A FUNCTIONAL LAW OF THE ITERATED LOGARITHM FOR SAMPLE SEQUENCES

By

TOSHIO MORI and HIROSHI OODAIRA

(Received October 5, 1974)

1. Introduction.

Let $X_1, X_2, \cdots$ be a sequence of independent random variables with common distribution function F. Suppose F satisfies

$$\lim_{n\to\infty} F^n(a_n x+b_n)=\exp(-e^{-x}), \quad x\in R,$$

for a pair of sequences $\{a_n\}$ and $\{b_n\}$ with $a_n>0$. Let $Z_n=\max(X_1, \cdots, X_n)$ and $Y_n(t)=(Z_{[nt]}-b_n)/a_n$, $t>0$. *Lamperti* [6] proved that the sequence $\{Y_n\}$ of random elements in the Skorohod space $D[r, s]$, $0<r<s<\infty$, converges in distribution to an extremal process. Under some additional assumptions on F *Pickands* [7], *de Haan* and *Hordijk* [4] obtained a law of the iterated logarithm for sample maxima:

$$\limsup_{n\to\infty} Y_n(1)/\log_2 n=1, \quad \liminf_{n\to\infty} Y_n(1)/\log_2 n=0, \quad \text{w.p.1},$$

where $\log_2=\log\log$. The main purpose of this paper is to give a functional form of this law of the iterated logarithm. According to the idea of *Pickands* [8] we formulate the problem in terms of two-dimensional point process. The proof of principal results follows the pattern of Strassen's proof ([3], [9]) of his functional law of the iterated logarithm for partial sums of i.i.d. random variables.

In Section 3 we introduce a two-dimensional Poisson point process P which plays the same role in our theory as the Brownian motion does in Strassen's invariance principle. In Section 4 we consider a sequence of two-dimensional point processes Φ_n, $n\geq 1$, each consisting of random points $(j/n, (X_j-b_n)/a_n)$, $j\geq 1$. It is shown that the sequence $\{\Phi_n\}$ converges weakly to P. This corresponds to Lamperti's weak convergence result and implies it.

Our main results are given in last two sections. Let Ψ_n be a point process on $S^+=\{(t, x);\ 0\leq t<\infty,\ 0<x\leq\infty\}$ consisting of random points $(j/n, (X_j-b_n)/(a_n\log_2 n))\in S^+$, $j\geq 1$. Then Ψ_n is a random element taking values in the space $N(S^+)$ of all locally finite non-negative integer valued measures on S^+ endowed

with the vague topology. A theorem in Section 6 states that under some additional conditions on F the sequence $\{\Psi_n\}_{n\geq 8}$ is relatively compact in $N(S^+)$ and the set of its limit points is K, where K is the set of $\mu\in N(S^+)$ such that $\int_{S^+} x\mu(dtdx)\leq 1$. In Section 5 we prove a corresponding result for the two-dimensional Poisson point process P, which is applied to obtain results in Section 6. As a corollary a functional law of the iterated logarithm for the sequence $\{Y_n/\log_2 n\}_{n\geq 8}$ is obtained.

After completing this work the authors have learned a recent paper of *Wichura* [10]. Our principal result (Theorem 5.1) seems to be essentially the same as the one obtained by Wichura (Theorem 1B of [10]) except for a few difference in presentation. However the point process approach adopted in this paper seems to have some advantage. For example with this formulation it is possible to obtain multi-parameter analogues of theorems in Section 6.

2. Notations and preliminaries

Let $\mathfrak{B}(S)$ be the σ-algebra of Borel subsets of a locally compact separable metric space (S,ρ) and $\mathfrak{B}_0(S)$ the ring of bounded Borel sets. Let $C_0(S)$ denote the space of all real valued continuous functions f on S with compact supports and $C_0^+(S)$ the subset of $C_0(S)$ consisting of non-negative f. Let $M(S)$ denote the space of all locally finite non-negative Borel measures φ on S and $N(S)$ the subspace of $M(S)$ consisting of integer valued measures. Endowed with vague topology $M(S)$ and $N(S)$ are Polish spaces.

Let $\{f_k\}$ be a sequence of functions $f_k\in C_0^+(S)$ which is dense in $C_0^+(S)$ in the following sense: for every $f\in C_0^+(S)$ there exist a compact set $E\subset S$ and a subsequence $\{f_{k'}\}$ of $\{f_k\}$ such that $\operatorname{supp}[f]\subset E$, $\operatorname{supp}[f_{k'}]\subset E$ and $f_{k'}\to f$ uniformly. Then the topology of $M(S)$ coincides with that derived from a metric d defined by

$$d(\varphi,\psi)=\sum_{k=1}^{\infty}2^{-k}\frac{|\varphi f_k-\psi f_k|}{1+|\varphi f_k-\psi f_k|}, \quad \varphi,\psi\in M(S) \tag{2.1}$$

where $\varphi f=\int f d\varphi$.

Lemma 2.1. *Let $\varphi\in N(S)$ and $A_j\in\mathfrak{B}_0(S)$, $1\leq j\leq n$, be disjoint. If $\varphi(\partial A_j)=0$, $1\leq j\leq n$, then there exists a neighborhood U of φ in $N(S)$ such that $\psi(A_j)=\varphi(A_j)$, $1\leq j\leq n$, for $\psi\in U$.*

Proof. By assumption for each j there exist an open G_j and a closed F_j

such that $F_j \subset A_j^\circ \equiv (\bar{A}_j^c)^c$, $G_j \supset \bar{A}_j$ and $\varphi(G_j - F_j) = 0$. Let f_j, $g_j \in C_0^+(S)$ be such that $\chi_F \leq f_j \leq \chi_A \leq g_j \leq \chi_G$, where χ_A denotes the indicator of a set A. Then $U = \{\psi \in N(S);\ |\psi f_j - \varphi f_j| < 1,\ |\psi g_j - \varphi g_j| < 1,\ 1 \leq j \leq n\}$ is a neighborhood satisfying the condition of the lemma.

