ON BIRECURRENT ALMOST PRODUCT AND ALMOST DECOMPOSABLE MANIFOLDS

By

M. P. S. RATHORE

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Abstract: In a recent paper [2], the author and Mishra have defined (1), (1, 2) and (1, 2, 3) birecurrent manifolds in an almost Hermite manifold. In the present paper the author has defined first order generalized birecurrent manifolds for almost product manifold and various properties have been discussed.

1. Introduction.

We consider an n-dimensional manifold M_n of differentiability class c^∞ endowed with a real vector valued linear function F such that for arbitrary vector fields X, Y, Z, etc.

(1.1)a $$\bar{\bar{X}}=X\,,$$

where

(1.1)b $$\bar{X}=F(X)\,.$$

Let us further suppose that in M_n there is given a positive definite Riemannian metric tensor g, such that

(1.2) $$g(\bar{X},\ \bar{Y})=g(X,\ Y)\,.$$

Then M_n is said to be an almost product manifold. Let us put

(1.3) $$'F(X,\ Y)=g(\bar{X},\ Y)\,.$$

Then from (1.1), (1.2) and (1.3), we get

(1.4)a $$'F(X,\ Y)=g(\bar{X},\ Y)=g(X,\ \bar{Y})='F(Y,\ X)\,,$$

that is, $'F$ is symmetric and

(1.4)b $$'F(\bar{X},\ \bar{Y})=g(X,\ \bar{Y})='F(X,\ Y)\,,$$

that is, hybrid.

Let D be a Riemannian connexion in M_n:

(1.5) $$D_XY-D_YX=[X,\ Y]\,,\quad (D_Xg)(Y,\ Z)=0\,.$$

Then if

(1.6)a $$(\nabla F)(Y, X)=0 ,$$

where

(1.6)b $$(\nabla F)(Y, X)=(D_X F)(Y) ,$$

we say that M_n is almost product and almost decomposable.

Let K be the curvature tensor of M_n. Then in an almost product and almost decomposable manifold, we have

(1.7)a $$K(X, Y, \bar{Z})=\overline{K(X, Y, Z)} ,$$

(1.7)b $$K(\bar{X}, \bar{Y}, Z)=K(X, Y, Z) .$$

The Weyl projective curvature tensor W^* is given by

(1.8)a $$W^*(X, Y, Z)=K(X, Y, Z)-\frac{1}{(n-1)}[\mathrm{Ric}(Y, Z)X-\mathrm{Ric}(X, Z)Y] ,$$

where Ric is the Ricci tensor defined by

(1.8)b $$\mathrm{Ric}(Y, Z)=(c_1^1 K)(Y, Z) ,$$

and c_1^1 is the contraction in the first slot.

The almost product and almost decomposable manifold M_n is called Q-recurrent if

(1.9) $$(\nabla Q)(X, Y, Z, U)=\alpha(U)Q(X, Y, Z) ,$$

where Q is any of the curvature tensor and α is the 1-form.

2. Birecurrent almost product manifold.

Definition (2.1). An almost product manifold M_n will be called Q-birecurrent manifold if

(2.1) $$(\nabla\nabla Q)(Z, T, W, X, Y)=b(X, Y)Q(Z, T, W) ,$$

where b is a C^∞ scalar valued function. The manifold M_n will be called Ricci birecurrent manifold if

(2.2) $$(\nabla\nabla\,\mathrm{Ric})(T, W, X, Y)=b(X, Y)\,\mathrm{Ric}(T, W) .$$

The almost product manifold M_n will be called first order Q-birecurrent manifold if

$$(2.3)\qquad (\nabla\nabla Q(\bar{Z}, T, W, X, Y)+(\nabla Q)((\nabla F)(Z, Y), T, W, X)$$
$$+(\nabla Q)((\nabla F)(Z, X), T, W, Y)+Q((\nabla\nabla F)(Z, X, Y), T, W)$$
$$=b(X, Y)Q(\bar{Z}, T, W)\ .$$

Theorem (2.1). *If an almost product manifold is first order projective birecurrent and first order birecurrent (K-birecurrent) manifold for the same scalar valued function b, then it is also Ricci birecurrent manifold provided*

$$(2.4)\text{a}\qquad (\nabla\nabla F)(Z, X, Y)\,\text{Ric}\,(T, W)+(\nabla F)(Z, X)(\nabla\,\text{Ric})(T, W, Y)$$
$$+(\nabla F)(Z, Y)(\nabla\,\text{Ric})(T, W, X)=0\ ,$$

or

$$(2.4)\text{b}\qquad K(Y, X, \bar{Z})=\overline{K(Y, X, \bar{Z})}\ .$$

Proof. From (1.8), we have

$$(2.5)\qquad W^*(\bar{Z}, T, W)=K(\bar{Z}, T, W)-\frac{1}{(n-1)}[\text{Ric}\,(T, W)\bar{Z}-\text{Ric}\,(\bar{Z}, W)T]\ .$$

