# CLOSED VECTOR MEASURES

By

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## 1. Introduction.

Let $S$ be a set, $R$ a ring of subsets of $S$ with $S \notin R$, $\Sigma$ the algebra generated by $R$, $X$ a Banach space and $m: R \to X$ a set function. Define $\bar{m}: \Sigma \to X$ by $\bar{m}(E)=m(E)$ if $E \in R$ and $\bar{m}(E)=-m(S-E)$ if $S-E \in R$. Then we have

(1) if $m$ is finitely additive, then so is $\bar{m}$.

(2) if $m$ is bounded, then so is $\bar{m}$.

(3) if $m$ is $s$-bounded, then so is $\bar{m}$. (for example, see [2]).

But we shall note that the countable additivity of $m$ does not imply the countable additivity of $\bar{m}$ (Example 2). In this paper we shall discuss the countable additivity of $\bar{m}$. In § 2 we shall introduce the notion of closed measure. In § 3 we shall consider some of its applications.

## 2. Closed measures.

Let $S$ be a set, $R$ a ring of subsets of $S$, $X$ a Banach space and $m: R \to X$ a set function. We define an order $A_1 \leqq A_2$ if and only if $A_1 \subset A_2$ for every sets $A_1, A_2 \in R$. Then $R$ is a directed set with the order $\leqq$.

**Definition 1.** *A set function $m: R \to X$ is called closed if the image set $\{m(A): A \in R\}$ of the directed set $R$ converges in $X$.*

**Proposition 1.** *Let $R$ be a ring of subsets of $S$, $X$ a Banach space and $m: R \to X$ a finitely additive set function. Then the following conditions are equivalent.*

(1) *$m$ is closed.*

(2) *For every number $\varepsilon>0$ there exists a set $E_0 \in R$ such that for every set $E \in R$ with $E \subset S-E_0$ we have $\|m(E)\|<\varepsilon$.*

**Proof.** (1)$\Rightarrow$(2). By hypothesis $x_0=\lim\{m(A): A \in R\} \in X$ exists. Then for every number $\varepsilon>0$ there exists a set $E_0 \in R$ such that for every set $E \in R$ with $E_0 \subset E$ we have $\|m(E)-x_0\|<\varepsilon$. Since for every set $E \in R$ with $E \subset S-E_0$ we have $E_0 \subset E \cup E_0$, $\|m(E \cup E_0)-x_0\|<\varepsilon$. Since $m(E \cup E_0)=m(E)+m(E_0)$ and $\|m(E_0)-x_0\|<\varepsilon$, we have $\|m(E)\|=\|m(E \cup E_0)-m(E_0)\| \leqq \|m(E \cup E_0)-x_0\|+\|x_0-m(E_0)\|<2\varepsilon$.

(2)⇒(1). It is easy to show that $\{m(A): A \in R\}$ is a Cauchy net in $X$. See Oberle ([4] Proposition 2).

**Definition 2.** *A set function $m: R \to X$ is called strongly bounded (s-bounded) if for every sequence $\{E_n\}$ of mutually disjoint sets of $R$ we have*

$$\lim_{n\to\infty} \|m(E_n)\| = 0 .$$

**Proposition 2.** *Let $R$ be a ring of subsets of $S$, $X$ a Banach space and $m: R \to X$ a finitely additive set function. If $m$ is s-bounded, then $m$ is closed.*

**Proof.** If it were false, then there exist a number $\varepsilon > 0$ and an increasing sequence $\{E_n\}$ of sets of $R$ such that $\|m(E_{n+1}) - m(E_n)\| > \varepsilon$ for all $n$. We put $F_n = E_{n+1} - E_n (n=1, 2, \cdots)$. Then $\{F_n\}$ is a mutually disjoint sets of $R$ such that $\|m(F_n)\| > \varepsilon$ for all $n$. Therefore we have a contradiction.

The converse of the above mentioned proposition is not true.

