ON FRAMED METRIC SUBMANIFOLDS

By

M. P. S. RATHORE and R. S. MISHRA

(Received April 9, 1974)

Abstract: In the present paper, we have obtained the necessary and sufficient conditions for framed metric submanifolds of almost Hermite manifold to posses a certain structure. Also the conditions for framed metric submanifolds of Kähler and almost Tachibana manifolds have been derived. Further the framed umbilical submanifolds have been defined and various results have been discussed.

1. Introduction.

In the sequence, the enveloping manifold V_{2m} will be an almost Hermite manifold with the structure (F, G) satisfying:

(1.1) $$F(F(\underline{X}))+\underline{X}=0$$

(1.2) $$G(F(\underline{X}), F(\underline{Y}))=G(\underline{X}, \underline{Y}) ,$$

for arbitrary vector fields $\underline{X}, \underline{Y}\ \underline{Z}, \underline{U}, \underline{V}$, etc. in V_{2m}.

Let us put

(1.3) $$'F(\underline{X}, \underline{Y})\overset{\text{def}}{=}G(F(\underline{X}), \underline{Y}) .$$

Then $'F$ is skew symmetric. An almost Hermite manifold for which

(1.4) $$(E_{\underline{X}}F)(\underline{Y})=0 ,$$

or

(1.5) $$(E_{\underline{X}}F)(\underline{Y})+(E_{\underline{Y}}F)(\underline{X})=0 ,$$

or

(1.6) $$(E_{\underline{X}}'F)(\underline{Y}, \underline{Z})+(E_{\underline{Y}}'F)(\underline{Z}, \underline{X})+(E_{\underline{Z}}'F)(\underline{X}, \underline{Y})=0 ,$$

or

(1.7) $$(E_{\underline{X}}F)(\underline{Y})+(E_{F(\underline{X})}F)(F(\underline{Y}))=0 ,$$

where E is the Riemannian connexion in V_{2m} is satisfied, is called a Kähler, an almost Tachibana, an almost Kähler and an almost O^*-manifold respectively.

On the other hand, *K. Yano* [3], [4] has introduced the notion of an f-structure on a C^∞-manifold V_n, say $(n=2m-s)$, that is, the notion of a tensor field f of type (1,1) and rank $2m$ satisfying $f^3+f=0$, the existence of which is equivalent to a reduction of the structure group of the tangent bundle to $U(m)\otimes O(s)$. Almost complex ($s=0$) and almost contact ($s=1$) structures are well-known examples of f-structure.

Let V_{2m-s} be a manifold with an f-structure of rank $2m$. If there exist on V_{2m-s} vector field $\underset{x}{t}$, such that if $\underset{x}{a}$ are dual 1-forms, then

$$f(f(X))+X=\sum \underset{x}{a}(X)\underset{x}{t} ,$$

$$\underset{x}{a}(f(X))=0 , \qquad \underset{x}{a}(\underset{y}{t})=\underset{xy}{\delta} , \qquad f(\underset{x}{t})=0 ,$$

where $x, y=1, \cdots, s$. We say that the f-structure has complemented frames and V_{2m-s} is said to be a globally framed f-manifold or simply, a framed manifold [5]. If V_{2m-s} has an f-structure with complemented frames, there exists on V_{2m-s} a Riemannian metric g such that

$$g(X, Y)=g(f(X), f(Y))+\sum \underset{x}{a}(X)\underset{x}{a}(Y) .$$

Then the framed manifold $(f, \underset{x}{t}, \underset{x}{a})$ is called a framed metric manifold $(f, \underset{x}{t}, \underset{x}{a}, g)$. Further we say an f-structure is normal if it has complemented frames and

$$N(X, Y)+\sum \underset{x}{t}\otimes(d\underset{x}{a})(X, Y)=0 ,$$

where N is the Nijenhuis tensor of f. Finally a metric f-structure which is normal and has closed fundamental 2-form $'f$ is said to be a K-structure and V_{2m-s} a K-manifold. Let V_{2m-s} be a submanifold of V_{2m} and let

$$b\colon V_{2m-s}\longrightarrow V_{2m} ,$$

be the inclusion map. The differential db of the map b will be denoted by B, so that to a vector X in V_{2m} there corresponds a vector BX in V_{2m}. If g is the induced metric tensor in V_{2m-s}, we have

