ON SIMILARITIES OF OPERATORS

By

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(Received May 8, 1973)
(Revised December 17, 1974)

The object of the present note is to develop further the ideas initiated by *Embry* [4].

In what follows, by an operator on a complex Hilbert space H, we mean a bounded linear transformation on H. Let $\sigma(A)$, $W(A)$ and $W_e(A)$ denote respectively, the spectrum, the numerical range and the essential numerical range of an operator A. We write $Cl\,W(A)$ to denote the closure of $W(A)$. The notation D will be used for the collection of all operators A such that either $0 \notin W(A)$ or $\sigma(A) \cap \sigma(-A) = \emptyset$. A unitary operator is called a cramed unitary operator if its spectrum is contained in an arc of the unit circle with central angle less than π. If P is a positive operator such that $\langle P_x, x \rangle > 0$ for all x in H, then it is called a strictly positive operator. The operator having zero kernel (or null space) and dense range is defined to be quasi-invertible or quasi-regular. It is obvious that strictly positive operators are quasi-regular. A regular positive operator is said to be positive definite.

In [4], Embry proved the following

Theorem A. *If F and G are commuting normal operators and $AF = GA$ where $0 \notin W(A)$, then $F = G$.*

Our first result will show that under the suitable relaxation of hypothesis in Theorem A, $F-G$ turns out to be compact. Before proving this result, we state the following Lemma whose proof is kindly provided by Professor C. R. Putnam in his private communication to the author. Throughout the present note, α will denote the Borel set in the complex plane.

Lemma. *Let f and g denote the spectral resolutions of normal operators F and G respectively. Then $AF - GA$ is compact whenever $Af(\alpha) - g(\alpha)A$ is compact for each α. However, the converse may not hold.*

Proof. We know that F=uniform limit of (finite) sum $S_F = \sum z_i f(\alpha_i)$ and G= uniform limit of (finite) sum $S_G = z_i g(\alpha_i)$. Then, by our hypothesis, $AS_F - S_G A$ is compact. Consequently, being the uniform limit of compact operators, $AF-GA$ is a compact operator.

To show that the converse may not be true, we produce the following example: Let H be the Hilbert space of square summable sequences. Let $A=I$, $F=\mathrm{diag}\{1, 1/2, 1/3, \cdots\}$ and $G=0$ on H. Then $AF-GA=F$ which is compact. But if $\alpha=\{0\}$, then $Af(\alpha)=0$ and $g(\alpha)A=I$. Thus, for $\alpha=\{0\}$, $Af(\alpha)-g(\alpha)A$ fails to be compact.

Theorem 1. *Let F and G be normal operators with spectral resolutions f and g. If* (1) *$f(\alpha)g(\alpha)-g(\alpha)f(\alpha)$ is compact for each α, and* (2) *there exists an operator A such that $0\notin W_e(A)$ and $AF=GA$, then $F-G$ is compact.*

Proof. We shall use the argument similar to that given in Theorem 1 [4]. Since $AF=GA$, $Af(\alpha)-g(\alpha)A$ for each α and so by (1),

(1*) $p(\alpha)^*Ap(\alpha)$ and $q(\alpha)^*Aq(\alpha)$ are compact for each α, where

(2*) $p(\alpha)=(I-f(\alpha))Af(\alpha)$ and $q(\alpha)=f(\alpha)A(I-f(\alpha))$, are compact.

We claim that $p(\alpha)$ and $q(\alpha)$ are compact for each α. Assume to the contrary that for some α, $p(\alpha)$ fails to be compact. Then, by virtue of [6, Corollary to Theorem 2.5], one can find an orthonormal sequence $\{e_n\}$ $(n=1, 2, 3, \cdots)$ in H such that $\|p(\alpha)e_n\|>M$ for some positive real number M. Let $x_n=p(\alpha)e_n/\|p(\alpha)e_n\|$. Then, since $e_n\to 0$ weakly in H, we have

$$\begin{aligned}|\langle x_n, x\rangle| &= |\langle e_n, p(\alpha)x\rangle|/\|p(\alpha)e_n\| \\ &< (1/M)|\langle e_n, p(\alpha)x\rangle|\to 0 \text{ for each } x.\end{aligned}$$