**Example 1.** Let $S$ be the interval $[0, 1]$, $R$ the ring generated by the intervals $(a, b]$ $(0 \leqq a < b \leqq 1)$ and $F_n$ the real valued function defined by

$$F_n(x) = \begin{cases} 2nx & \text{if } 0 \leqq x < 1/2n^{-1} \\ 1 - (2nx - 1) & \text{if } 1/2n^{-1} \leqq x < n^{-1} \\ 0 & \text{if } n^{-1} \leqq x \leqq 1 \qquad (n = 1, 2, \cdots). \end{cases}$$

We put $m_n((a, b]) = F_n(b) - F_n(a)$ $(0 \leqq a < b \leqq 1)$ and $m(A) = (m_n(A))_{n=1}^{\infty}$ for every set $A \in R$. Then $m: R \to c_0$ is finitely additive. Since $\|m((1/2^{n+1}, 1/2^n])\| = |m_{2^n}((1/2^{n+1}, 1/2^n])| = 1$ $(n = 1, 2, \cdots)$, $m$ is not s-bounded. For every number $\varepsilon > 0$ we put $E_0 = (0, 1 - \varepsilon/2]$. Then for every set $E \in R$ with $E \subset S - E_0$ we have $\|m(E)\| \leqq \sup\{\|m(A)\|: A \subset E, A \in R\} = \varepsilon/2$. Therefore $m$ is closed.

**Remark.** Note that if $R$ is a $\delta$-ring, then any closed measure is s-bounded ([4] Proposition 1).

**Proposition 3.** *Let $R$ be a ring of subsets of $S$, $X$ a Banach space and $m: R \to X$ a finitely additive set function. If $m$ is closed and $\sup\{\|m(B)\|: B \subset A, B \in R\} < +\infty$ for every set $A \in R$, then $m$ is bounded.*

**Proof.** Since $m$ is closed, there exists a set $E_0 \in R$ such that for every set $E \in R$ with $E \subset S - E_0$ we have $\|m(E)\| \leqq 1$. There exists a number $M > 0$ such that for every set $E \in R$ with $E \subset E_0$ we have $\|m(E)\| \leqq M$. Since for every set $E \in R$ we have $E = (E \cap E_0) \cup (E - E_0)$, we have $\|m(E)\| \leqq \|m(E \cap E_0)\| + \|m(E - E_0)\|$

$\leqq M+1$. The proof is complete.

**Corollary.** *Let $R$, $X$ and $m: R\to X$ are as in Proposition 3. If $X$ has no subspace isomorphic to $c_0$, then the followings are equivalent.*

(1) $m$ is $s$-bounded.

(2) $m$ is closed and $\sup\{\|m(B)\|: B\subset A,\ B\in R\}<+\infty$ for every set $A\in R$.

(3) $m$ is bounded.

**Proof.** (1)⇒(2)⇒(3). We can prove without the condition "$X$ has no subspace isomorphic to $c_0$".

(3)⇒(1). If it were false, then there exist a number $\varepsilon>0$ and a mutually disjoint sets $\{E_n\}$ of $R$ such that $\|m(E_n)\|>\varepsilon$ for all $n$. Since $X$ has no subspace isomorphic to $c_0$, there exists for every number $K>0$ a finite subsequence $\{E_{n_r}\}$ of $\{E_n\}$ such that $\|\sum_r m(E_{n_r})\|=\|m(\bigcup_r E_{n_r})\|>K$. Since $\bigcup_r E_{n_r}\in R$ and $m$ is bounded, we have a contradiction.

We put $\mathfrak{m}=\{A\subset S$: for every set $E\in R$ we have $E\cap A\in R\}$. Then $\mathfrak{m}$ is an algebra containing $R$. We say that $\mathfrak{m}$ is the locally measurable sets. Note that if $S\in R$, then we have $\mathfrak{m}=R$.