Lemma 2.2. *Let $S_\varepsilon(x)$ be the open ε-neighborhood of $x \in S$. If $\varphi \in N(S)$ has only a finite number of atoms $x_1, \cdots, x_n$, then for every neighborhood U of φ there exists $\varepsilon > 0$ such that $S_\varepsilon(x_j)$, $1 \leq j \leq n$, are disjoint and $V_\varepsilon = \{\psi;\ \psi(S_\varepsilon(x_j)) = \varphi(S_\varepsilon(x_j))$, $1 \leq j \leq n$, and $\psi(S - \bigcup_{j=1}^n S(x_j)) = 0$ is contained in U.*

Proof. It suffices to prove assuming that

$$U = \{\psi;\ \psi \in N(S)\ ,\ |\psi f_k - \varphi f_k| < \varepsilon_k\ ,\ 1 \leq k \leq m\}\ ,$$

where $f_k \in C_0^+(S)$, $\varepsilon_k > 0$. Let $\varepsilon > 0$ be so chosen that $S_\varepsilon(x_j)$, $1 \leq j \leq n$, are disjoint and for any j and k,

$$|f_k(x) - f_k(x_j)| < \varepsilon_k (n\varphi(\{x_j\}))^{-1} \quad \text{if} \quad d(x, x_j) < \varepsilon\ .$$

Then this ε satisfies the conditions of the lemma.

The σ-algebra $\mathfrak{N}(S)$ of Borel subsets of $N(S)$ coincides with the smallest σ-algebra with respect to which every mapping $\varphi \to \varphi(A)$, $A \in \mathfrak{B}_0(S)$, is measurable.

A probability measure P on $(N(S), \mathfrak{N}(S))$ is called a point process on S. For every $\mu \in M(S)$ there exists a unique point process P on S such that for every disjoint $B_j \in \mathfrak{B}_0(S)$ and integer $k_j \geq 0$, $1 \leq j \leq n$,

$$P\{\varphi;\ \varphi(B_j) = k_j\ ,\ 1 \leq j \leq n\} = \prod_{j=1}^n e^{-\mu(B_j)} \frac{\mu(B_j)^{k_j}}{k_j!}\ .$$

We call P a Poisson point process and μ its intensity measure.

3. A Poisson point process.

Throughout the rest let $S = [0, \infty) \times (-\infty, \infty] = \{(t, x);\ 0 \leq t < \infty,\ -\infty < x \leq \infty\}$. Let P be a Poisson point process on S with intensity measure π where

$$\pi(dt\,dx) = e^{-x} dt\,dx\ ,\quad t \geq 0\ ,\quad -\infty < x < \infty\ ,$$
$$\pi([0, \infty) \times \{\infty\}) = 0\ . \tag{3.1}$$

Let N_t, $t \geq 0$, be a subset of $N(S)$ defined by

$$N_t = \{\varphi;\ \varphi \in N(S)\ ,\quad \varphi(\{t\} \times (-\infty, \infty]) = 0\}\ ,$$

and let N' be the set of all $\varphi \in N(S)$ satisfying the following four conditions:

(i) $\varphi(\{t\}\times(-\infty, \infty])\leq 1$ for all $t\geq 0$,
(ii) $\varphi([0, \infty)\times\{x\})\leq 1$ for all $x\in(-\infty, \infty)$,
(iii) $\varphi([0, \infty)\times\{\infty\})=0$,
(iv) $\varphi([0, t]\times(-\infty, \infty])=\infty$ for all $t>0$.

Then it is easy to see that $P(N_t)=1$, $t\geq 0$, and $P(N')=1$.

For $s>0$ let T_s be a continuous mapping from S to S which sends (t, x) to $(t/s, x-\log s)$ and (t, ∞) to $(t/s, \infty)$. A mapping from $N(S)$ to $N(S)$ induced by T_s is also denoted by T_s, i.e. $(T_s\varphi)(A)=\varphi(T_s^{-1}A)$, $\varphi\in N(S)$, $A\in\mathfrak{B}(S)$. Each T_s is a measurable mapping from $N(S)$ to itself. It follows from (3.1) and the invariance of π under T_s that P is invariant under T_s:

$$PT_s^{-1}=P \quad \text{for } s>0 . \tag{3.2}$$

For every integer $k\geq 1$ and $\varphi\in N(S)$ let $\zeta^{(k)}=\zeta^{(k)}[\varphi]$ be a function on $(0, \infty)$ defined by

$$\zeta^{(k)}(t)=\sup\{y;\ \varphi([0, t]\times(y, \infty])\geq k\} . \tag{3.3}$$

Write ζ for $\zeta^{(1)}$.

Lemma 3.1. *For each k and $t>0$ the function $\zeta^{(k)}[\cdot](t)$ on $N(S)$ is continuous at $\varphi\in N_t\cap N'$.*

Proof. Let $\varphi\in N_t\cap N'$, $\alpha=\zeta^{(k)}[\varphi](t)$ and $r=\varphi([0, t]\times\{\alpha\})$. Since $\varphi\in N'$, α is finite and $1\leq r\leq k$. For any $\varepsilon>0$ there exist a and b such that $\alpha-\varepsilon<a<\alpha<b<\alpha+\varepsilon$ and $\varphi(A_1)=r$, $\varphi(A_2)=k-r$, $\varphi(\partial A_1)=\varphi(\partial A_2)=0$, where $A_1=[0, t]\times(a, b)$ and $A_2=[0, t]\times(b, \infty]$. Thus by Lemma 2.1 there exists a neighborhood U of φ such that $\psi(A_1)=r$, $\psi(A_2)=k-r$ for $\psi\in U$. This shows that $|\zeta^{(k)}[\psi](t)-\alpha|<\varepsilon$ for $\psi\in U$.

For each $t>0$ the function $\zeta[\cdot](t)$ on $N(S)$ is $\mathfrak{N}(S)$-measurable and finite P-a.e. The stochastic process $\zeta(t)$, $t>0$, is an extremal process introduced by *Dwass* [2] and *Lamperti* [6], i.e. a Markov process such that

$$P\{\varphi;\ \zeta[\varphi](t)\leq y\}=\exp(-te^{-y}) , \tag{3.4}$$

and

$$P\{\zeta(s+t)\leq y\,|\,\zeta(s)=x\}=\begin{cases}\exp(-te^{-y}) , & y\geq x ,\\ 0, & y<x ,\end{cases} \tag{3.5}$$

for $s, t>0$.

Let $D[r, s]$, $0<r<s<\infty$ be the space of right continuous functions on $[r, s]$ with left limits endowed with the Skorohod topology [1]. If $\varphi\in N'$ then the restriction of $\zeta(t)$ on $[r, s]$ is an element of $D[r, s]$ which will also be denoted by $\zeta[\varphi]$. If $\varphi \notin N'$ then it is possible that $\zeta[\varphi](t)$ is not finite for some $t \in [r, s]$. For such a φ let us modify $\zeta[\varphi]$ to be a fixed element of $D[r, s]$.

