ON A PRIME SURFACE OF GENUS 2 AND HOMEOMORPHIC SPLITTING OF 3-SPHERE

By

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The following considerations are based upon the semi-linear point of view. Throughout this paper a *surface* means a connected closed 2-submanifold in Euclidean 3-space E^3 unless otherwise stated. The author studied the *isotopy sum* of surfaces and its prime decomposition, and proved that the prime decomposition of any surface of genus 2 is unique up to isomorphism [2].

The purpose of this paper is to prove the theorem 1, which gives a necessary and sufficient condition for a surface of genus 2 to be *prime*. Theorem 1 is also an affirmative answer for special case $n=2$ of the conjecture [2, (7, 2)]. In § 4 we discuss homeomorphic splitting of 3-sphere S^3 by a prime surface and prove theorem 2. Finally in § 5 we give an interesting example, which makes clear a little the relation of this field between knot types and homeomorphic complementary domains.

Theorem 1. *Any surface M of genus 2 is prime if and only if either group $\pi_1(\mathrm{Int}M)$ or $\pi_1(\mathrm{Ext}M)$ is indecompesable with respect to free product.*

Corollary to theorem 1. *There is no prime surface of genus 2 in S^3 which separates S^3 homeomorphically.*

Theorem 2. *For any integer $n\neq 2$, there exists a surface of genus n in S^3 which splits S^3 homeomorphically.*

1. Definitions and notations

In this paper we will use the same definitions and notations as [2]. We describe here some of them. The *isotopy sum* $M \# M'$ of two surfaces M and M' is a surface which is connected of M and M' by a thin pipe in natural way [2, definition 1]. Surfaces M and M' are said to be *isomorphic*, denoted by $M \approx M'$, if there exists an isotopy of E^3 throwing M onto M'. A surface M is *trivial* if M is isomorphic to 2-sphere S^2 in E^3. A surface M is *prime* if either M_1 or M_2

is trivial for any decomposition $M \approx M_1 \# M_2$ of M. We use Int$M$ and Ext$M$ to be denoted the closures of bounded and un-bounded components, respectively, of E^3-M for a surface M. Also denote $\mathring{\text{Int}}M=\text{Int}M-M$, $\mathring{\text{Ext}}M=\text{Ext}M-M$ and $\tilde{\text{Ext}}M$ means a one-point compactification of Ext$M$. A surface M is *I-free* (*or E-free*) if $\pi_1(\text{Int}M)$ (or $\pi_1(\text{Ext}M)$, respectively) is a free group.

A loop (simple closed polygonal curve) J on a surface M is said to be I-*unknotted* or E-*unknotted* if J is trivial (i.e. $J \simeq 1$) in Int$M$ or in Ext$M$, respectively, and bi-unknotted if both I- and E- unknotted. By *Dehn's lemma*, an unknotted loop bounds a non-singular proper 2-disk in respective region. Using these termes, we can prove that *a non-trivial surface M is prime if and only if any bi-unknotted loop on M is trivial on M* [2, (3, 2)]. Two sets X and Y of disjoint loops on a surface M are said to be *normal* on M, if the number of points in $X \cap Y$ is minimum with respect to any isotopy of X in M. $\simeq$, $\sim$, and $\cong$ mean homotopic, homologues, and homeomorphic or algebraicaly isomorphic, respectively. ∂X, $\mathring{X}$, and $\bar{X}(cl(X))$ mean boundary, interior, and closure, respectively, of X. $\#(X)$ means the number of connected components of a set X.

Suppose D is a 2-disk and $\mathscr{B}=\{\beta_1, \beta_2, \cdots, \beta_k\}$ is a finite set of disjoint proper arcs in D. $\mathscr{B}$ separates D into interior disjoint 2-disks $D_0, D_1, \cdots, D_k$. We say an arc β_i (and a disk D_j) to be outer most in D if $D_j \cap \mathscr{B}=\partial D_j \cap \mathscr{B}=\beta_i$.