**Theorem 1.** *Let $R$ be a ring of subsets of $S$ with $S\notin R$, $\mathfrak{m}$ the locally measurable sets, $X$ a Banach space and $m: R\to X$ a countably additive set function. If $m$ is closed, then $m$ can be extended to a countably additive set function $m_1: \mathfrak{m}\to X$.*

**Proof.** Let $A$ be any set of $\mathfrak{m}$. It is easy to show that the set $\{m(E\cap A)$: $E\in R\}$ is a Cauchy net in $X$. Since $X$ is complete, define $m_1(A)=\lim\{m(E\cap A)$: $E\in R\}\in X$. The finite additivity of $m_1$ is obvious. We shall prove that $m_1$ is countably additive. Let $\{A_n\}$ be a mutually disjoint sets of $\mathfrak{m}$ such that $A=\bigcup_{n=1}^{\infty} A_n\in\mathfrak{m}$. By definition of $m_1$ there exists for every number $\varepsilon>0$ a set $E\in R$ such that $\|m_1(A)-m(A\cap E)\|<\varepsilon$ and $\|m(E')\|<\varepsilon$ for every set $E'\in R$ with $E'\subset S-E$. Since $A\cap E=\bigcup_{n=1}^{\infty} A_n\cap E\in R$, we have $m(A\cap E)=\sum_{n=1}^{\infty} m(A_n\cap E)$. Then there exists a positive integer $n_0$ such that $\|m(A\cap E)-\sum_{i=1}^{n_0} m(A_i\cap E)\|<\varepsilon$. For each positive integer $i(1\leqq i\leqq n_0)$ there exists a set $E_i\in R$ such that $E\subset E_i$ and $\|m_1(A_i)-m(A_i\cap E_i)\|<(1/n_0)\varepsilon$. $\sum_{i=1}^{n_0}(m(A_i\cap E_i)-m(A_i\cap E))=\sum_{i=1}^{n_0} m(A_i\cap(E_i-E))=m(\bigcup_{i=1}^{n_0}(A_c\cap(E_i-E))$ and $\bigcup_{i=1}^{n_0} A_i\cap(E_i-E)\subset S-E$. Hence we have $\|\sum_{i=1}^{n_0}(m(A_i\cap E_i)-m(A_i\cap E))\|<\varepsilon$. Therefore we have $\|m_1(A)-\sum_{i=1}^{n_0} m_1(A_i)\|\leqq\|m_1(A)-m(A\cap E)\|$

$+\|m(A\cap E)-\sum_{i=1}^{n_0} m(A_i\cap E)\|+\|\sum_{i=1}^{n_0}(m(A_i\cap E)-m(A_i\cap E_i))\|+\sum_{i=1}^{n_0}\|m(A_i\cap E_i)-m_1(A_i)\|<4\varepsilon$. The proof is complete.

We put $\Sigma=\{A\subset S\colon A\in R$ or $S-A\in R\}$. Then $\Sigma$ is the smallest algebra containing $R$ and $\Sigma\subset\mathfrak{m}$. We say that $\Sigma$ is the algebra generated by $R$.

**Proposition 4.** *Let $R$ be a ring of subsets of $S$ with $S\notin R$, $\Sigma$ the algebra generated by $R$, $X$ a Banach space and $m\colon R\to X$ a countably additive set function. If $m$ is closed, then $m$ can be extended to a countably additive set function $\bar{m}\colon \Sigma\to X$. Further $\bar{m}(A)=m(A)$ if $A\in R$ and $\bar{m}(A)=x_0-m(S-A)$ if $S-A\in R$ (where $x_0=\lim\{m(E)\colon E\in R\}\in X$).* ([4] *Proposition* 2).

