SOME REMARKS ON THE CROSSED PRODUCT

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Let R be a ring with unit and J its Jacobson radical. If a finite group of automorphisms G acts as a Galois group on R/J, then G is a Galois group of R over S, S is the fixed ring of R under G. Let Δ be the crossed product of R and G.

Let G act as a completely outer group of automorphisms on R/J. If R is semiprimary, primary, Artinian, left perfect or a commutative semiperfect ring, (respectively), then Δ is semiprimary, primary, Artinian, left perfect or semi-perfect ring (respectively). Similar results hold for R and S.

Let R be a ring with identity. We denote the Jacobson radical of R by J. We assume that $G=\{1, \sigma, \cdots, \tau\}$ is a finite group of automorphisms of R, which acts as a Galois group on $R/J(R)$. In other words, there are $\bar{a}_1, \cdots, \bar{a}_n; \overline{a_1^*}, \cdots, \overline{a_n^*}$ in R/J such that $\sum_{i=1}^{n} \overline{a_i\sigma(a_i^*)}=\overline{\delta_{1,\sigma}}$ for all $\sigma \in G$, where $\bar{a}_i=a_i+J$, $a_i \in R$ and $\delta_{1,\sigma}$ is the Kronecker delta.

Let $S=R^G=\{r \in R | \sigma(r)=r$ for all $\sigma \in G\}$, S is the fixed ring of R under G. The crossed product of R/J with G, $\Delta(R/J, G)$ is $\sum_{\sigma \in G} R/Ju_\sigma$ with $\bar{x}u_\sigma \cdot \bar{y}u_\tau=\overline{xy^\sigma}u_{\sigma\tau}$ where $\overline{y^\sigma}=\sigma(y)+J$ and $\bar{x}=x+J$, $x, y \in R$. The crossed product of R with $G, \Delta(R, G)$ is defined analogously.

Proposition 1. *Let G act as a Galois group on R/J, then R over S is G-Galois.*

Proof. Define π: $\Delta(R: G)\to\Delta(R/J, G)$ as follows $\pi(\sum_{\sigma \in G} x_\sigma u_\sigma)=\sum_{\sigma \in G} \bar{x}_\sigma u_\sigma$, where $\bar{x}_\sigma=x_\sigma+J$, $x_\sigma \in R$. Then π is an R epimorphism with kernel $N=Ju_1+Ju_\sigma+\cdots+Ju_\tau$. Since each summand Ru_σ of N is small in Ru_σ, we conclude π is a minimal left R epimorphism.

Let $u=1+\sigma+\cdots+\tau$, the trace. Since we have assumed G acts as a Galois group on R/J; $R/J\ u\ R/J=(R/J: G)$. See *T. Kanzaki* ([2], Proposition 2, p. 108).

Since $\pi(RuR)=R/J\ u\ R/J$ and π is a minimal epimorphism, we conclude $RuR=\Delta(R: G)$. Thus there exists $a_1, \cdots, a_n; a_1^*, \cdots, a_n^* \in R$ such that $\sum_i a_i u a_i^*=u_1$ or $\sum_i a_i\sigma(a_i^*)=\delta_{1,\sigma}$. Thus R is G-Galois over S.

If R over S is G-Galois, it is clear that G acts as a Galois group on R/J.

We can view R as a bi $\Delta-S$ module by defining a left Δ action to be $xu_\sigma\cdot r=x\sigma(r)$ for $x, r\in R$ and the obvious right S action.

We say G acts in a completely outer way on R/J, if G induces a completely outer group on R/J. See *Y. Miyashita* ([3], p. 127).

Proposition 2. *Assume G acts in a completely outer way on R/J and R/J is an Artinian ring. Then R has a normal basis over S.*

Proof. By Proposition 1, R over S is G-Galois. Thus by Propositions 2 and 3 of [6], we conclude R has a normal basis.

Corollary. *Let R/J be a commutative ring and G acts as a Galois group on R/J, then R has a normal basis.*

Also, if R/J is simple and G acts as a Galois group on R/J which contains no inner automorphisms, then R has a normal basis.

Proof. See ([3], Theorem 6.6, p. 128) and ([3], Corollary of Proposition 6.4, p. 128).

From now on, we assume R/J is an Artinian ring and G acts as a completely outer group of automorphisms on R/J. We call R semiprimary, primary or local, if R/J is Artinian, simple and Artinian or a division ring (respectively).