Lemma 3.2. *The mapping from N to $D[r, s]$ which sends φ to $\zeta[\varphi]$ is continuous at every $\varphi \in N' \cap N_r \cap N_s$.*

Proof. Let $\varphi \in N' \cap N_r \cap N_s$ and $\alpha = \zeta[\varphi](r)$. Let (t_1, x_1) be an atom of φ such that $t_1 < r$, $\zeta[\varphi](t_1) = \alpha = x_1$. Let $t_2 < \cdots < t_k$ be successive jump points of $\zeta[\varphi](t)$ in $[r, s]$ and $x_j = \zeta[\varphi](t_j)$. Then (t_j, x_j), $1 \leq j \leq k$, are atoms of φ contained in $A = [0, s] \times [\alpha - 1, \infty]$. Let $(t_{k+1}, x_{k+1}), \cdots, (t_N, x_N)$ be remaining atoms of φ in A if they exist.

By assumption for given $\varepsilon > 0$ there exist rectangular neighborhoods $A_j = I_j \times J_j$ of (t_j, x_j), $1 \leq j \leq N$, such that $\{I_j\}$ and $\{J_j\}$ are two families of disjoint bounded open intervals of length $< \varepsilon$ and such that every I_j contains neither r nor s.

It follows from Lemma 2.1 that there exists a neighborhood U of φ such that $\psi \in U$ implies $\psi(A_j) = 1$, $1 \leq j \leq N$, and $\psi(A \cap (\bigcup_{j=1}^{N} A_j)) = 0$. Let $\psi \in U$ and let $(t'_j, x'_j) \in A_j$ be an atom of ψ. Then $t'_1 < r < t'_2 < \cdots < t'_k < s$, and $\zeta[\psi](t) = x'_1$ for $r \leq t < t'_2$, $\zeta[\psi](t) = x'_j$ for $t_j \leq t < t_{j+1}$, $2 \leq j \leq k-1$, $\zeta[\psi](t) = x'_k$ for $t'_k \leq t \leq s$.

Let λ be a continuous increasing function on $[r, s]$ such that $\lambda(r) = r$, $\lambda(s) = s$, $\lambda(t_j) = t'_j$, $2 \leq j \leq k$, and linearly interpolated on intervals (r, t_2), $(t_2, t_3), \cdots,$ (t_k, s) respectively. Then we have

$$\sup_{r \leq t \leq s} |\lambda(t) - t| = \sup_{2 \leq j \leq k} |t_j - t'_j| < \varepsilon$$

and

$$\sup_{r \leq t \leq s} |\zeta[\psi](\lambda(t)) - \zeta[\varphi](t)| = \sup_{1 \leq j \leq k} |x_j - x'_j| < \varepsilon .$$

These inequalities prove the lemma.

4. Weak convergence of sample sequences.

Let $X_1, X_2, \cdots$ be i.i.d. random variables on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ with common distribution function F. Assume that there exist two sequences $\{a_n\}$ and $\{b_n\}$, $a_n > 0$, of reals such that

$$\text{(4.1)} \qquad \lim_{n \to \infty} F^n(a_n x + b_n) = \exp(-e^{-x}) , \quad x \in R ,$$

or equivalently

$$\text{(4.2)} \qquad \lim_{n \to \infty} n\{1 - F(a_n x + b_n)\} = e^{-x} , \quad x \in R .$$

For $A \in \mathfrak{B}$ let $\Phi_n(A, \omega)$ be the number of $i \geq 1$ such that $(i/n, (X_i(\omega) - b_n)/a_n) \in A$.

Then for each $\omega\in\Omega$, $\Phi_n(\omega)=\Phi_n(\cdot,\omega)$ is a measure belonging to $N(S)$, and the mapping Φ_n is a measurable mapping from $(\Omega,\mathfrak{F})$ to $(N(S),\mathfrak{N}(S))$. Let $P_n=\mathbf{P}\Phi_n^{-1}$ be the distribution of Φ_n. Then we have

Theorem 4.1. *P_n converges weakly to P.*

Proof. Let $K_1,\cdots,K_m$ be any finite family of disjoint compact intervals of S. By a result of *Kallenberg* [5] it suffices to prove that

$$\lim_{n\to\infty}\mathbf{P}\{\Phi_n(K_1)=a_1,\cdots,\Phi_n(K_m)=a_m\}=P\{\varphi;\varphi(K_1)=a_1,\cdots,\varphi(K_m)=a_m\}, \tag{4.3}$$

where $a_j\ge 0$, $1\le j\le m$, are integers. We prove (4.3) assuming that $K_1=I\times J_1,\cdots,K_m=I\times J_m$ where I is a compact interval of $[0,\infty)$, $J_1=[\alpha_1,\beta_1],\cdots,J_m=[\alpha_m,\beta_m]$ are disjoint compact intervals of $(-\infty,\infty]$. The general case easily follows from this case. Let n' be the number of i such that $i\in nJ$, then $n'/n\to|J|$ as $n\to\infty$. The random vector $(\Phi_n(I\times J_1),\cdots,\Phi_n(I\times J_m),\ \Phi_n(I\times L))$, where $L=(-\infty,\infty]-\bigcup_{j=1}^m J_j$, is multinomially distributed with parameters $n';p_{n1},\cdots,p_{nm},\ p_{n,m+1}$ where $p_{nj}=F(a_n\beta_j+b_n)-F(a_n\alpha_j+b_n)$, $1\le j\le m$, $p_{n,m+1}=1-\sum_{j=1}^m p_{nj}$. Since $\lim_{n\to\infty}n'p_{nj}=(e^{-\alpha_j}-e^{-\beta_j})\cdot|J|$ by (4.2), the joint characteristic function $\{1-\sum_{j=1}^m p_{nj}(1-e^{it_j})\}^{n'}$ of $\Phi_n(I\times J_1),\cdots,\Phi_n(I\times J_m)$ converges to

$$\exp\{-\sum_{j=1}^m(e^{-\alpha_j}-e^{-\beta_j})\cdot|J|\cdot(1-e^{it_j})\}=\int\prod_{j=1}^m e^{it_j\varphi(I_j)}P(d\varphi)\ .$$

This proves (4.3).