(1.8) $$g(X, Y)=(G(BX, BY))\circ b .$$

Let $\underset{x}{N}$ be a system of C^∞ mutually orthogonal unit vector fields to V_{2m-s} at p. Then we have

(1.9) $$\text{a)}\ \ (G(\underset{x}{N}, BX))\circ b=0 , \qquad \text{b)}\ \ (G(\underset{x}{N}, \underset{y}{N}))\circ b=\underset{xy}{\delta} .$$

Let D be the induced Riemannian connexion with respect to V_{2m-s}. Then we have Gauss equations

(1.10) $$E_{BX}BY=BD_XY+\underset{x}{H}(X, Y)\underset{x}{N} ,$$

where $\underset{x}{H}$ are the symmetric bilinear functions in V_{2m-s}. The equations of Weingarten's in V_{2m-s} are given by

(1.11) a) $$E_{BX}\underset{x}{N}=-B'\underset{x}{H}(X)\,,$$
where
(1.11) b) $$g('\underset{x}{H}(X),\,Y)=\underset{x}{H}(X,\,Y)\,.$$

2. Framed metric submanifolds.

Theorem (2.1). *The necessary and sufficient conditions that V_{2m-s} be framed metric submanifold with the structure $(f,\,\underset{x}{t},\,\underset{x}{a},\,g)$ in the almost Hermite manifold V_{2m} are*

(2.1) a) $$F(BX)=B\bar{X}+\underset{x}{a}(X)\underset{x}{N}\,,$$
where
(2.1) b) $$\bar{X}\overset{\text{def}}{=}f(X)\,,$$
and
(2.2) $$F(\underset{x}{N})=-B\underset{x}{t}\,.$$

Proof. Let us put
(2.3) a) $$F(BX)=B\bar{X}+\underset{x}{a}(X)\underset{x}{N}\,,$$
b) $$F(\underset{x}{N})=-B\underset{x}{t}+\underset{xy}{J}\underset{y}{N}\,.$$

Premultiplying (2.3) by F, we have

$$FF(BX)=F(B\bar{X})+\underset{x}{a}(X)F(\underset{x}{N})\,,$$

$$FF(\underset{x}{N})=-F(B\underset{x}{t})+\underset{xy}{J}F(\underset{y}{N})\,.$$

Using (1.1) and (2.3) in the above equations, we have

$$-BX=B\bar{\bar{X}}+\underset{x}{a}(\bar{X})\underset{x}{N}+\underset{x}{a}(X)(-B\underset{x}{t}+\underset{xy}{J}\underset{y}{N})\,,$$

$$-\underset{x}{N}=-B\bar{\underset{x}{t}}-\underset{y}{a}(\underset{x}{t})\underset{y}{N}+\underset{xy}{J}(-B\underset{y}{t}+\underset{yx}{J}\cdot\underset{x}{N})\,.$$

Separating tangential and normal parts from the above equations, we get

(2.4) a) $\bar{\bar{X}}+X=\underset{x}{a}(X)\underset{x}{t}\,,$ b) $\underset{y}{a}(\bar{X})+\underset{xy}{J}\underset{x}{a}(X)=0\,,$

c) $\bar{\underset{x}{t}}+\underset{xy}{J}\underset{y}{t}=0\,,$ d) $\underset{y}{a}(\underset{x}{t})-\underset{xy}{J}\underset{xy}{J}=\underset{xy}{\delta}\,.$

In consequence of (1.2), (1.8), (1.9) and (2.3), we have

(2.5) $$g(X,\,Y)=G(BX,\,BY)\circ b=(G(F(BX),\,F(BY)))\circ b$$
$$=g(X,\,Y)+\underset{x}{a}(X)\underset{x}{a}(Y)\,.$$