**Proof.** By Theorem 1 $m$ can be extended to a countably additive set function $m_1\colon \mathfrak{m}\to X$. Let $\bar{m}$ be the restriction of $m_1$ to $\Sigma$. We put $x_0=\lim\{m(E)\colon E\in R\}$. It is easy to show that $\bar{m}(A)=m(A)$ if $A\in R$ $\bar{m}(A)=x_0-m(S-A)$ if $S-A\in R$.

**Example 2.** Let $S$ be the set of all positive integers and $R$ the ring of all finite subsets of $S$. Define $m\colon R\to\{0,1\}$ by $m(A)=1$ if $1\in A\in R$ and $m(A)=0$ if $1\notin A\in R$. Then $m$ is countably additive and $s$-bounded. Let $\Sigma$ be the algebra generated by $R$. Define $\bar{m}\colon \Sigma\to\{-1,0,1\}$ by $\bar{m}(A)=m(A)$ if $A\in R$ and $\bar{m}(A)=-m(S-A)$ if $S-A\in R$. Then $\bar{m}$ is not countably additive. For, let $A_n$ be the singleton set $\{n\}$, $n\in S$ $(n\geqq 2)$ and put $A=\bigcup_{n=2}^{\infty} A_n$. Since $S-A=\{1\}\in R$, we have $A\in\Sigma$ and $\bar{m}(A)=-1$. On the other hand, $\sum_{n=2}^{\infty}\bar{m}(A_n)=\sum_{n=2}^{\infty} m(A_n)=0$.

**Corollary.** *Let $R$, $\Sigma$, $X$ and $m\colon R\to X$ are as in Proposion 4. Suppose that $S\notin R_\sigma$. We define $\bar{m}\colon \Sigma\to X$ by $\bar{m}(A)=m(A)$ if $A\in R$ and $\bar{m}(A)=x_0-m(S-A)$ if $S-A\in R$. Then $\bar{m}$ is countably additive (where $R_\sigma$ is the set of all countable unions of sets of $R$ and $x_0$ is any element of $X$).*

**Proof.** We note that if $A\cap B=\phi$, $A,B\in\Sigma$, then $A\in R$ or $B\in R$. Let $\{A_n\}$ be a mutually disjoint sets of $\Sigma$ such that $A=\bigcup_{n=1}^{\infty} A_n\in\Sigma$.

Case 1. $A_n\in R$ $(n=1,2,\cdots)$ and $A\in R$. It is obvious.

Case 2. $S-A_1\in R$ and $S-A\in R$. Since $(S-A_1)-(S-A)=\bigcup_{n=2}^{\infty} A_n\in R$, we have $m(S-A_1)-m(S-A)=\sum_{n=2}^{\infty} m(A_n)$. Then $\bar{m}(A)=x_0-m(S-A)=x_0-m(S-A_1)+\sum_{n=2}^{\infty} m(A_n)=\sum_{n=1}^{\infty}\bar{m}(A_n)$.

Case 3. $A_n\in R$ $(n=1,2,\cdots)$ and $S-A\in R$. Since $S=(S-A)\cup\bigcup_{n=1}^{\infty} A_n\in R_\sigma$, we have a contradiction.

**3. Applications.**

Let $S$ be a set, $R$ a ring of subsets of $S$, $X$ a Banach space and $m: R \to X$ a finitely additive set function. Let $R_0$ be a subfamily of $R$ such that

(1) $A, B \in R_0 \Rightarrow A \cup B \in R_0$.

(2) $A \in R$, $B \in R_0$ and $A \subset B \Rightarrow A \in R_0$.

Then $R_0$ is a subring and is a directed set with the order $\leqq$ in § 2. The following theorem is a generalization of [5] Theorem 1.

**Theorem 2.** *Let $R$ be a ring of subsets of $S$, $X$ a Banach space and $m: R \to X$ a finitely additive set function. Suppose that $R_0$ be a subfamily of $R$ such that*

(i) $A,\ B \in R_0 \Rightarrow A \cup B \in R_0$.