From the above equation, we have

$$(2.6)\qquad (\nabla\nabla W^*)(\bar{Z}, T, W, X, Y)+(\nabla W^*)((\nabla F)(Z, Y), T, W, X)$$
$$+(\nabla W^*)((\nabla F)(Z, X), T, W, Y)+W^*((\nabla\nabla F)(Z, X, Y), T, W)$$
$$=(\nabla\nabla K)(\bar{Z}, T, W, X, Y)+(\nabla K)((\nabla F)(Z, Y), T, W, X)$$
$$+(\nabla K)((\nabla F)(Z, X), T, W, Y)+K((\nabla\nabla F)(Z, X, Y), T, W)$$
$$-\frac{1}{(n-1)}[(\nabla\nabla F)(Z, X, Y)\,\text{Ric}\,(T, W)+(\nabla F)(Z, X)(\nabla\,\text{Ric})(T,W,Y)$$
$$+(\nabla F)(Z, Y)(\nabla\,\text{Ric})(T, W, X)+(\nabla\nabla\,\text{Ric})(T, W, X, Y)\bar{Z}$$
$$-((\nabla\nabla\,\text{Ric})(\bar{Z}, W, X, Y)+(\nabla\,\text{Ric})((\nabla F)(Z, Y), W, X)$$
$$+(\nabla\,\text{Ric})((\nabla F)(Z, X), W, Y)+\text{Ric}\,((\nabla\nabla F)(Z, X, Y), W))T]\ .$$

Let the manifold M_n be first order birecurrent manifold. Then putting K for Q in (2.3) and contracting it, we get

$$(2.7)\qquad (\nabla\nabla\,\text{Ric})(\bar{Z}, W, X, Y)+(\nabla\,\text{Ric})((\nabla F)(Z, Y), W, X)$$
$$+(\nabla\,\text{Ric})((\nabla F)(Z, X), W, Y)+\text{Ric}\,((\nabla\nabla F)(Z, X, Y), W)$$
$$=b(X, Y)\,\text{Ric}\,(\bar{Z}, W)\ .$$

Let the manifold M_n be first order projective birecurrent and first order birecurrent manifold for the same birecurrence scalar valued function b. Then using (2.3) and (2.7) in (2.6), we get

$$(2.8)\qquad ((\nabla\nabla\,\text{Ric})(T, W, X, Y)-b(X, Y)\,\text{Ric}\,(T, W))\bar{Z}$$

$$+(\nabla\nabla F)(Z, X, Y)\,\mathrm{Ric}\,(T, W)+(\nabla F)(Z, X)(\nabla\,\mathrm{Ric})(T, W, Y)$$
$$+(\nabla F)(Z, Y)(\nabla\,\mathrm{Ric})(T, W, X)=0\,.$$

Making use of (2.4)a in the above equation, we get

$$(\nabla\nabla\,\mathrm{Ric})(T, W, X, Y)=b(X, Y)\,\mathrm{Ric}\,(T, W)\,. \tag{2.9)a}$$

Interchanging X and Y in (2.4)a and subtracting the resulting equation thus obtained from (2.4)a, we get

$$(\nabla\nabla F)(Z, X, Y)-(\nabla\nabla F)(Z, Y, X)=0\,, \tag{2.9)b}$$

which in consequence of Ricci identity implies (2.4)b. Hence the statement.

Theorem (2.2). *For the first order Q-birecurrent almost product manifold with the Killing structure tensor F, we have*

$$(\nabla\nabla Q)(Z, T, W, \bar{X}, \bar{Y})+(\nabla\nabla Q)(Y, T, W, \bar{X}, \bar{Z}) \tag{2.10}$$
$$+(\nabla\nabla Q)(X, T, W, \bar{Z}, \bar{Y})+Q((\nabla\nabla F)(\bar{Y}, \bar{X}, Z), T, W)$$
$$=b(\bar{X}, \bar{Y})Q(Z, T, W)+b(\bar{X}, \bar{Z})Q(Y, T, W)+b(\bar{Z}, \bar{Y})Q(X, T, W)\,.$$

Proof. Since F is Killing:

$$(\nabla F)(X, Z)+(\nabla F)(Z, X)=0\,. \tag{2.11)a}$$

From (2.11)a, we get

$$(\nabla\nabla F)(X, Z, Y)+(\nabla\nabla F)(Z, X, Y)=0\,. \tag{2.11)b}$$

Interchanging Y and Z in (2.3) and then X and Z in (2.3) and adding these two equations thus obtained with (2.3) itself and making use of (2.11)a,b, we get

$$(\nabla\nabla Q)(\bar{Z}, T, W, X, Y)+(\nabla\nabla Q)(\bar{Y}, T, W, X, Z) \tag{2.12}$$
$$+(\nabla\nabla Q)(\bar{X}, T, W, Z, Y)Q(((\nabla\nabla F)(Y, X, Z), T, W)=b(X, Y)Q(\bar{Z}, T, W)$$
$$+b(Z, Y)Q(\bar{X}, T, W)+b(X, Z)Q(\bar{Y}, T, W)$$

Barring X, Y and Z in (2.12) and using (1.1), we get (2.10).