Corollary 4.1. *Let $y_1<\cdots<y_k$ then*

$$\lim_{n\to\infty}\mathbf{P}\{(Z_n^{(j)}-b_n)/a_n\le y_j\ ,\quad 1\le j\le k\}=\exp(-e^{-y_k})\ \Sigma^*\prod_{j=1}^k\{(e^{-y_{j+1}}-e^{-y_j})^{a_j}/(a_j!)\}\ , \tag{4.4}$$

where Σ^* denote the summation over all k-tuples $(a_1,\cdots,a_k)$ of non-negative integers such that $a_1\le 1$, $a_1+a_2\le 2,\cdots,\ a_1+\cdots+a_k\le k$.

Proof. The set of discontinuity points of the function $\zeta^{(j)}[\varphi](1)$ of φ has P-measure zero by Lemma 3.1. Since $Z_n^{(j)}(\omega)=\zeta^{(j)}[\Phi_n(\omega)](1)$ it follows from Theorem 4.1 and Theorem 5.1 of [1] that the limit on the left of (4.4) is equal to $P\{\varphi;\zeta^{(j)}[\varphi](1)\le y_j,\ 1\le j\le k\}$ whose value is given by the right side of (4.4).

By Lemma 3.2 the set of discontinuity points of the mapping $\zeta:\varphi\to\zeta[\varphi]\in D[r,s]$ has P-measure zero. Thus it follows from Theorem 4.1 that $\zeta[\Phi_n(\omega)]$

converges in distribution to an extremal process $\zeta[\varphi]$ characterized by (3.4) and (3.5). Since

$$\zeta[\Phi_n(\omega)](t)=Y_n(t,\omega)\equiv(Z_{nt}(\omega)-b_n)/a_n \tag{4.5}$$

we have the following:

Corollary 4.2. (*Lamperti* [6]) *The random element* $Y_n(t,\omega)$ *in* $D[r,s]$ *defined by* (4.5) *converges in distribution to the extremal process characterized by* (3.4) *and* (3.5).

5. A law of the iterated logarithm for a Poisson point process.

Let $S^+=[0,\infty)\times(0,\infty]=\{(t,x);\ 0\le t<\infty,\ 0<x\le\infty\}$. Let K denote a subset of $N(S^+)$ consisting of all $\mu\in N(S^+)$ such that $\int_{S^+}x\mu(dt\,dx)\le 1$. Then K is a compact subset of $N(S^+)$. In fact it is easy to see that K is closed. Applying a standard argument with Riesz's representation theorem one can prove that K is relatively compact. Let $S_m^+=[0,m]\times(0,\infty]\subset S^+$ and let $K^{(m)}=\{\mu\in N(S^+);\ \int_{S_m^+}x\mu(dt\,dx)\le 1+1/m\}$. Then $K^{(m)}$ is closed and $K=\bigcap_{m=1}^{\infty}K^{(m)}$.

For real $s>e$ let T_s^* be a mapping from S to S which sends (t,x) to $(t/s,\ (x-\log s)/\log_2 s)$. A mapping from $N(S)$ to itself induced by T_s^* is also denoted by T_s^*.

For $\varphi\in N(S)$ and $s>e$ let us define $\mu_s=\mu_s[\varphi]\in N(S^+)$ by $\mu_s(A)=(T_s^*\varphi)(A)=\varphi(T_s^{*-1}A)$, $A\in\mathfrak{B}(S^+)$. In this section we prove the following:

Theorem 5.1. *There exists a subset* N_0 *of* $N(S)$ *such that* $P(N_0)=1$ *and for each* $\varphi\in N_0$ *the sequence* $\{\mu_n[\varphi]\}_{n\ge 3}$ *is relatively compact in* $N(S^+)$ *and the set of its limit points coincides with* K.

Lemma 5.1. *Let* $X_a(\varphi)=\int_{S_a^+}x\varphi(dt\,dx)$, *where* $\varphi\in N(S)$, $0<a<\infty$. *Then as* $y\to\infty$ $P\{X_a(\varphi)>y\}=o(e^{-\alpha y})$ *for every fixed* a *and* $\alpha<1$.

Proof. Let $(\tau_1,\xi_1),\cdots,(\tau_\nu,\xi_\nu)$, $0<\tau_1<\cdots<\tau_\nu<a$, be all atoms of φ contained in S_a^+. Then ξ_j's are exponentially distributed with mean one, ν is independent of ξ_j's and Poisson distributed with mean a. Applying the inequality

$$\sum_{n=j+1}^{\infty}\frac{x^n}{n!}\le\frac{x^{j+1}}{(j+1)!}\left\{1+\frac{x}{2!}+\frac{x^2}{3!}+\cdots\right\}=C\frac{x^{j+1}}{(j+1)!},\quad j\ge 0,$$

where $C=(e^x-1)/x$, we have

$$P\{X_a(\varphi)>y\}=P\{\xi_1+\cdots+\xi_\nu>y\}=\sum_{n=0}^{\infty}e^{-a-y}\frac{a^n}{n!}\sum_{j=0}^{n-1}\frac{y^j}{j!}$$
$$=e^{-a-y}\sum_{j=0}^{\infty}\frac{y^j}{j!}\sum_{n=j+1}^{\infty}\frac{a^n}{n!}\leq \text{const.}\times e^{-y}\sum_{j=0}^{\infty}\frac{(ay)^j}{j!(j+1)!}$$
$$=\text{const.}\times e^{-y}(ay)^{-1/2}I_1(2\sqrt{ay})<\text{const.}\times e^{-\alpha y}\,,\quad \alpha<1\,.$$

Lemma 5.2. *Let $b>1$. For P-a.e. φ the sequence $\{\mu_{b^j}[\varphi]\}_{j\geq 1}$ is relatively compact and every limit point of it is contained in K.*

Proof. It is seen that given $\varepsilon>0$ $K^{(m)}$ is contained in the ε-neighborhood of K for large m. Hence it suffices to prove that for every $m\geq 1$ and for P-a.e. φ $\mu_{b^j}[\varphi]\in K^{(m)}$ ultimately. Let $1<\alpha<1+1/m$. By (3.2) and Lemma 5.1 we have

$$P\left\{\int_{S_m^+}x\mu_{b^j}(dt\,dx)>1+1/m\right\}=P\{X_m(\varphi)>(1+1/m)\log_2 b^j\}$$
$$=\text{const.}\times e^{-\alpha\log_2 b^j}=\text{const.}\times j^{-\alpha}\qquad\text{for large } j.$$