2. Preliminary lemmas

Lemma (2, 1). *If there exist disjoint proper 2-disks D and C in* Int$M$ *and* Ext$M$, *respectively, for a surface M of genus 2 such that $\partial D \not\sim 0$ and $\partial C \not\sim 0$ on M, then M is non-prime.*

Proof. It is obvious that there are disjoint loops J and J' on M such that J and J' are crossing ∂D and ∂C, respectively, at a point and $(J \cup \partial D) \cap (J' \cup \partial C) = \emptyset$. Let $N=N(J \cup \partial D; M)$ and $N'=N'(J' \cup \partial C; M)$ be disjoint regular neighborhoods of $J \cup \partial D$ and $J' \cup \partial C$, respectively, in M. N and N' are both surfaces of genus 1 with connected boundaries. Then $N''=\overline{M-N-N'}$ is an annulus and $\partial N \simeq \partial N'$ on M. Obviously $\partial N \simeq 1$ in Int$M$ and $\partial N' \simeq 1$ in Ext$M$. Hence ∂N is a non-trivial bi-unknotted loop on M and M is non-prime.

Following two lemmas are useful in some special situations. Their proofs are essencially the same to [2, (6. 2)] and [2, (6, 3)], and we drop them here.

Lemma (2, 2). If $\pi_1(\text{Int}M) \cong G_1 * G_2$ is a non-trivial free product, where G_i is indecomposable and $G_i \not\cong Z$, $i=1, 2$, then there exists a unique proper 2-disk D in Int$M$, up to isotopy, so that $\partial D \sim 0$ but $\partial D \not\simeq 1$ on M.

Lemma (2, 3). If $\pi_1(\mathrm{Int}M)\cong G*Z$ (or $\pi_1(\mathrm{Ext}M)\cong G*Z$) is non-trivial free product, where G is indecomposable and $G\not\cong Z$, then there exists a unique proper 2-disk D in $\mathrm{Int}M$ (or in $\mathrm{Ext}M$), up to isotopy, so that $\partial D\not\sim 0$ on M.

For $\mathrm{Ext}M$ we get a unique proper 2-disk D, up to isotopy of ∂D in M, in (2, 2). The isotopy classes of D in $\mathrm{Ext}M$ are just two.

Lemma (2, 4). Suppose both $\pi_1(\mathrm{Int}M)\cong A_1*A_2$ and $\pi_1(\mathrm{Ext}M)\cong A_3*A_4$ are non-trivial free products for a surface M of genus 2. If $A_i\not\cong Z$, $i=1,2$, then M is non-prime.

Proof. From (2, 2) there must be a unique proper 2-disk D in $\mathrm{Int}M$ such that $\partial D\not\simeq 1$ but $\partial D\sim 0$ on M. If $\partial D\simeq 1$ in $\mathrm{Ext}M$, ∂D is a non-trivial bi-unknotted loop on M and M is non-prime. Hence, from now on, we suppose $\partial D\not\simeq 1$ in $\mathrm{Ext}M$.

Let $h{:}D\times I\to \mathrm{Int}M$ be an embedding such that $h(D\times\{1/2\})=D$ and $h(D\times I)\cap M=h(\partial D\times I)$. $\mathrm{Int}M-h(D\times\mathring{I})$ must consist of two components of 3-manifolds, say V_1 and V_2, so that $h(D\times\{0\})\subset\partial V_1$ and $h(D\times\{1\})\subset\partial V_2$. $M_i\equiv\partial V_i$ is an I-nonfree surface of genus 1, $i=1,2$, since we may assume that $\pi_1(\mathrm{Int}M_i)\cong A_i$, $i=1,2$. Let us note that M_1 and M_2 are separated in E^3 (that is, there is a 3-ball B^3 in E^3 such that $\mathrm{Int}M_1\subset\mathring{B}^3$ and $B^3\cap\mathrm{Int}M_2=\emptyset$), for $V_1\cap V_2=\emptyset$ and $A_1\not\cong Z\not\cong A_2$. Then there exists a unique proper 2-disk E_i in $\mathrm{Ext}M_i$ up to isotopy, $i=1,2$. We can take E_1 and E_2 so that $E_1\cap(\mathrm{Int}M_2\cup E_2)=\emptyset=E_2\cap(\mathrm{Int}M_1\cup E_1)$.