(2.4) and (2.5) yield

(2.6) a) $\bar{\bar{X}}+X=\underset{x}{a}(X)\underset{x}{t}$, b) $\underset{x}{a}(\bar{X})=0$,

c) $\underset{x}{\bar{t}}=0$, d) $\underset{y}{a}(\underset{x}{t})=\underset{xy}{\delta}$,

e) $g(\bar{X},\ \bar{Y})=g(X,\ Y)-\underset{x}{a}(X)\underset{x}{a}(Y)$,

if

(2.7) $\underset{xy}{J}=0$.

Putting this value in (2.3)b, we obtained (2.1) and (2.2). Equation (2.6)a-e) show that the conditions (2.1) and (2.2) are sufficient.

To prove that the conditions are necessary, let us put

(2.8) a) $F(BX)=Bh(X)+\underset{x}{c}(X)\underset{x}{M}$, b) $F(\underset{x}{M})=-B\underset{x}{p}$.

Then as above, we have

(2.9) a) $h(h(X))+X=\underset{x}{c}(X)\underset{x}{p}$,

b) $\underset{x}{c}(h(X))=0$, c) $h(\underset{x}{p})=0$,

d) $\underset{x}{c}(\underset{x}{p})=1$,

e) $g(h(X),\ h(Y))=g(X,\ Y)-\underset{x}{c}(X),\ \underset{x}{c}(Y)$.

Substituting from (2.8)b) in

$$G(F(\underset{x}{M}),\ F(\underset{x}{M}))=G(\underset{x}{M},\ \underset{x}{M})=1 .$$

We get

(2.9) f) $g(\underset{x}{p},\ \underset{x}{p})=1$.

In order that equations (2.9) be equivalent to (2.7), we must have

$$h=f\ ,\qquad \underset{x}{c}=\underset{x}{a}\ ,\qquad \underset{x}{p}=\underset{x}{t}\ .$$

Theorem (2.2). *The conditions that* V_{2m-s} *be a framed metric submanifold with the structure* $(f,\underset{x}{t},\underset{x}{a},g)$ *in the Kähler manifold* V_{2m} *are* (2.1), (2.2) *and*

(2.10) a) $D_X\bar{Y}-\underset{x}{a}(Y)'\underset{x}{H}(X)=\overline{D_X Y}-\underset{x}{H}(X,\ Y)\underset{x}{t}$,

b) $H(X,\ \bar{Y})+(D_X\underset{x}{a})(Y)=0$.

Proof. For a Kähler manifold V_{2m}, we have (1.4) which implies

(2.11) a) $(E_{BX}F)(BY)=0$,

or

(2.11) $$\text{b)}\quad (E_{BX}F(BY))=F(E_{BX}BY)\,.$$

Substituting from (2.1) and (2.2) in (2.11)b, we have

$$E_{BX}B\bar{Y}+E_{BX}(\underset{x}{a}(Y)\underset{x}{N})=F(BD_XY+\underset{x}{H}(X,\,Y)\underset{x}{N})\,.$$

By making use of (1.10) and (1.11) ,we have

$$BD_X\bar{Y}+\underset{x}{H}(X,\,\bar{Y})\underset{x}{N}+(D_X\underset{x}{a})(Y)\underset{x}{N}-B'\underset{x}{H}(X)\underset{x}{a}(Y)=\overline{BD_XY}-\underset{x}{H}(X,\,Y)B\underset{x}{t}\,.$$

From the above equation, we get (2.10).