Thus by the first Borel-Cantelli lemma we obtain

$$P\{\mu_{b^j}\notin K^{(m)}\ \text{i.o.}\}=0\,.$$

Lemma 5.3. *For any $f\in C_0^+(S^+)$ and for any $\delta>0$ there exists $b_0>1$ such that if $1<b<b_0$ then*

$$P\{\sup_{b^{j-1}<n\leq b^j}|\mu_n f-\mu_{b^j}f|>\delta\quad\text{i.o.}\}=0\,.$$

Proof. Let $a>0$ and $b>1$ be such that $f(t,x)=0$ if either $t>a$ or $x<\log b$. Let $b^{j-1}<n\leq b^j$ then as $j\to\infty$

$$1\leq\log_2 b^j/\log_2 n\leq\log_2 b^j/\log_2 b^{j-1}\to 1\,,$$

and if $x\geq-\log(b^j/n)+(\log b)(\log_2 n)$ then

$$0<x^{-1}\log(b^j/n)\leq(\log_2 b^{j-1}-1)^{-1}\to 0\,.$$

Hence given $\delta>0$ one can choose $b_0>1$ such that for any b, $1<b<b_0$, there exists j_0 satisfying

$$\left|\frac{x+\log(b^j/n)}{x}\,\frac{\log_2 b^j}{\log_2 n}\,f\left(\frac{b^j}{n}t,\ \frac{x+\log(b^j/n)}{\log_2 n}\right)-f\left(t,\ \frac{x}{\log_2 b^j}\right)\right|<\frac{\delta}{3}\tag{5.1}$$

for every $t\geq 0$, $x>0$, $b^{j-1}<n\leq b^j$ and $j\geq j_0$.

Let $1<b<b_0$. By the definition of μ_n and μ_{b^j} we have

$$\sup_{b^{j-1}<n\leq b^j}|\mu_n f-\mu_{b^j}f|\tag{5.2}$$

$$= \sup_{b^{j-1}<n\leq b^j} \left| \int_S \frac{x-\log n}{\log_2 n} f\left(\frac{t}{n}, \frac{x-\log n}{\log_2 n}\right)\varphi(dt\,dx) - \int_S \frac{x-\log b^j}{\log_2 b^j} f\left(\frac{t}{b^j}, \frac{x-\log b^j}{\log_2 b^j}\right)\varphi(dt\,dx)\right| .$$

It follows from (3.2) that (5.2) has the same distribution as

$$\sup_{b^{j-1}<n\leq b^j} \left| \int_S \frac{x+\log (b^j/n)}{\log_2 n} f\left(\frac{b^j}{n}t, \frac{x+\log (b^j/n)}{\log_2 n}\right)\varphi(dt\,dx) - \int_S \frac{x}{\log_2 b^j} f\left(t, \frac{x}{\log_2 b^j}\right)\varphi(dt\,dx)\right| . \tag{5.3}$$

By (5.1), (5.3) is dominated by $(3\log_2 b^j)^{-1}\delta X_a(\varphi)$ for large j. It follows from Lemma 5.1 that

$$P\{(3\log_2 b^j)^{-1}\delta X_a(\varphi)>\delta\}\leq \text{const.}\times j^{-2} .$$

Therefore

$$P\{\sup_{b^{j-1}<n\leq b^j} |\mu_n f-\mu_{b^j} f|>\delta\}\leq \text{const.}\times j^{-2}$$

for large j. Thus the lemma follows from the Borel-Cantelli lemma.

Let d be a metric in $N(S^+)$ defined by (2.1). Then we have from Lemma 5.3 the following:

Lemma 5.4. *For any $\delta>0$ there exists $b>1$ such that*

$$P\{\sup_{b^{j-1}<n\leq b^j} d(\mu_n, \mu_{b^j})>\delta \text{ i.o.}\}=0 .$$

From Lemma 5.2 and Lemma 5.4 we have immediately the following:

Lemma 5.5. *For P-a.e. φ the sequence $\{\mu_n[\varphi]\}_{n\geq 8}$ is relatively compact and the set of its limit points is contained in K.*

For $n\geq 1$ let C_n be the set of points $(j/2^n, k/2^n)\in S^+$, where j and k are positive integers. Let D_n be the set of $\mu\in N(S^+)$ satisfying (i) supp $[\mu]\subset C_n$, (ii) $\mu\{(t, x)\}=1$ for every atom (t, x) of μ, and (iii) $\int_{S^+} x\mu(dt\,dx)=1$. Then $D_\infty=\bigcup_{n=1}^{\infty} D_n$ is a countable dense subset of K.

Lemma 5.6. *For every $\mu\in K$*

$$P\{\liminf_{n\to\infty} d(\mu_n, \mu)=0\}=1 .$$

Proof. It suffices to prove assuming that $\mu\in D_\infty$. Let $(t_1, x_1), \cdots, (t_l, x_l)$ be atoms of $\mu \in D_\infty$ and let $S_{j,\varepsilon}=\{(t, x);\ |t-t_j|<\varepsilon,\ |x-x_j|<\varepsilon\}$. By Lemma 2.2 given $\delta>0$ one can choose $\varepsilon>0$ so small that $S_{j,\varepsilon}$, $1\leq j\leq l$ are disjoint and

$$V_\varepsilon=\{\nu;\ \nu\in N(S^+)\ ,\ \ \nu(S_{j,\varepsilon})=1\ ,\ \ 1\leq j\leq l\ ,\ \text{ and }\ \nu(S^+-\bigcup_{j=1}^{l} S_{j,\varepsilon})=0\}$$
$$\subset \{\nu;\ d(\mu, \nu)<\delta\}\ .$$

Let b be such that $b^{-1/2}+\varepsilon<t_j<b^{1/2}-\varepsilon$ for every j. For each $\varphi\in N(S)$ let us define $\hat{\mu}_n=\hat{\mu}_n[\varphi]\in N(S^+)$ by $\hat{\mu}_n(A)=\mu_n(A\cap S')$, where $S'=(b^{-1/2},\ b^{1/2}]\times(0, \infty]$. Write for simplicity $\nu_j=\mu_{b^j}$ and $\hat{\nu}_j=\hat{\mu}_{b^j}$. Then

$$\{\varphi;\ \hat{\nu}_j[\varphi]\in V_\varepsilon\}=\{\varphi;\ \varphi(B_k^{(j)})=1\ ,\ \ 1\leq k\leq l\ ,\ \ \varphi(B^{(j)}-\bigcup_{k=1}^{l} B_k^{(j)})=0\}\ ,$$

where $B_k^{(j)}=\{(t, x);\ |t/b^j-t_k|<\varepsilon,\ |(x-\log b^j)/\log_2 b^j-x_k|<\varepsilon\}$ and $B^{(j)}=(b^{j-1/2},\ b^{j+1/2}]\times(\log b^j,\ \infty]$. Therefore the events $\{\varphi;\ \hat{\nu}_j[\varphi]\in V_\varepsilon\}$, $j\geq 1$, are independent and

$$P\{\hat{\nu}_j\in V_\varepsilon\}=\exp\{-\pi(B^{(j)}-\bigcup_{k=1}^{l} B_k^{(j)})\prod_{k=1}^{l}\pi(B_k^{(j)})\exp\{-\pi(B_k^{(j)})\}\ .$$