If $\mathring{E}_j\cap\mathrm{Int}M=E_j\cap h(D\times I)=\emptyset$, for some $j=1$ or 2, the regular neighborhood $N=N(E_j\cup\mathrm{Int}M_j;E^3)$ of $E_j\cup\mathrm{Int}M_j$ in E^3 is a 3-ball so that $\partial N\cap M$ is a loop which is a boundary of a regular neighborhood of $(M_j\cap M)$ in M. This loop is non-trivial bi-unknotted, so M is non-prime.

Hence we assume that $(\mathring{E}_1\cup\mathring{E}_2)\cap\mathrm{Int}M=(E_1\cup E_2)\cap h(D\times I)$ consists of finite number $\neq 0$ of disjoint 2-disks $H_1, H_2,\cdots, H_m$ which are all isotopic to above D

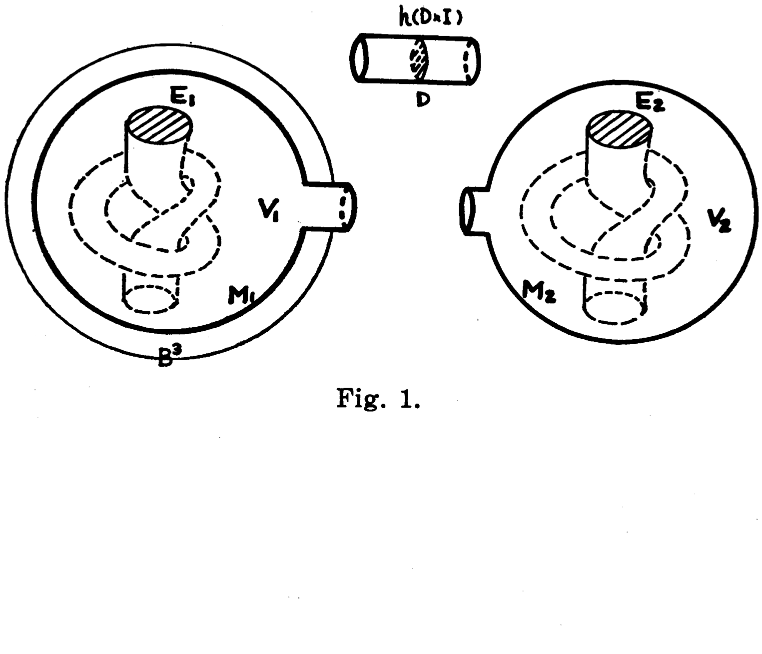

Fig. 1.

in Int$M$. $(\mathring{E}_1 \cup \mathring{E}_2)$ cut Int$M$ into $m+1$ interior disjoint connected 3-manifolds $W_1, W_2, \cdots, W_{m+1}$. W_j's are all 3-balls for $j \neq 1, 2$, and W_i is isotopic to IntM_i in E^3, $i=1, 2$.

On the other hand, from [2, (4, 2)] there exists a proper 2-disk C in Ext$M$ such that $\partial C \not\simeq 1$ but $\partial C \sim 0$ on M. If $\partial D \cap \partial C = \emptyset$ we will get $\partial D \simeq \partial C$ on M as in (2, 1) and this is a contradiction. Hence we suppose that ∂D and ∂C are normal and also ∂C and $\{\partial H_1, \cdots, \partial H_m\}$ are normal on M so that $\partial C \cap \partial H_i$ has a same number of crossing points for all $i=1, 2, \cdots, m$. Then we may assume that $C \cap (E_1 \cup E_2 - h(\mathring{D} \times I))$ consists of finite number$\neq 0$ of disjoint proper arcs $\beta_1, \beta_2, \cdots, \beta_n$, since loops are negligible by the general way as in [2, (4, 6)]. We may also assume that ∂C and $\{\partial E_1, \partial E_2\}$ are normal on M.

Now, if M is prime, there exist D, E_1, E_2, C, and h as above so that $\#\{(\mathring{E}_1 \cup \mathring{E}_2) \cap \text{Int}M\} = \#\{(E_1 \cup E_2) \cap h(D \times I)\} \geqq 2$ is minimum.

β_i's separate C into interior disjoint finite 2-disks $C_1, C_2, \cdots, C_{n+1}$. There must be at least two pairs of outer-most arc and disk, say β_1 and C_1 to be one of them; $C_1 \cap (E_1 \cup E_2) = \partial C_1 \cap E_1 = \beta_1$. Following four cases are considerable for β_1.