Thus $x_n\to 0$ weakly in H. Next $|\langle Ax_n, x_n\rangle|=|\langle Ap(\alpha)e_n, p(\alpha)e_n\rangle|/\|p(\alpha)e_n\|^2<(1/M^2)|\langle p(\alpha)^*Ap(\alpha)e_n, e_n\rangle|$. Consequently the compactness of $p(\alpha)^*Ap(\alpha)$ implies $\langle Ax_n, x_n\rangle \to 0$. Thus we are able to find out a sequence $\{x_n\}$ of unit vectors such that $x_n\to 0$ weakly in H and $\langle Ax_n, x_n\rangle\to 0$. But, then in view of [10, Theorem 9], $0\in W_e(A)$, which contradicts our hypothesis and hence we conclude that $p(\alpha)$ is compact for each α. Similarly, it can be shown that $q(\alpha)$ is compact for each α. Therefore, by (2*), $Af(\alpha)-f(\alpha)A$ is compact and hence invoking (2), $(g(\alpha)-f(\alpha))A$ is compact for each α. Since $0\notin W_e(A)$, there exists a compact operator K such that $0\notin Cl\,W(A+K)$ [10, Theorem 9]. If $B=A+K$, then the compactness of $(g(\alpha)-f(\alpha))A$ implies $(g(\alpha)-f(\alpha))B$ is compact for each α. Since B is non-singular, we conclude that $g(\alpha)-f(\alpha)$ is compact for each α and hence an application of Lemma shows that $F-G$ is compact.

Remark. It is now easy to show that in the preceeding result, one can replace the hypothesis "$AF=GA$" by the another hypothesis "$Af(\alpha)-g(\alpha)A$ is compact for each α" without affecting the conclusion.

Next, we shall obtain several results parallel to those established in [4].

Theorem 2. *If for a non-singular operator E, there exists an operator A in D such that $AE=E^{-1}A$, where either E is normal or A is a non-singular normal operator, then $E^2=I$.*

Proof. Since $A\in D$, either $0\notin W(A)$ or $\sigma(A)\cap\sigma(-A)=\emptyset$.

Assume first that $0\notin W(A)$. If E is normal, then the result follows immediately from Theorem A. If A is a non-singular normal operator, then by hypothesis, $E^{-1}A^{-1}=A^{-1}E$. Therefore $AE^{-1}A=AE^2$ and so on substitution $E^{-1}A=AE$, we obtain $A^2E=EA^2$. By [4, Corollary 5], $AE=EA$. Since $AE=E^{-1}A$, we have $E^2=I$.

Next, assume that $\sigma(A)\cap\sigma(-A)=\emptyset$. Obviously then A is non-singular. Again arguing as before, one has $A^2E=EA^2$ and hence by [4], $AE=EA$. This equation together with that given in hypothesis yields $E^2=I$. This finishes the proof of theorem.

Remark. The conclusion of the above result cannot be strengthend to $E=\pm I$. To see this, consider $A=I$ and $E=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

Next, we shall obtain several corollaries of Theorem 2.

Corollary 1. *If for a non-singular operator E, there exists an operator A in D such that $AE^{-1}=E^*A$ and $AE^*=E^{-1}A$, then E is a unitary operator.*

Proof. Since $AE^{-1}=E^*A$, we have

$$(1)\quad AEA^{-1}=(EE^*)A.$$

Now the second equation of hypothesis is equivalent to

$$(2)\quad EA=AE^{*-1}$$

Then by (1) and (2), we have $(EE^*)A=A(EE^*)^{-1}$. An application of Theorem 2 yields $(EE^*)^2=I$ and hence $EE^*=I$ or $E^*=E^{-1}$. This shows that E is unitary.

Recently, U. N. Singh and Kanta Mangla have shown in [9] that if E is a non-singular operator for which there exists a cramed unitary operator A such that $AE^*=E^{-1}A$, then E is unitary. To put this result in a more general form, we prove

Corollary 2. *If for a non-singular operator E, $AE^*=E^{-1}A$ where $A\in D$ and either A is unitary or E is normal, then E is unitary.*

Proof. If A is unitary, then the equation $AE^*=E^{-1}A$ given in hypothesis reduces to $EA=AE^{*-1}$. Multiplying both sides of this by A^* and then taking adjoint, we get $E^*A=AE^{-1}$. If E is normal, then in view of [8], the equation given in hypothesis takes the form $AE=E^{*-1}A$ and hence $E^*A=AE^{-1}$. Thus in either case, Corollary 1 is applicable.

Remark. If $A\notin D$, then Corollary 2 is not valid even if A is unitary or E is normal. To illustrate this point, we sketch a simple example as follows: Let $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $E=\begin{bmatrix}2 & 0\\ 0 & 1/2\end{bmatrix}$. Then $A\notin D$ and $AE^*=E^{-1}A$. However, as $\sigma(E)$ does not lie on the unit circle, E cannot be unitary.