Proposition 3. *a) R is semiprimary if and only if $\Delta(R, G)$ and S are semiprimary. See* ([3], *Proposition* 7.3, *p.* 130)

b) If R is primary, then $\Delta(R\colon G)$ and S are primary.

c) If R is local, then S is local and $\Delta(R, G)$ is primary.

Proof. The Jacobson radical of $\Delta(R; G)$, $J(\Delta)$ is $J\cdot\Delta(R; G)=\Delta(R; G)J$, where J is the Jacobson radical of R. See (6, Proposition 1). Thus $\Delta(R; G)/J(\Delta)$ is isomorphic to $\Delta(R/J; G)$. Hence $\Delta(R/J, G)/J(\Delta)$ is a finitely generated R/J module. Thus Δ is semiprimary.

Since G acts as a completely outer group on R/J, $J\cap S\subseteq J(S)$, where $J(S)$ is the Jacobson radical of S. See (6, Proposition 1).

Now $\bar{R}$ is a finitely generated projective $\bar{\Delta}$ module, since $\bar{\Delta}$ is semisimple, Artinian. Since $\mathrm{Hom}_{\bar{\Delta}}(\bar{R}, \bar{\Delta})\subseteq\mathrm{Hom}_{\bar{R}}(\bar{R}, \bar{\Delta})\subseteq\bar{\Delta}$, we conclude $\mathrm{Hom}_{\bar{\Delta}}(\bar{R}, \bar{\Delta})=\sum_{\sigma\in G}u_\sigma\bar{R}$. See (3, Lemma 2.5, p. 128). Thus there exist $f_1, \cdots, f_n\in\sum_{\sigma\in G}u_\sigma\bar{R}$ and $\bar{x}_1, \cdots, \bar{x}_n\in\bar{R}$ such that for all $\bar{x}\in\bar{R}$, $\sum_{i=1}^n f_i(\bar{x})\bar{x}_i=\bar{x}$. If $f_i(\bar{x})=\bar{x}\sum u_\sigma\bar{r}_i$, then $\bar{x}=\bar{x}\sum_i\sum_\sigma(\bar{r}_i\bar{x}_i)^\sigma$, for all $\bar{x}\in\bar{R}$. Thus $\bar{1}=\sum_i\sum_\sigma(\bar{r}_i\bar{x}_i)^\sigma$, let $\bar{d}=\sum_i(\bar{r}_i\bar{x}_i)$, then $\mathrm{tr}\,\bar{d}=\bar{1}$. So $\mathrm{tr}\,d-1\in J(R)\cap S\subseteq J(S)$. Thus $\mathrm{tr}\,R+J(S)=S$, but $J(S)$ is small. Thus $\mathrm{tr}\,(R)=S$ or there is a

c in R such that tr $c=1$. We conclude that tr: $R\to S\to 0$ splits; hence S is a left S direct summand of R.

Now Δ is isomorphic to End R_S, since R over S is G-Galois, by Proposition 1. We conclude that ${}_\Delta R$ is a finitely generated projective module. See *K. Morita* ([4], Lemma 3.3, p. 100).

Clearly, the $n\times n$ matrices over Δ and $e\Delta e$, $e^2=e\in\Delta$ are semiprimary rings. Thus the endomorphism ring of a projective module over Δ is semiprimary. For example, $\mathrm{End}_\Delta(R)\simeq S$ is semiprimary.

Since $\Delta(R, G)/J(\Delta)\simeq\Delta(R/J, G)$, if $\Delta(R, G)$ is semiprimary R is semiprimary. Also R is a finitely generated projective right S module, so R is semiprimary, if S is semiprimary.

Proof of b). As in the proof of a) R is left Δ projective. By (6, Proposition 6), $S/J(S)$ is the fixed ring of R/J under G. Furthermore, $S/J(S)$ is a direct summand, as an $S/J(S)$ module, of R/J. See *T. Nakayama* ([5], Lemma 4, p. 207).

There is a one-to-one correspondence between ideals of $S/J(S)$, G invariant ideals of $R/J(R)$ and ideals of $\Delta/J(\Delta)$. See (3, p. 132). Since $R/J(R)$ is Artinian, Δ and S are primary.

Proof of c). Let $s\in S$, then if s is a unit in R, it is a unit in S. For assume there is an $r\in R$ such that $rs=sr=1$. Then $1=\sigma(r)s=s\sigma(r)$ for all $\sigma\in G$. So $\sigma(r)=r$ for all $\sigma\in G$ or $r\in S$. Thus if R is local, S is local.