Since $\pi(B_k^{(j)})=2\varepsilon\{(j\log b)^{-x_k+\varepsilon}-(j\log b)^{-x_k-\varepsilon}\}\geq \text{const.}\times j^{-x_k}$ and since $\lim_{j\to\infty}\pi(B^{(j)}-\bigcup_{k=1}^{l} B_k^{(j)})=\lim_{j\to\infty}\pi(B^{(j)})=b^{1/2}-b^{-1/2}$, we have

$$P\{\hat{\nu}_j\in V_\varepsilon\}\geq cj^{-\sum_{k=1}^{l} x_k}=cj^{-1}\ ,$$

with some constant $c>0$. Thus it follows from the second Borel-Cantelli lemma that

$$P\{\hat{\nu}_j\in V_\varepsilon \text{ i.o.}\}=1\ ,$$

and therefore

$$P\{d(\hat{\nu}_j, \mu)<\delta \text{ i.o.}\}=1\ .$$

Since δ was arbitrary we must have

$$\liminf_{j\to\infty} d(\hat{\nu}_j, \mu)=0\ . \tag{5.4}$$

On the other hand by Lemma 5.2 we have for P-a.e.

$$\lim_{j\to\infty} d(\nu_j, K)=0\ . \tag{5.5}$$

Let φ satisfy (5.4) and (5.5). Let $\{\hat{\nu}_{j'}[\varphi]\}$ be a subsequence of $\{\nu_j[\varphi]\}$ tending to μ, and let μ^* be a limit point of $\{\nu_{j'}\}$. Then $\nu_{j'}f\geq\hat{\nu}_{j'}f$, $f\in C_0^+(S^+)$, implies $\mu^*\geq\mu$. Since $\int_{S^+} x\mu^*(dt\,dx)=\int_{S^+} x\mu(dt\,dx)=1$ we must have $\mu^*=\mu$. This proves the lemma.

Proof of Theorem 5.1. Immediate from Lemma 5.5 and Lemma 5.6.

Let $0<r<\infty$ be fixed and let d_r be a metric in $N(S_r^+)$, defined by (2.1) with a countable dense family $\{f_k\}$ of functions in $C_0^+(S_r^+)$. For $\mu\in N(S^+)$ let $(\mu)_r\in N(S_r^+)$ denote the restriction of μ to S_r^+. Then the mapping $\mu\to(\mu)_r$ maps K onto the compact set $K_r=\{\nu\in N(S_r^+); \int_{S_r^+} x\nu(dt\,dx)\le 1\}$ of $N(S_r^+)$. This mapping is continuous at every $\mu\in N'$, where N' is the set of $\mu\in N(S^+)$ such that $\mu\{(r,x);\ 0<x\le\infty\}=0$.

Lemma 5.7. *Let $\{\lambda_n\}$ be a sequence of measures $\lambda_n\in N(S^+)$. If $\{\lambda_n\}$ is relatively compact and if the set of its limit points is K, then the sequence $\{(\lambda_n)_r\}_{n\ge1}$ is relatively compact in $N(S_r^+)$ and the set of its limit points is K_r.*

Proof. By assumption for every $m\ge1$ there exists $n_0=n_0(m)$ such that $\lambda_n\in K^{(m)}$ for $n\ge n_0$. Hence if $n\ge n_0(m)$ then $(\lambda_n)_r\in K_r^{(m)}=\{\nu\in N(S_r^+); \int_{S_r^+} x\nu(dt\,dx) <1+1/m\}$. Since $K_r^{(m)}$ is a compact subset of $N(S_r^+)$, every limit point of $\{(\lambda_n)_r\}$ is in $\bigcap_{m=1}^{\infty} K_r^{(m)}=K_r$. If $\nu\in K_r\cap N_r'$ where N_r' is the set of $\nu\in N(S_r^+)$ such that $\nu\{(r,x); 0<x\le\infty\}=0$, then there exists $\mu\in K\cap N'$ satisfying $(\mu)_r=\nu$. It follows from the assumption that $\liminf_{n\to\infty} d(\lambda_n,\mu)=0$ and therefore $\liminf_{n\to\infty} d_r((\lambda_n)_r,\nu)=0$. Since $K_r\cap N_r'$ is dense in K_r every $\nu\in K_r$ is a limit point of $\{(\lambda_n)_r\}$.

Theorem 5.2. *For each $r>0$ and P-a.e. φ the sequence $\{(\mu_n[\varphi])_r\}_{n\ge3}$ is relatively compact in $N(S_r^+)$ and the set of its limit points is K_r.*

Proof. Immediate from Theorem 5.1 and Lemma 5.7.

Corollary 5.1. *For $r>0$ and P-a.e. φ*

$$\limsup_{n\to\infty}\int_{S_r^+} x(\mu_n[\varphi])_r(dt\,dx)=1\ ,\qquad \liminf_{n\to\infty}\int_{S_r^+} x(\mu_n[\varphi])_r(dt\,dx)=0\ .$$

Proof. The mapping from $N(S_r^+)$ to $[0,\infty]$ which sends ν to $\int_{S_r^+} x\nu(dt\,dx)$ is continuous and maps K_r onto $[0,1]$.

