ON ORDER AND TYPE OF AN ENTIRE DIRICHLET SERIES OF SEVERAL COMPLEX VARIABLES

By

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1. Introduction.

The concepts of order and type of an everywhere convergent power series (entire function) were extended to an everywhere absolutely convergent Dirichlet series (entire Dirichlet series): $f(s)=\sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ ($s=\sigma+it$, $\lambda_1>0$, $\lambda_{n+1}>\lambda_n\to\infty$ with n) by *Ritt* [5], who also estimated the order and the type, so introduced, in terms of the coefficients of the series. The result of *Ritt*, expressing the order in terms of the coefficients, was improved significantly by *Azpeitia* [1] who replaced the condition: $\liminf_{n\to\infty} (\lambda_n/\log n)>0$ by the weaker condition: $\lim_{n\to\infty} (\lambda_n \log \lambda_n/\log n)=\infty$. In his subsequent paper, *Azpeitia* [2] derived inequalities for the order valid even without this restriction. Also, following the lines of *Azpeitia* [2], *Jain* and *Gupta* (P.N.) [3] improved considerably on a result of *Kamthan* [4], and consequently the corresponding results of *Yu* [6] and *Ritt* [5] which expressed the type of $f(s)$ in terms of the coefficients.

In the present paper, we have extended the results of *Ritt* [5] and *Yu* [6] to the entire Dirichlet series of several complex variables. It would be of much interest if our results could be improved on the lines of *Azpeitia* [2] and *Jain* and *Gupta* [3], in fact, we, at present, find it difficult. However, our interest shall continue to obtain the improved forms of these results. In the last section of this paper we state without proofs the results for a class of functions of infinite order analogues to our Theorem 1 and 2.

For simplicity, we consider here only two variables, though our results can easily be extended to several complex variables.

2. Consider a double entire Dirichlet series

$$f(s_1, s_2)=\sum_{n,m=1}^{\infty} a_{nm} \exp(s_1\lambda_n+s_2\mu_m)\ , \qquad (s_j=\sigma_j+it_j\,;\ j=1, 2)$$

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where

$$0 \leqslant \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty ;\ 0 \leqslant \mu_0 < \mu_m \to \infty ,$$

and

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n} = 0 = \lim_{m\to\infty} \frac{\log m}{\mu_m} ,$$

Let

$$M(\sigma_1, \sigma_2) = \sup \{ |f(\sigma_1 + it_1, \sigma_2 + it_2)| : -\infty < t_i < \infty,\ j = 1, 2\} ,$$

be the maximum modulus of $f(s_1, s_2)$ on the tube $\mathrm{Re}\, s_j = \sigma_j$, $j = 1, 2$.

We define the order $\rho (0 \leqslant \rho \leqslant \infty)$ of $f(s_1, s_2)$ as:

$$\limsup_{\sigma_1, \sigma_2 \to \infty} \frac{\log M(\sigma_1, \sigma_2)}{\log (e^{\sigma_1} + e^{\sigma_2})} = \rho .$$

Further, if $0 < \rho < \infty$, then the type $T (0 \leqslant T \leqslant \infty)$ of $f(s_1, s_2)$ is defined as:

$$\limsup_{\sigma_1, \sigma_2 \to \infty} \frac{\log M(\sigma_1, \sigma_2)}{e^{\rho\sigma_1} + e^{\rho\sigma_2}} = T .$$

Now, we estimate ρ and T in terms of the sequences $\{a_{nm}\}$, $\{\lambda_n\}$ and $\{\mu_m\}$.