(Case 1) β_1 connects H_i and H_j in $\mathring{E}_1 \cap \text{Int}M$, $i \neq j$.

The arc $\overline{\partial C_1 - \beta_1}$ must be contained in $\partial W_k \cap M$ for some k. Let $N = N(C_1 \cup W_k; E^3)$ be a regular neighborhood of $C_1 \cup W_k$ in E^3 relative to E_1 so that $N \cap E_1 = \partial N \cap \mathring{E}_1$ is a regular neighborhood of $(C_1 \cup W_k) \cap E_1 = H_i \cup H_j \cup \beta_1$ in E_1. $E_1' = \overline{(E_1 - N)} \overline{\cup (\partial N - E_1)}$ is a 2-disk so that $\partial E_1' = \partial E_1$ and $E_1' \cap (\text{Int}M_2 \cup E_2) = \emptyset$. It is noted that $\#(E_1' \cap h(D \times I)) \leqq \#(E_1 \cap h(D \times I)) - 2$. Let $E_2' = E_2$, so we have new pair of disks E_1' and E_2' so that $\#\{(E_1' \cup E_2') \cap h(D \times I)\} \leqq \#\{(E_1 \cup E_2) \cap h(D \times I)\} - 2$. (See figure 2).

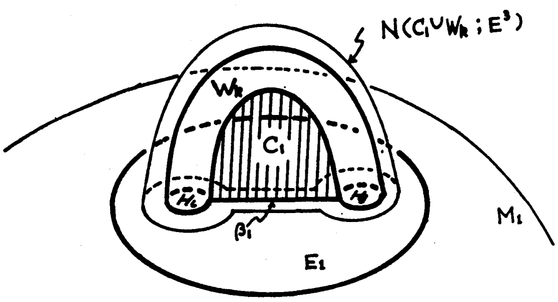

Fig. 2.

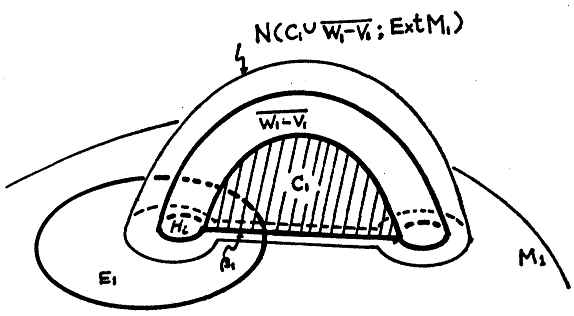

Fig. 3.

(Case 2) β_1 connects some H_i in $\mathring{E}_1 \cap \text{Int}M$ and ∂E_1.

We note that the arc $\overline{\partial C_1 - \beta_1}$ is contained in $\partial W_1 \cap M$ and $\overline{W_1 - V_1}$ is a 3-ball. Let $N = N(C_1 \cup \overline{W_1 - V_1}; \text{Ext}M_1)$ be a regular neighborhood of $C_1 \cup \overline{W_1 - V_1}$ in ExtM_1

relative to E_1 so that $N\cap E_1=\partial N\cap E_1$ is a regular neighborhood of $(C_1\cup\overline{W_1-V_1})\cap E_1=(H_i\cup\beta_1)$ in E_1 relative to ∂E_1. Then $\overline{\partial N-(E_1\cup V_1)}$ is a 2-disk and $\partial N\cap\overline{E_1-\partial N}$ is a proper arc in E_1. $E_1'=\overline{(E_1-N)}\cup\overline{(\partial N-(E_1\cup V_1))}$ is a proper 2-disk in Ext M_1 and $E_1'\cap(V_2\cup E_2)=\emptyset$. Since M_1 is I-nonfree of genus 1 and $\partial E_1'\not\sim 0$, $\partial E_1'$ is isotopic to ∂E_1 in M_1 (also in M). Let $E_2'=E_2$, then it is obvious that $\#\{(E_1'\cup E_2')\cap h(D\times I)\}\leqq\#\{(E_1\cup E_2)\cap h(D\times I)\}-1$. (See figure 3.)

(**Case** 3) $\partial\beta_1$ is contained in some ∂H_j.