Theorem (2.3). *The first order Q-birecurrent almost product and almost decomposable manifold is Q-birecurrent manifold.*

Proof. From (2.3) and (1.6), we get

$$(\nabla\nabla Q)(\bar{Z}, T, W, X, Y)=b(X, Y)Q(\bar{Z}, T, W)\,.$$

Barring Z in the above equation and using (1.1), we get the result.

Theorem (2.4). *The scalar curvature R in the birecurrent almost product* *manifold M_n is given by*

(2.13)a $$R=((\nabla\nabla R)(X, Y)/b(X, Y),$$

and the birecurrence scalar valued function b is symmetric:

(2.13)b $$b(X, Y)=b(Y, X).$$

Proof. For birecurrent almost product manifold, we get

(2.14)a $$(\nabla\nabla K)(Z, T, W, X, Y)=b(X, Y)K(Z, T, W).$$

Contracting (2.14)a, we get

(2.14)b $$(\nabla\nabla \operatorname{Ric})(Z, W, X, Y)=b(X, Y)\operatorname{Ric}(Z, W).$$

From (2.14)b, we get

(2.14)c $$(\nabla\nabla r)(Z, X, Y)=b(X, Y)r(Z),$$

where

(2.14)d $$\operatorname{Ric}(Y, Z)\overset{\text{def}}{=}g(r(Y), Z).$$

Contracting (2.14)c, we get (2.13)a. Interchanging X and Y in (2.13)a and subtracting the resulting equation from (2.13)a itself, we get (2.13)b.

Theorem (2.5). *In birecurrent almost product and almost decomposable manifold, the 2-form B is hybrid in both the slots:*

(2.15)a $$B(\bar{X}, \bar{Y})=B(X, Y),$$

where

(2.15)b $$B(X, Y)=b(X, Y)-b(Y, X).$$

Proof. Interchanging X and Y in (2.14)c and subtracting the resulting equation thus obtained from (2.14)c and applying Ricci identity, we get

(2.16) $$K(Y, X, r(Z))-r(K(Y, X, Z))=B(X, Y)r(Z).$$

Barring X and Y in (2.16) and using (1.7)b, we get

(2.17) $$K(Y, X, r(Z))-r(K(Y, X, Z))=B(\bar{X}, \bar{Y})r(Z).$$

Comparing (2.16) and (2.17), we get (2.15)a.

Theorem (2.6). *Every recurrent manifold M_n with the 1-form α satisfying*

(2.18) $$(\nabla\alpha)(X, Y)+\alpha(X)\alpha(Y)\not\equiv 0,$$

is a birecurrent manifold M_n but the converse is not true in general.

Proof. From (1.9), we get

$$(\nabla\nabla Q)(Z, T, W, X, Y)=((\nabla\alpha)(X, Y)+\alpha(X)\alpha(Y))Q(Z, T, W) . \tag{2.19}$$

Comparing (2.19) and (2.1), we get

$$b(X, Y)=(\nabla\alpha)(X, Y)+\alpha(X)\alpha(Y) , \tag{2.20}$$

which proves the statement.

Theorem (2.7). *In M_n, 1-form α and 2-form B satisfy the relation:*

$$(\nabla B)(X, Y, Z)=\alpha(Z)B(X, Y) . \tag{2.21}$$

Proof. From (2.16), we have

$$\begin{aligned}(\nabla K)(Y, X, r(T), Z)+K(Y, X, (\nabla r)(T, Z))-(\nabla r)(K(Y, X, T), Z)\\ -r((\nabla K)(Y, X, T, Z))=(\nabla B)(X, Y, Z)r(T)+B(X, Y)(\nabla r)(T, Z) .\end{aligned} \tag{2.22}$$

For a recurrent manifold M_n, we have (1.9), which implies

$$(\nabla r)(T, Z)=\alpha(Z)r(T) . \tag{2.23}$$

Substituting from (2.23) and (1.9) in (2.22) and using (2.16), we get (2.21).

Theorem (2.8). *In the birecurrent manifold M_n, we get*

$$\alpha(\bar{X})B(Y, Z)+\alpha(\bar{Y})B(Z, X)+\alpha(\bar{Z})B(X, Y)=0 . \tag{2.24}$$

Proof. From (2.20), we get

$$B(X, Y)=(\nabla\alpha)(X, Y)-(\nabla\alpha)(Y, X) . \tag{2.25}$$

From (2.25), we get

$$(\nabla B)(X, Y, Z)=(\nabla\nabla\alpha)(X, Y, Z)-(\nabla\nabla\alpha)(Y, X, Z) . \tag{2.26}$$

From (2.26), we have

$$(\nabla B)(X, Y, Z)+(\nabla B)(Y, Z, X)+(\nabla B)(Z, X, Y)=0 . \tag{2.27}$$

From (2.21) and (2.27), we get

$$\alpha(Z)B(X, Y)+\alpha(X)B(Y, Z)+\alpha(Y)B(Z, X)=0 . \tag{2.28}$$

Barring X, Y and Z in (2.28) and using (2.25)a we get (2.24).

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Department of Mathematics
Banaras Hindu University
Varanasi-5, India