Theorem 1. *If $f(s_1, s_2)$ is an entire function of order $\rho (0 \leqslant \rho \leqslant \infty)$, then*

$$\rho = \limsup_{n, m \to \infty} \frac{\log (\lambda_n^{\lambda_n} \mu_m^{\mu_m})}{\log |a_{nm}|^{-1}} .$$

Proof. Let

$$\mu = \limsup_{n, m \to \infty} \frac{\log (\lambda_n^{\lambda_n} \mu_m^{\mu_m})}{\log |a_{nm}|^{-1}} .$$

We first show that $\rho \geqslant \mu$. Let us assume that $\mu > 0$, for otherwise the result is trivially true. Then for a given $\varepsilon > 0$, we have two sequences $\{\lambda_{N_p}\}$ and $\{\mu_{M_q}\}$ with $N_p \to \infty$ as $p \to \infty$ and $M_q \to \infty$ and $q \to \infty$ such that

$$\log |a_{nm}| > -(\mu - \varepsilon)^{-1} (\lambda_n \log \lambda_n + \mu_m \log \mu_m) ,$$

for $n = N_p$ and $m = M_q$.

Since the inequality: $M(\sigma_1, \sigma_2) \geqslant |a_{nm}| \exp (\sigma_1 \lambda_n + \sigma_2 \mu_m)$ holds for all σ_1, σ_2 and n, m, it follows for all σ_1 and σ_2 and $n = N_p$, $m = M_q$ that

$$\log M(\sigma_1, \sigma_2) > \lambda_n (\sigma_1 - (\mu - \varepsilon)^{-1} \log \lambda_n) + \mu_m (\sigma_2 - (\mu - \varepsilon)^{-1} \log \mu_m) .$$

Taking

$$\sigma_{1,p} = (\mu - \varepsilon)^{-1} \log (e \lambda_{N_p}) \quad \text{and} \quad \sigma_{2,q} = (\mu - \varepsilon)^{-1} \log (e \mu_{M_q}) ,$$

in the above we find that

$$\log M(\sigma_1, \sigma_2) > \frac{\exp(\sigma_1(\mu-\varepsilon)) + \exp(\sigma_2(\mu-\varepsilon))}{e(\mu-\varepsilon)},$$

for $\sigma_1 = \sigma_{1,p}$ and $\sigma_2 = \sigma_{2,q}$. Hence $\rho \geqslant \mu$.

Further, to prove the counter part, we suppose that $\mu < \infty$, for otherwise the result is obviously true, so that

(2.1) $$|a_{nm}| < \lambda_n^{-\lambda_n/(\mu+\varepsilon)} \mu_m^{-\mu_m/(\mu+\varepsilon)}, \qquad n > n_0,\ m > m_0 .$$

Now

$$M(\sigma_1, \sigma_2) \leqslant \Big(\sum_{n=1}^{n_0} \sum_{m=1}^{m_0} + \sum_{n=n_0+1}^{\infty} \sum_{m=1}^{m_0} + \sum_{m=m_0+1}^{\infty} \sum_{n=1}^{n_0}$$

$$+ \sum_{n=n_0+1}^{\infty} \sum_{m=m_0+1}^{\infty} \Big) |a_{nm}| \exp(\sigma_1 \lambda_n + \sigma_2 \mu_m)$$

(2.2) $$= \sum_1 + \sum_2 + \sum_3 + \sum_4 , \text{ (say).}$$

Estimation of $\sum_i$ $(1 \leqslant i \leqslant 4)$:

Clearly

$$\sum_1 = O(\exp(\sigma_1 \lambda_{n_0} + \sigma_2 \mu_{m_0})) .$$

Also, in view of (2.1), we get

$$\sum_4 \leqslant \sum_{n>n_0} \sum_{m>m_0} \exp(\sigma_1 \lambda_n + \sigma_2 \mu_m - (\mu+\varepsilon)^{-1} (\lambda_n \log \lambda_n + \mu_m \log \mu_m))$$

$$\leqslant \max.\ \exp(\sigma_1 \lambda_n + \sigma_2 \mu_m - (\mu+2\varepsilon)^{-1} (\lambda_n \log \lambda_n + \mu_m \log \mu_m))$$

$$\times \sum_{n>n_0} \sum_{m>m_0} \exp\left(-\frac{\lambda_n \log \lambda_n + \mu_m \log \mu_m}{\varepsilon^{-1}(\mu+\varepsilon)(\mu+2\varepsilon)} \right).$$

