \in \sigma_l(\hat{B}^{-1})$, then by [5, Theorem 1.1.] there exists a sequence $\langle y_n \rangle$ in H of unit vectors such that $y_n \to 0$ weakly and $\|B^{-1}y_n - \mu y_n\| \to 0$ as $n \to \infty$. Let

$$1 > \varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots\cdots$$

be a sequence of positive real numbers converging to zero. $B(\Omega)$ is dense in H, there exists a sequence $\langle x_n \rangle$ in $B(\Omega)$ such that

$$\|x_n - y_n\| < \varepsilon_n .$$

Now $\langle x_n \rangle$ is also weakly convergent because

$$(x_n, x) = (x_n - y_n + y_n, x) = (x_n - y_n, x) + (y_n, x) \to 0 \text{ as } n \to \infty .$$

Also

$$\|B^{-1}x_n - \mu x_n\| = \|B^{-1}(x_n - y_n + y_n) - \mu(x_n - y_n + y_n)\|$$
$$\times \|B^{-1}\| \, \|x_n - y_n\| + \|B^{-1}y_n - \mu y_n\| + |\mu| \, \|x_n - y_n\| \to 0 \text{ as } n \to \infty .$$

Therefore $\|B^{-1}x_n - \mu x_n\| \to 0$ as $n \to \infty$. Hence

$$\|x_n\| = \|x_n - y_n + y_n\| \leq \|x_n - y_n\| + \|y_n\| < 1 + \varepsilon_n < 2 ,$$

and

$$\|x_n\| = \|y_n - (y_n - x_n)\| \geq |\|y_n\| - \|y_n - x_n\|| > 1 - \varepsilon_n > 1 - \varepsilon_1 > 0 .$$

But then

$$\begin{aligned} |(B^{-1}A_1x_n, x_n) - (A_2B^{-1}x_n, x_n)| &= |(A_1x_n, B^{-1}x_n) - (A_2B^{-1}x_n, x_n)| \\ &= |(A_1x_n, B^{-1}x_n - \mu x_n) + (A_1x_n, \mu x_n) - (A_2B^{-1}x_n - \mu A_2x_n, x_n) - (\mu A_2x_n, x_n)| \\ &\leq \|A_1\| \, \|B^{-1}x_n - \mu x_n\| \, \|x_n\| + \|A_2\| \, \|B^{-1}x_n - \mu x_n\| \, \|x_n\| + |\mu| \, \|(A_1 - A_2)x_n\| \, \|x_n\| \\ &\leq 2(\|A_1\| + \|A_2\|)\|B^{-1}x_n - \mu x_n\| + 2|\mu| \, \|(A_1 - A_2)x_n\| . \end{aligned}$$

Since $A_1 - A_2$ is compact, therefore $\|(A_1 - A_2)x_n\| \to 0$ as $n \to \infty$. Hence

$$|(CB^{-1}x_n, B^{-1}x_n)| = |(B^{-1}CB^{-1}x_n, x_n)| = |(B^{-1}A_1x_n, x_n) - (A_2B^{-1}x_n, x_n)| \to 0 .$$

Therefore

$$\alpha \|B^{-1}x_n\|^2 \leq |(CB^{-1}x_n, B^{-1}x_n)| \to 0 \text{ as } n \to \infty .$$

Since $\alpha \neq 0$, $B^{-1}x_n \to 0$, which implies $x_n \to 0$, a contradiction. Hence $B(\Omega)$ is not dense in H.

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