As in the proof of b) there is a one-to-one correspondence between ideals of $\Delta/J(\Delta)$ and ideals of $S/J(S)$. Thus Δ is primary.

Proposition 4. *Assume G acts as a completely outer group of automorphisms on $R/J(R)$. Then R is left Artinian if and only if $\Delta(R\colon G)$ and S are left Artinian.*

Proof. We have shown in the course of the proof of Proposition 3a, that S, as a right S module, is a direct summand of R. Thus if R is left Artinian, so is S.

Proposition 5. *Assume G acts as a completely outer group on R/J. Then if R is left perfect, then $\Delta(R\colon G)$ and S are left perfect.*

Proof. By Proposition 3, we know $\Delta(R\colon G)$ and S are semiprimary. Since $J(S)\subseteq J(R)$, ([6], Proposition 1), $J(S)$ is left T nilpotent. Thus S is left perfect.

We know that as a right S module R is finitely generated, projective and a generator. Also End R_S is isomorphic to $\Delta(R, G)$. Thus $\Delta(R, G)$ is semiperfect.

Let M be left Δ module, in order that M have a projective cover it suffices that for any left Δ module B requiring no more generators than $M, B=J(\Delta)B$ implies $B=0$. ([1], Lemma 2.6, p. 473). But $B=J(\Delta)B=J(R)B$ and R being left perfect implies $B=0$. ([1], Proposition 2.7, p. 474). Thus every left Δ module has a projective cover and Δ is left perfect.

Proposition 6. *Let G act as a completely outer Galois group on R/J. If R is a commutative, semiperfect ring, then $\Delta(R, G)$ and S are semiperfect.*

Proof. Let $1=e_1+\cdots+e_n$, where $e_1, \cdots, e_n$ are completely primitive orthogonal idempotents, Let $H_i=\{\sigma\in G|\sigma(e_i)=e_i\}$, then H_i is subgroup of G for $i=1, \cdots, n$.

By Δ_i, we mean the crossed product of e_iR and H_i. Now $\Delta_i=e_i\Delta(R, H_i)e_i=e_i\Delta(R, G)e_i$. We show the second equality.

Let $H_i=\{1=\rho_1, \cdots, \rho_r\}$ and $G=\{1=\rho_1, \rho_2, \cdots, \rho_r; \varepsilon_1, \cdots, \varepsilon_t\}$. Now $e_i\Delta(R, G)e_i=e_i(Ru_1+Ru_{\rho_2}+\cdots+Ru_{\rho_r}+Ru_{\varepsilon_1}+\cdots+Ru_{\varepsilon_t})e_i=e_i\Delta_ie_i+e_iRe_i^{\varepsilon_1}u_{\varepsilon_1}+\cdots+e_iRe_i^{\varepsilon_t}u_{\varepsilon_t}$. Now $1=\sum_{i=1}^{n} e_i^{\varepsilon_j}$ for each $j=1, \cdots, t$. By Azumaya's Theorem $e_i^{\varepsilon_j}=e_k$, $k\neq i$ for $\varepsilon_j\notin H_i$ for $j=1, \cdots, t$. Thus $e_iRe_i^{\varepsilon_j}=0$ for $j=1, \cdots, t$, since R is commutative.

Now $\Delta_i\cap\Delta_j=0$, if $i\neq j$. Also H_i acts as a completely outer group of automorphisms on Re_i/Je_i and Re_i is a local ring. Thus Δ_i is semiperfect, by Proposition 3c. Let $\Delta'=\sum_{i=1}^{n}\oplus\Delta_i$, Δ' is a semiperfect ring.

Thus $u_1=E_1+\cdots+E_k$, the E_i's are completely primitive orthogonal idempotents of Δ'. For each j, $E_j=(\sum_{i=1}^{n} e_iu_1)E_j$, then e_iE_j, $i=1, \cdots, n$, are orthogonal idempotents since the e_i's are central in Δ'. Now E_j is primitive, so $E_j=E_je_j$ (after renumbering).

Thus $E_j\Delta'E_j=E_je_j\Delta'e_jE_j=E_j\Delta_jE_j=E_je_j\Delta(R: G)e_jE_j=E_j\Delta(R; G)E_j$. Thus $E_j\Delta(R; G)E_j$ is a local ring, for each j, hence $\Delta(R, G)$ is semiperfect.

We conclude by asking if Proposition 6 is true for a noncommutative semiperfect ring.

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