The series on the right of the above inequality being convergent, we find that

$$\sum_4 \leqslant A \exp(e^{-1}(\mu+2\varepsilon)^{-1}(e^{\sigma_1(\mu+2\varepsilon)} + e^{\sigma_2(\mu+2\varepsilon)})) ,$$

where A is an absolute constant, since the maximum of the expression:

$$\exp(\sigma_1 \lambda_n + \sigma_2 \mu_m - (\mu+2\varepsilon)^{-1} (\lambda_n \log \lambda_n + \mu_m \log \mu_m))$$

is attained at

$$\lambda_n = e^{-1} \exp(\sigma_1(\mu+2\varepsilon))\,;\ \mu_m = e^{-1} \exp(\sigma_2(\mu+2\varepsilon)) ,$$

Further to estimate $\sum_2$, μ being finite, it is noted, for all values of n and m, that

$$\frac{\log(\lambda_n^{\lambda_n} \lambda_m^{\mu_m})}{\log |a_{nm}|^{-1}} \leq \zeta .$$

Therefore

$$\sum_2 \leqslant \sum_{n>n_0} \sum_{m=1}^{m_0} \exp(\sigma_1\lambda_n+\sigma_2\mu_m-\zeta^{-1}\lambda_n \log \lambda_n-\zeta^{-1}\mu_m \log \mu_m))$$
$$=O(\exp(\sigma_2\mu_{m_0})) \sum_{n>n_0} \exp(\sigma_1\lambda_n-\zeta^{-1}\lambda_n \log \lambda_n)$$
$$\leqslant O(\exp(\sigma_2\mu_{m_0}) \max. \exp(\sigma_1\lambda_n-(\zeta+\varepsilon)^{-1}\lambda_n \log \lambda_n)$$
$$\times \sum_{n>n_0} \exp(-\varepsilon\zeta^{-1}(\zeta+\varepsilon)^{-1}\lambda_n \log \lambda_n)$$
$$=O(\exp(\sigma_2\mu_{m_0}) \max. \exp(\sigma_1\lambda_n-(\zeta+\varepsilon)^{-1}\lambda_n \log \lambda_n) .$$

But the maximum of

$$\exp(\sigma_1\lambda_n-(\zeta+\varepsilon)^{-1}\lambda_n \log \lambda_n)$$

is attained at

$$\lambda_n=e^{-1} \exp(\sigma_1(\zeta+\varepsilon)) .$$

Hence

$$\sum_2 \leqslant O(\exp(\sigma_2\mu_{m_0})) \exp(e^{-1}(\zeta+\varepsilon)^{-1} \exp(\sigma_1(\zeta+\varepsilon))) .$$

Similarly

$$\sum_3 \leqslant O(\exp(\sigma_1\lambda_{n_0})) \exp(e^{-1}(\zeta+\varepsilon)^{-1} \exp(\sigma_2(\zeta+\varepsilon))) .$$

Substituting these values of $\sum_i (1\leqslant i\leqslant 4)$ in (2.2), we have

$$\log \log M(\sigma_1, \sigma_2)\leqslant \log(\exp(\sigma_1(\mu+2\varepsilon))+\exp(\sigma_2(\mu+2\varepsilon)))+O(1) ,$$

which gives $\rho\leqslant\mu$.

Hence the theorem follows.

Theorem 2. *If* $f(s_1, s_2)$ *is an entire function of order* $\rho(0<\rho<\infty)$ *and type* $T(0\leqslant T\leqslant\infty)$. *Then*

$$\limsup_{n,m\to\infty} (\lambda_n^{\lambda_n}\mu_m^{\mu_m}|a_{nm}|^\rho)^{1/(\lambda_n+\mu_m)}=e\rho T .$$

Proof. We shall merely sketch the proof, since it follows on the lines of the proof of Theorem 1. We note, for $n=n_p$ and $m=m_q$, that

$$\log M(\sigma_1, \sigma_2)>\lambda_n\left(\sigma_1+\rho^{-1} \log \frac{\alpha-\varepsilon}{\lambda_n}\right)+\mu_m\left(\sigma_2+\rho^{-1} \log \frac{\alpha-\varepsilon}{\mu_m}\right), \tag{2.3}$$

where α is assumed to be positive and is given by

$$\alpha=\limsup_{n,m\to\infty} (\lambda_n^{\lambda_n}\mu_m^{\mu_m}|a_{nm}|^\rho)^{1/(\lambda_n+\mu_m)} .$$