Theorem (2.3). *For a framed metric submanifold V_{2m-s} of Kähler manifold V_{2m}, we have*

(2.12) $$\text{a)}\quad \overline{{}'\underset{x}{H}(X)}=D_X\underset{x}{t}\,,$$

$$\text{b)}\quad \underset{y}{H}(X,\,\underset{x}{t})=\underset{y}{a}({}'\underset{x}{H}(X))\,.$$

Proof. For a Kähler manifold V_{2m} from (2.2), we get

(2.13) $$F(E_{BX}\underset{x}{N})=-E_{BX}B\underset{x}{t}\,.$$

Substituting from (1.10) and (1.11) in (2.13), we have

$$-F(B'\underset{x}{H}(X))=-BD_X\underset{x}{t}-\underset{y}{H}(X,\,\underset{x}{t})\underset{y}{N}\,.$$

Using (2.1)a in this equation, we have

$$B'\overline{\underset{x}{H}(X)}+\underset{y}{a}({}'\underset{x}{H}(X))\underset{y}{N}=BD_X\underset{x}{t}+\underset{y}{H}(X,\,\underset{x}{t})\underset{y}{N}\,.$$

From above equation, we get (2.12)a)b).

Theorem (2.4). *Let us put*

(2.14) $$D_X\underset{x}{t}\overset{\text{def}}{=}\underset{x}{\lambda}f(X)\,.$$

Then for a framed metric submanifold V_{2m-s} of Kähler manifold V_{2m}, we have

(2.15) $$\underset{x}{H}(X,\,Y)=\underset{x}{\lambda}g(X,\,Y)+(\underset{x}{H}(\underset{x}{t},\,\underset{x}{t})-\underset{x}{\lambda})\underset{x}{a}(X)\underset{x}{a}(Y)\,.$$

Such a framed metric submanifold will be called framed umbilicalsu bmanifold.

Proof. Substituting from (2.14) in (2.12)a, we have

$$'\overline{\underset{x}{H}(X)}=\lambda\underset{x}{\bar{X}}\,.$$

Barring the above equation and using (2.6)a, we have

$$-'\underset{x}{H}(X)+\underset{y}{a}('\underset{x}{H}(X))\underset{y}{t}=\lambda(-X+\underset{y}{a}(X)\underset{y}{t})\,,$$

that is,

$$\underset{x}{H}(X,Y)=\underset{x}{\lambda}g(X,Y)+(\underset{y}{a}(\underset{x}{H}(X))-\underset{x}{\lambda}\,\underset{y}{a}(X))\underset{y}{a}(Y)\,.$$

Using (2.12)b and putting $X=\underset{x}{t}$, we get (2.15).

Theorem (2.5). *The conditions that V_{2m-s} be a framed metric submanifold of an almost Tachibana manifold V_{2m} are* (2.1), (2.2) *and*

(2.16) a) $D_X\bar{Y}+D_Y\bar{X}-\underset{x}{a}(Y)'\underset{x}{H}(X)-\underset{x}{a}(X)'H(Y)=\overline{D_XY}+\overline{D_YX}-2\underset{x}{H}(X,Y)\underset{x}{t}\,,$

b) $\underset{x}{H}(X,\bar{Y})+\underset{x}{H}(Y,\bar{X})+(D_X\underset{x}{a})(Y)+(D_Y\underset{x}{a})(X)=0\,.$

Proof. The proof follows the pattern of the proof of theorem (2.2).

Theorem (2.6). *In a framed umbilical submanifold of an almost Tachibana manifold, the induced vector valued linear function f is a conformal Killing tensor when C^∞ vector fields $\underset{x}{t}$ and the dual 1-forms $\underset{x}{a}$ are Killing.*

Proof. Substituting from (2.15) in (2.16), we get

(2.17) a) $(D_Xf)(Y)+(D_Yf)(X)=\underset{x}{\lambda}(\underset{x}{a}(X)Y+\underset{x}{a}(Y)X-2g(X,Y)\underset{x}{t})\,,$

b) $(D_X\underset{x}{a})(Y)+(D_Y\underset{x}{a})(X)=0\,,$

which proves the statement.