Corollary 5.2. *Let*

$$\tilde\zeta_n^{(j)}=\tilde\zeta_n^{(j)}[\varphi]=(\zeta^{(j)}[\varphi](n)-\log n)/\log_2 n$$

for $\varphi\in N(S)$, $j\ge1$, $n\ge3$, and let $\tilde\zeta_n=(\tilde\zeta_n^{(1)},\cdots,\tilde\zeta_n^{(k)})$. For P-a.e. φ the sequence

$\{\tilde{\zeta}_n\}$ *is relatively compact in* R^k *and the set of its limit points is* $\{(x_1, \cdots, x_k);$ $x_1 \geq \cdots \geq x_k \geq 0,\ \sum_{j=1}^{k} x_j \leq 1\}$. *In particular*

$$\limsup_{n\to\infty} \tilde{\zeta}_n^{(k)} = 1/k\ , \qquad \liminf_{n\to\infty} \tilde{\zeta}_n^{(k)} = 0\ .$$

Proof. For $\nu \in N(S_1^+)$ and $j \geq 1$ let

$$m^{(j)}(\nu) = \inf\{x;\ x \geq 0\ ,\quad \nu([0,1]\times(x,\infty]) \leq j-1\}\ ,$$

and $m(\nu) = (m^{(1)}(\nu), \cdots, m^{(k)}(\nu))$. Then m is a continuous mapping from $N(S_1^+)$ to R^k (see the proof of Lemma 3.1) and $\hat{\zeta}_n[\varphi] = m((\mu_n)_1[\varphi])$, where $\hat{\zeta}_n = (\hat{\zeta}_n^{(1)}, \cdots, \hat{\zeta}_n^{(k)})$, $\hat{\zeta}_n^{(j)} = \max(\tilde{\zeta}_n^{(j)}, 0)$. Thus Theorem 5.2 implies that for P-a.e. φ the sequence $\{\hat{\zeta}_n[\varphi]\}$ is relatively compact in R^k and the set of its limit points is $m(K_1) = \{(x_1, \cdots, x_k);\ x_1 \geq \cdots \geq x_k \geq 0,\ \sum_{j=1}^{k} x_j \leq 1\}$. Since by Lemma 5.8 $\liminf_{n\to\infty} \tilde{\zeta}_n^{(k)}[\varphi] = 0$ for P-a.e. φ we have $\lim_{n\to\infty}(\hat{\zeta}_n^{(k)} - \tilde{\zeta}_n^{(k)}) = 0$, P-a.e. Hence $\{\tilde{\zeta}_n[\varphi]\}$ is relatively compact with the limit point set $m(K_1)$.

Let $0 < r < s < \infty$ and let $F[r,s]$ denote the space of nonnegative bounded right-continuous non-decreasing functions on $[r,s]$ endowed with the Skorohod topology. The same space with the weak topology will be denoted by $F_w[r,s]$. A sequence $\{f_n\}$ converges to f in $F_w[r,s]$ iff $f_n(t)$ converges to $f(t)$ at every continuity point of f. For $\nu \in N(S_s^+)$ let $f = \theta(\nu)$ be a function in $F[r,s]$ defined by $f(t) = \inf\{x;\ x \geq 0,\ \nu([0,t]\times[x,\infty]) = 0\}$, $t \in [r,s]$. The mapping θ from $N(S_s^+)$ to $F[r,s]$ is not continuous. However it is continuous as a mapping to $F_w[r,s]$. The image $\theta(K_s)$ of K_s is the set of $f \in F[r,s]$ satisfying:

(i) the set C of increasing points of f is countable,

(ii) $f(r) + \sum_{x\in C} f(x) \leq 1$.

Let us define $\zeta_n^* = \zeta_n^*[\varphi] \in F_w[r,s]$ by $\zeta_n^*(t) = (\log_2 n)^{-1}(\zeta(nt) - \log n)$, $t \in [r,s]$, $n \geq 3$. Then $(\theta((\mu_n)_s))(t) = \zeta_n^*(t)$, $t \in [r,s]$. Hence we have

Corollary 5.3. *For* P-a.e. φ *the sequence* $\{\zeta_n^*[\varphi]\}$ *is relatively compact in* $F_w[r,s]$ *and the set of its limit points is* $\theta(K_s)$.

Let $h_n = h_n[\varphi]$ be the maximum of jumps of the function $\zeta_n^*(t)$, $t \in [r,s]$. It is easy to derive from Theorem 5.2 the following

Corollary 5.4. *For P-a.e.* φ

$$\limsup_{n\to\infty} h_n[\varphi] = 1, \quad \liminf_{n\to\infty} h_n[\varphi] = 0.$$

6. A law of the iterated logarithm for sample sequences.

Suppose the common distribution function F of independent random variables $X_1, X_2, \cdots$ has the positive derivative $F'(x)$ for all sufficiently large x. Let

$$u(x)=\frac{1-F(x)}{F'(x)}$$

and

$$v(x)=\frac{\{1-F(x)\}\log_2\{1/(1-F(x))\}}{F'(x)}.$$

Define two sequences $\{a_n\}$ and $\{b_n\}$ by

$$F(b_n)=1-n^{-1}\,, \quad a_n=u(b_n)\,. \tag{6.1}$$

Throughout the rest we assume that F is twice differentiable and satisfies

$$\lim_{x\to\infty} v'(x)=0\,. \tag{6.2}$$

The following lemma is a slightly modified form of Lemma 3 of de Haan and Hordijk [4].

Lemma 6.1. *If F satisfies* (6.2) *then*

$$\lim_{n\to\infty}\frac{-\log n-\log\{1-F(b_n+a_n x\log_2 n)\}}{x\log_2 n}=1$$

uniformly on every x-interval of the form $(0, c]$.

Let $Y_j=-\log\{1-F(X_j)\}$. Then $Y_1, Y_2, \cdots$ are independent and there exists x_0 such that $P\{Y_j\geq x\}=e^{-x}$ for $x\geq x_0$. Let $\tau_1, \tau_2, \cdots$ be independently exponentially distributed with mean one and independent of X_n, and let $\sigma_n=\tau_1+\cdots+\tau_n$.

Let

$$\xi_{nj}=\frac{X_j-b_n}{a_n\log_2 n}$$

and

$$\eta_{nj}=\frac{Y_j-\log n}{\log_2 n}=\frac{-\log n-\log\{1-F(b_n+a_n\xi_{nj}\log_2 n)\}}{\log_2 n}\,.$$

Let $P_{nj}=(j/n, \xi_{nj})$ and $Q_{nj}=(\sigma_j/n, \eta_{nj})$ be random points of S. Let us define $\Psi_n(\omega)=\Psi_n(\cdot, \omega)\in N(S^+)$ and $\Psi'_n(\omega)=\Psi'_n(\cdot, \omega)\in N(S^+)$ by $\Psi_n(A)=$no. of $P_{nj}\in A$ and $\Psi'_n(A)=$no. of $Q_{nj}\in A$, $A\in\mathfrak{B}(S^+)$, respectively.