The arc $\overline{\partial C_1-\beta_1}$ must be contained in $M\cap\partial W_k$ for some k, but k can not be $3,4,\cdots$, or $m+1$. First we suppose $k=1$ (figure 4-1). $H_j\cup\beta_1$ cuts a 2-disk U off from $\overline{E_1-H_j}$ so that $\partial E_1\cap U=\emptyset$ and $\partial U\subset\beta_1\cup\partial H_j$. $U\cup C_1$ is a proper 2-disk in $E^3-\mathring{W}_1$. It is noted that $\#(E_1\cap h(D\times I))=\#\{(E_1-(H_j\cup U))\cap h(D\times I)\}+\#(\mathring{U}\cap h(D\times I))+1$. We take E_1' by slight deformation of $U\cup C_1$ away from $C\cup\overline{W_1-V_1}$ so that $\partial E_1'\subset M_1$. From the normality of ∂C and ∂E_1, $\partial E_1'\not\sim 0$ on M_1. Hence, let $E_2'=E_2$, then $\#\{(E_1'\cup E_2')\cap h(D\times I)\}\leqq\#\{(E_1\cup E_2)\cap h(D\times I)\}-1$.

Secondary, suppose $k=2$ (figure 4-2). Since W_2 is isotopic to IntM_2 in E^3 and ∂D and ∂C are normal on M, $\partial(U\cup C_1)$ is isotopic to ∂E_2 on $M\cap\partial W_2$. If $\#(E_2\cap h(D\times I))>\#(\mathring{U}\cap h(D\times I))$, we take E_2' by slightly deforming $C_1\cup U$ away from $\overline{W_2-V_2}\cup C$. Then $\partial E_2'\not\sim 0$ on M_2 and $\#(E_2'\cap h(D\times I))=\#(\mathring{U}\cap h(D\times I))<\#(E_2\cap h(D\times I))$. Also if $\#(E_2\cap h(D\times I))\leqq\#(\mathring{U}\cap h(D\times I))$, there is a 3-ball B_0^3 in $E^3-\mathring{W}_1-\mathring{W}_2$ such that $\partial B_0^3\subset U\cup C_1\cup E_2\cup(\partial W_2\cap M)$, since $\partial(U\cup C_1)$ is isotopic to ∂E_2 on $\partial W_2\cap M$. Let $N=N(W_2\cup B_0^3;E^3)$ be a regular neighborhood of $W_2\cup B_0^3$ in E^3 relative to E_1 so that $N\cap E_1=\partial N\cap\mathring{E}_1$ is a regular neighborhood of $U\cup H_j$ in E_1. Set $E_1'=\overline{E_1-(\partial N\cap E_1)}\cup\overline{\partial N-E_1}$. Then $\partial E_1'=\partial E_1$ and $\#(E_1'\cap h(D\times I))=\#\{(E_1-\overline{U\cup H_j})\cap h(D\times I)\}+\#(E_2\cap h(D\times I))\leqq\#(E_1\cap h(D\times I))-1$. Obviously, E_1' is a proper 2-disk, since N is a 3-ball. Also, set $E_2'=E_2$. Hence, $\#\{(E_1'\cup E_2')\cap h(D\times I)\}\leqq\#\{(E_1\cup E_2)\cap h(D\times I)\}-1$.

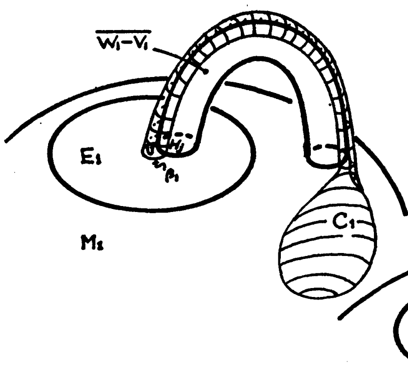

Fig. 4-1.

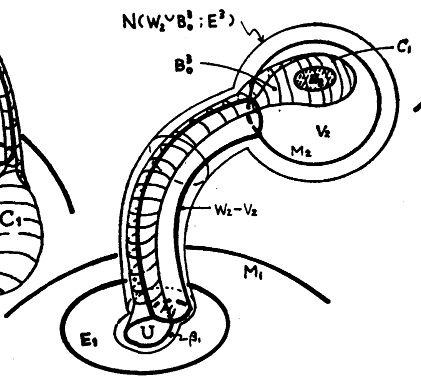

Fig. 4-2.