As an immediate consequence of Lemma 4.1 of [2] together with the preceeding corollary, one has the following result which is parallel to that obtained by Beck and Putnam [1].

Corollary 3. *Let A be a quasi-invertible operator with the polar decomposition UP where P is positive and U is a unitary operator such that $U\in D$. If E is a non-singular normal operator such that $AE^*=E^{-1}A$, then E is unitary.*

For an another application of Corollary 2, we prove

Corollary 4. *Let A be a non-singular normal operator with the polar decomposition UP, where P is positive definite and U is a unitary operator in D. If E is a non-singular operator such that $AE^{*n}=E^{-n}A$ for some integer n, then E is similar to a unitary operator.*

Proof. Since P is positive definite, it has the unique positive definite square root Q. Therefore, as A is normal, $UP=PU$ and hence $UQ=QU$. Now, under the hypothesis, $E^{*n}=A^{-1}E^{-n}A=P^{-1}(U^*E^{-n}U)P=Q^{-1}(U^*Q^{-1}E^{-n}QU)Q$. Consequently, $(Q^{-1}E^nQ)^*=U^*(Q^{-1}E^nQ)^{-1}U$. Invoking Corollary 2, we come to the conclusion that $Q^{-1}E^nQ$ is unitary. By [7, Corollary 1], $Q^{-1}EQ$ is similar to a unitary operator and hence the result follows.

In [9], the following result is established.

Theorem B. *If E is a non-singular operator such that $AE^*=E^{-1}A$, where $0\notin Cl\,W(A)$, then E is similar to a unitary operator.*

Here we would like to note that the condition '$0\notin Cl\,W(A)$' cannot be replaced by the weaker condition '$\sigma(A)\cap\sigma(-A)=\emptyset$', even if A is normal. To see this, consider the following example:

Let $A=\begin{bmatrix}1 & -1\\ -1 & 0\end{bmatrix}$ and $E=\begin{bmatrix}2 & 3/4\\ 0 & 1/2\end{bmatrix}$.

Since $\sigma(E)$ is not on the unit circle, E cannot be similar to a unitary operator.

With the disposal of the case $\sigma(A)\cap\sigma(-A)=\emptyset$, it is natural to raise the question whether the another condition '$0\notin W(A)$' is sufficiently strong in Theorem B to guarantee that E is similar to a unitary operator. Here, we are able to solve this question only under the particular situation when E is a spectral operator and A is self adjoint.

Theorem 3. *Let E be a non-singular spectral operator. If $AE^*=E^{-1}A$, where A is self adjoint such that $0\notin W(A)$, then E is similar to a unitary operator.*

Proof. Since $0\notin W(A)$, A is strictly positive. Let Y be the positive square root of A. Then Y is also strictly positive and hence, in particular, it is quasi-invertible. Now the hypothesis $AE^*=E^{-1}A$ implies $EAE^*=A$. Then

$$\|YE^*x\|^2=\langle Y^2E^*x, E^*x\rangle=\langle AE^*x, E^*x\rangle=\langle EAE^*x, x\rangle=\|Yx\|^2\ .$$

In particular, for every x in the domain of Y^{-1}, $\|YE^*Y^{-1}x\|=\|x\|$. Since this domain is dense in H, YE^*Y^{-1} can be extended by continuity to an isometry U on H such that $YE^*=UY$. This equation together with the non-singularity of E yields

$$U(H)=U(\overline{Y(H)})=\overline{UY(H)}=\overline{YE^*(H)}=\overline{Y(\overline{E^*(H)})}=\overline{Y(H)}=H\ ;$$

thus U is onto and hence it turns out to be unitary. Let $Y=U^*$. Then equation $YE^*=UY$ reduces to

(1) $EY=YV$.

Since E is spectral, it has the unique canonical decomposition of the form $N+S$ where N is the scalar part and S is a quasinilpotent operator commuting with N. Applying [6, Theorem 2.2] to (1), we have $N=R^{-1}VR$ for some non-singular operator R and $SY=0$. Since Y is quasi-invertible, $SY=0$ implies $S=0$. Thus $E=N=R^{-1}VR$, which completes the proof.

The author wishes to thank Dr. *B. S. Yadav* for his encouragement and guidance. Also he wishes to express his gratitude to the referee for suggesting some improvement over the original text of this paper.

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