Choosing in (2.3) the sequences of the values of σ_1 and σ_2 by

$$\sigma_1=\rho^{-1} \log\left(\frac{e\lambda_{n_p}}{\alpha-\varepsilon}\right), \quad \sigma_2=\rho^{-1} \log\left(\frac{e\mu_{m_q}}{\alpha-\varepsilon}\right),$$

it follows that $e\rho T \geqslant \alpha$. This assertion is trivially true for the case when $\alpha=0$.

The counter part follows by using the following estimations of $\sum_i$'s in (2.2):

$$\sum_1 \leqslant O(\exp(\sigma_1\lambda_{n_0}+\sigma_2\mu_{m_0}))\ ,$$

$$\sum_2 \leqslant O(\exp \sigma_1\lambda_{n_0}) \exp\left(\frac{\zeta+\varepsilon}{e\rho} e^{\rho\sigma_2}\right)\ ,$$

$$\sum_3 \leqslant O(\exp \sigma_2\mu_{m_0}) \exp\left(\frac{\zeta+\varepsilon}{e\rho} e^{\rho\sigma_1}\right)\ ,$$

$$\sum_4 \leqslant A_1 \exp\left(\frac{\alpha+2\varepsilon}{e\rho}(e^{\rho\sigma_1}+e^{\rho\sigma_2})\right)\ ,$$

where A_1 is an absolute constant.

3. Functions of order infinity.

In this section, we consider a class of entire Dirichlet series of two variables of order infinity i.e. for which $\rho=\infty$. To have more precise description of the growth relations of such a class of functions, we define the k-th order $\rho_k(0\leqslant\rho_k\leqslant\infty)$ of $f(s_1, s_2)$ as:

$$\rho_k = \limsup_{\sigma_1,\sigma_2\to\infty} \frac{\iota_k M(\sigma_1, \sigma_2)}{\log(e^{\sigma_1}+e^{\sigma_2})}\ , \qquad k>2\ ,$$

where $\rho_{k-1}=\infty$ and $\iota_k X=\iota_{k-1}(\iota_1 X)$, $\iota_1 X=\log X$, $\iota_2 X=\log\log X$.

Purther if $0<\rho_k<\infty$, then the k-th type $\tau_k(0\leqslant\tau_k\leqslant\infty)$ of $f(s_1, s_2)$ is defined as:

$$\tau_k = \limsup_{\sigma_1,\sigma_2\to\infty} \frac{\iota_{k-1} M(\sigma_1, \sigma_2)}{(e^{\rho_k\sigma_1}+e^{\rho_k\sigma_2})}\ , \qquad k>2\ .$$

In the following theorems, we have estimated ρ_k and τ_k in terms of the sequences $\{a_{nm}\}$, $\{\lambda_n\}$ and $\{\mu_m\}$:

Theorem 3. *If $f(s_1, s_2)$ is an entire function of k-th order $\rho_k(0\leqslant\rho_k\leqslant\infty)$, then*

$$\rho_k = \limsup_{n,m\to\infty} \frac{\lambda_n\iota_{k-1}\lambda_n+\mu_m\iota_{k-1}\mu_m}{\log|a_{nm}|^{-1}}\ .$$

Theorem 4. *If $f(s_1, s_2)$ is an entire function of k-th order $\rho_k(0<\rho_k<\infty)$ and k-th type $\tau_k(0\leqslant\rho_k\leqslant\infty)$, then*

$$\tau_k = \limsup_{n,m\to\infty} \{(\iota_{k-2}\lambda_n)^{\lambda_n}(\iota_{k-2}\mu_m)^{\mu_m}|a_{nm}|^{\rho_k}\}^{1/(\lambda_n+\mu_m)}\ .$$

For conciseness the proofs of the Theorems 3 and 4 are omitted as these follow on the lines of Theorems 1 and 2 with slight modifications in the calculations.

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