Lemma 6.2. *W.p.*1 *the sequence* $\{\Psi'_n\}$ *is relatively compact in* $N(S^+)$ *and* *the set of its limit points is K.*

Proof. Since for large n the distribution of $\{\Psi'_n\}$ coincides with that of $\{\mu_n\}$ in the preceding section the lemma follows from Theorem 5.1.

Lemma 6.3.

$$\lim_{n\to\infty} d(\Psi_n, \Psi'_n)=0 \quad \text{w.p.1.}$$

Proof. It suffices to prove that for every $f\in C_0^+(S^+)$

$$\lim_{n\to\infty} |\Psi_n f-\Psi'_n f|=0 \quad \text{w.p.1.} \tag{6.3}$$

Let ρ be a metric in S^+ defined by

$$\rho((t_1, x_1),\ (t_2, x_2))=|t_2-t_1|+|e^{-x_2}-e^{-x_1}|\ .$$

Let $F_0=[0, a]\times[c, \infty]$, $0<a, c<\infty$, be such that supp $[f]\subset F_0$ and let $F_1=[0, a+1]\times[c/2, \infty]$. Given $\varepsilon>0$ choose $\delta>0$ so small that $e^{-c/2}-e^{-c}>\delta$ and $|f(P)-f(Q)|<c\varepsilon/4$ if $\rho(P, Q)<\delta$. By the strong law of large numbers and by Lemma 6.1 w.p.l there exists $n_0(\omega)$ such that $\rho(P_{nj}, Q_{nj})<\delta$ for every $n\geq n_0(\omega)$ and $j\leq n(a+1)$. Thus we have $|f(P_{nj})-f(Q_{nj})|<c\varepsilon/4$ for every $n\geq n_0$ and $j\geq 1$. By Corollary 5.2 w.p.l there exists $n_1=n_1(\omega)\geq n_0(\omega)$ such that $\sum_{j=1}^{\infty}\eta_{nj}<2$ for $n\geq n_1$. Therefore if $n\geq n_1$ then the number of $Q_{nj}\in F_1$ is less than $c/4$. Hence the number of j such that $f(P_{nj})-f(Q_{nj})\neq 0$ is less than $4/c$ if $n\geq n_1$. Thus

$$|\Psi_n f-\Psi'_n f|=|\sum_j f(P_{nj})-\sum_j f(Q_{nj})|\leq \sum_j |f(P_{nj})-f(Q_{nj})|$$

$<4/c\cdot c\varepsilon/4=\varepsilon$ for $n\geq n_1$. This proves (6.3).

Theorem 6.1. *W.p.1 the sequence $\{\Psi_n\}$ is relatively compact in $N(S^+)$ and the set of its limit points is K.*

Proof. Apply Lemma 6.2 and Lemma 6.3.

Theorem 6.2. *Let $0<r<\infty$. W.p.1 the sequence $\{(\Psi_n)_r\}$ is relatively compact in $N(S_r^+)$ and the set of its limit points is K_r.*

Proof. Immediate from Theorem 6.1 and Lemma 5.7.

The following corollaries correspond to those in the preceding section and are proved by the same method.

Corollary 6.1. *Let $0<r<\infty$. Then*

$$\limsup_{n\to\infty}\sum_{j=1}^{[nr]}\left(\frac{X_j-b_n}{a_n\log_2 n}\right)^+=1\,,\quad \liminf_{n\to\infty}\sum_{j=1}^{[nr]}\left(\frac{X_j-b_n}{a_n\log_2 n}\right)^+=0\quad \text{w.p.1.}$$

Corollary 6.2. *Let* $\tilde{Z}_n^{(j)}=(Z_n^{(j)}-b_n)/(a_n\log_2 n)$ *and* $\tilde{Z}_n=(\tilde{Z}_n^{(1)},\cdots,\tilde{Z}_n^{(k)})$. *Then w.p.*1 *the sequence* $\{\tilde{Z}_n\}$ *is relatively compact in* R^k *and the set of its limit points is* $m(K_1)$. *In particular*

$$\limsup_{n\to\infty}\tilde{Z}_n^{(k)}=1/k\,,\qquad \liminf_{n\to\infty}\tilde{Z}_n^{(k)}=0\quad \text{w.p.1.}$$

Corollary 6.3. *Let* $Y_n(t)=(Z_{[nt]}-b_n)/a_n$, $t\in[r,s]$, $0<r<s<\infty$. *Then w.p.*1 *the sequence* $\{Y_n/\log_2 n\}$ *is relatively compact in* $F_w[r,s]$ *and the set of its limit points is* $\theta(K_s)$.

Corollary 6.4. *Let* $V_n=\max_{1\le l\le n}(Z_l-Z_{l-1})/(a_n\log_2 n)$. *Then*

$$\limsup_{n\to\infty}V_n=1\,,\qquad \liminf_{n\to\infty}V_n=0\quad \text{w.p.1.}$$

REFERENCES

[1] Billingsley, P.: *Convergence of Probability Measures.* New-York: Wiley 1968.

[2] Dwass, M.: *Extremal processes.* Ann. Math. Statist. 35, 1718–1725 (1964).

[3] Freedman, D.: *Brownian Motion and Diffusion.* San Francisco: Holden Day 1971.

[4] de Haan, L., Hordijk, A.: *The rate of growth of sample maxima.* Ann. Math. Statist. 43, 1185–1196 (1972).

[5] Kallenberg, O.: *Characterization and convergence of random measures and point processes.* Z. Wahrscheinlichkeitstheorie verw. Gebiete 27, 9–21 (1973).

[6] Lamperti, J.: *On extreme order statistics.* Ann. Math. Statist. 35, 1726–1737 (1964).

[7] Pickands III, J.: *Sample sequences of maxima.* Ann. Math. Statist. 38, 1570–1574 (1967).

[8] Pickands III, J.: *The two dimensional Poisson process and extremal processes.* J. Appl. Prob. 8, 745–756 (1971).

[9] Strassen, V.: *An invariance principle for the law of the iterated logarithm.* Z. Wahrscheinlichkeitstheorie verw. Gebiete 3, 211–226 (1964).

[10] Wichura, M. J.: *On the functional form of the law of the iterated logarithm for the partial maxima of independent identically distributed random variables.* Ann. Prob. 2, 202–230 (1974).

Department of Mathematics
Yokohama City University
4646 Mutsuura-cho, Kanazawa-ku,
Yokohama 236, Japan

Department of Applied Mathematics
Yokohama National University
Ooka, Minami-ku
Yokohama 233, Japan