(ii) $A \in R$, $B \in R_0$ and $A \subset B \Rightarrow A \in R_0$.

*and*

(iii) *the image set $\{m(A): A \in R_0\}$ of the directed set $R_0$ converges in $X$.*

*Then there exist two set function $m_1: R \to X$ and $m_2: R \to X$ such that*

(1) $m = m_1 + m_2$

(2) $A \in R_0 \to m_1(A) = 0$

(3) *for every set $A \in R$ with $m_2(A) \neq 0$ there exists a set $B \in R_0$ such that $B \subset A$ and $m_2(B) \neq 0$.*

(4) *$m_1$ and $m_2$ are finitely additive.*

**Proof.** By the condition (iii) there exists for every number $\varepsilon > 0$ a set $A \in R_0$ such that for every set $B \in R_0$ with $B \subset S - A$ we have $\|m(B)\| < \varepsilon$. Then for every set $E \in R$ the set $\{m(E-B): B \in R_0\}$ is a Cauchy net in $X$, since for every sets $B, C \in R_0$ with $A \subset B$ and $A \subset C$ we have $\|m(E-B) - m(E-C)\| = \|m(E \cap C - B) - m(E \cap B - C)\| \leqq \|m(E \cap C - B)\| + \|m(E \cap B - C)\| < 2\varepsilon$. Similarly, the set $\{m(E \cap B): B \in R_0\}$ is a Cauchy net in $X$. Since $X$ is complete, $\lim\{m(E-B): B \in R_0\} \in X$ and $\lim\{m(E \cap B): B \in R_0\} \in X$ exist. Then we put $m_1(E) = \lim\{m(E-B): B \in R_0\}$ and $m_2(E) = \lim\{m(E \cap B): B \in R_0\}$. By the definition of $m_1$ and $m_2$ the properties (1)—(4) are obvious.

**Theorem 3.** (*The Lebesgue decomposition theorem*). *Let $\gamma$ be a $\sigma$-ring of subsets of $S$, $X$ a Banach space, $m: \gamma \to X$ a countably additive set function and $\mu$ a non-negative measure on $\gamma$. Then there exist unique countably additive set functions $m_1: \gamma \to X$ and $m_2: \gamma \to X$ such that*

(1) $m = m_1 + m_2$

(2) $\mu(A)=0 \Rightarrow m_1(A)=0$

(3) *there exists a locally measurable set* $A$ *such that* $m_2(E\cap A)=0$ *and* $\mu(E-A)=0$ *for every set* $E\in\gamma$.

**Proof.** We put $\gamma_0=\{A\in\gamma: \mu(A)=0\}$. Then $\gamma_0$ satisfies the conditions (i) and (ii) of Theorem 2. Since $m$ is $s$-bounded, $m$ satisfies the condition (iii) of Theorem 2. Then by Theorem 2 there exist $m_1$ and $m_2$ such that

(1′) $m=m_1+m_2$.

(2′) $A\in\gamma_0 \Rightarrow m_1(A)=0$.

(3′) for every set $A\in\gamma$ with $m_2(A)\neq 0$ there exists a set $B\in\gamma_0$ such that $B\subset A$ and $m_2(B)\neq 0$.

(1) and (2) are obvious. The countable additivity of $m_1$ and $m_2$ are obvious (for example, see the proof of Theorem 1). The proof of (3). Using [5] Proposition 2 we can prove that there exists a set $N\in\gamma_0$ such that $m_2(E-N)=0$ for every set $E\in\gamma$. Then we put $A=S-N$. Thus (3) is obvious. The uniqueness of the decomposition is obvious.

**Remark.** 1) In a recent paper, W. M. *Bogdanowicz* and R. A. *Oberle* [3] has given some decomposition theorems for vector measure.

2) Our definition of closed measure is different from Kluvanek's definition (see I. Kluvánek and G. Knowles: Vector Measures and Control Systems, North-Holland, Amsterdam (1976)).

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