Theorem (2.7). *Let V_{2m-s} be a framed metric submanifold of Kähler manifold V_{2m}. Then V_{2m-s} is normal iff V_{2m-s} is framed umbilical submanifold.*

Proof. From (2.10)a), we get

(2.18) $[\overline{X,Y}]\overset{\text{def}}{=}\overline{D_XY}-\overline{D_YX}=D_X\bar{Y}-D_Y\bar{X}-\underset{x}{a}(Y)'\underset{x}{H}(X)+\underset{x}{a}(X)'\underset{x}{H}(Y)\,.$

Also from (2.6)a, we get

(2.19) $D_X\bar{\bar{Y}}=-D_XY+((D_X\underset{x}{a})(Y)+\underset{x}{a}(D_XY))\underset{x}{t}+\underset{x}{a}(Y)D_X\underset{x}{t}\,.$

Consequently from (2.18) and (2.19) after simple manipulation, we get

(2.20) $N(X,Y)+((D_X\underset{x}{a})(Y)-(D_Y\underset{x}{a}))\underset{x}{t}+\underset{x}{a}(X)('\underset{x}{H}(\bar{Y})-D_Y\underset{x}{t})$

$-\underset{x}{a}(Y)('\underset{x}{H}('\bar{X})-D_X\underset{x}{t})=0\,.$

If V_{2m-s} is normal then from (2.20), we get

$$a_x(X)('H_x(\bar{Y})-D_Y t_x)=a_x(Y)('H_x(\bar{X})-D_X t_x) , \tag{2.21}$$

which in consequence of (2.14) yields (2.15). Conversely, if (2.14) and (2.15) hold, then from (2.20), the statement follows.

Theorem (2.8). *Let V_{2m-s} be a framed metric submanifold of Kähler manifold V_{2m}. Then V_{2m-s} is K-manifold iff V_{2m-s} is framed umbilical submanifold.*

Proof. From (2.10), we get

$$(D_X'f)(Y, Z)=g((D_X f)(Y), Z)=a_x(Y)H_x(X, Z)-a_x(Z)H_x(X, Y) . \tag{2.22}$$

Writing two other equations by cyclic permutation of X, Y, Z and adding we get

$$(D_X'f)(Y, Z)+(D_Y'f)(Z, X)+(D_Z'f)(X, Y)=0 . \tag{2.23}$$

From (2.23) and theorem (2.7), the proof follows.

Theorem (2.9). *Let V_{2m-s} be a framed metric submanifold of Kähler manifold V_{2m}. Then V_{2m-s} is cylindrical* [6] *iff f, t_x and a_x are covariant constant with respect to D.*

Proof. As f, t_x and a_x are covariant constant, then from (2.10)a)b) and (2.12)a), we get

$$\text{a)}\quad H_x(X, Y)=H_x(t_x, t_x)a_x(X)a_x(Y) . \tag{2.24}$$

Conversely if (2.24)a hold, that is, V_{2m-s} is cylindrical, then we get

$$\text{b)}\quad 'H_x(X)=H_x(t_x, t_x)a_x(X)t_x . \tag{2.24}$$

Substituting from (2.24)a)b) in (2.10)a)b) and (2.14)a), we get

$$D_X\bar{Y}=\overline{D_X Y} , \qquad (D_X a_x)(Y)=0 , \qquad D_X t_x=0 ,$$

which proves the theorem.

Theorem (2.10). *Let V_{2m-s} be a framed metric submanifold of Kähler manifold V_{2m}. Then V_{2m-s} is pseudo-umbilical* [6] *submanifold provided $H_x(t_x, t_x)=\lambda_x$, where λ_x is defined by* (2.14).

Proof. From (2.15), we get

$$\text{a)}\quad H_x(X, Y)=\lambda_x g(X, Y) , \tag{2.25}$$

where

(2.25) b) $\underset{x}{\lambda}=(C_1^1{}'\underset{x}{H})/2m-s$,

which proves the statement.

Acknowledgement

The authors are grateful to C.S.I.R. for financial assistance and the paper is a part of first author's doctoral dissertation.

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Department of Mathematics
Banaras Hindu University
Varanasi 221005, India