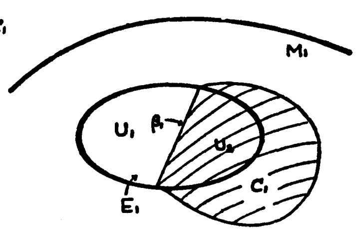

Fig. 5.

(**Case** 4) $\partial\beta_1$ is contained in ∂E_1.

In this case the arc $\overline{\partial C_1-\beta_1}$ is contained in $M\cap M_1$ and β_1 separates E_1 into 2-disks U_1 and U_2 such that $U_1\cup U_2=E_1$ and $U_1\cap U_2=\partial U_1\cap\partial U_2=\beta_1$ (figure 5). It is noted that $\#(E_1\cap h(D\times I))=\#(U_1\cap h(D\times I))+\#(U_2\cap h(D\times I))$. $\partial(U_i\cup C_1)$ is E-unknotted loop on M_1 and non-trivial on M, $i=1, 2$, for ∂E_1 and ∂C are normal on M. If $\partial(U_i\cup C_1)\not\simeq 0$ on M_1, then $\partial(U_j\cup C_1)\simeq\partial D$ on M, $i\neq j$, $i, j=1, 2$. Suppose $\#(U_2\cap h(D\times I))=0$, $\partial(U_2\cup C_1)$ is E-unknotted loop on M and also ∂D is. This is a contradiction. Hence, assume $\#(U_2\cap h(D\times I))\neq 0$. We will take E_1' by slightly deforming $U_1\cup C_1$ away from C. Then $\#(E_1'\cap h(D\times I))=\#(U_1\cap h(D\times I))\leqq\#(E_1\cap h(D\times I))-1$ and $\partial E_1'$ is isotopic to ∂E_1 on $M\cap M_1$.

For each of above cases, we get new pair of 2-disks E_1' and E_2' so that $\#\{(E_1'\cup E_2')\cap h(D\times I)\}<\#\{(E_1\cup E_2)\cap h(D\times I)\}$. If necessary, we can set again loops in normal position on M, where no new intersection appear. This is a contradiction to the minimality of the number $\#\{(E_1\cup E_2)\cap h(D\times I)\}$. Hence the proof of (2, 4) was completed.

Corollary (2, 5). *Suppose* $\pi_1(\mathrm{Int}M)\cong A_1*A_2$ *and* $\pi_1(\mathrm{Ext}M)\cong A_3*A_4$ *are both non-trivial free products for a surface* M *of genus* 2. *Then they are knot groups and at least two of them are infinite cyclic.*

Above corollary is an immediate consequence of (2, 4) but it is very basic to prove theorem 1.

3. Proof of the main theorem

Theorem (3, 1). *If both* $\pi_1(\mathrm{Int}M)$ *and* $\pi_1(\mathrm{Ext}M)$ *are non-trivial free products for a surface* M *of genus* 2, *then* M *is non-prime.*

Proof. From (2, 5) it is sufficient to prove the theorem for the following different four cases;

(3, 2)	$\pi_1(\mathrm{Int}M)\cong Z*Z$	$\pi_1(\mathrm{Ext}M)\cong Z*Z$
(3, 3)	$K*K'$	$Z*Z$
(3, 4)	$Z*K$	$Z*Z$
(3, 5)	$Z*K$	$Z*K'$

where K and K' mean any non-trivial knot groups, since the proof of the theorem for the case ($\pi_1(\mathrm{Int}M)\cong Z*Z$ and $\pi_1(\mathrm{Ext}M)\cong K*K'$) is the same as that for the case (3, 3) and so on.

For case (3, 2) the theorem is a direct consequence of the theorem (*Waldhausen*) [3] [2, § 7], and for case (3, 3) it was proved in (2, 4).

For every case of (3, 4) and (3, 5), from (2, 3) there exists a unique proper 2-disk D in Int$M$ up to isotopy such that $\partial D \not\sim 0$ on M. In case (3, 4), if there is a proper 2-disk A in Ext$M$ such that $\partial A \cap \partial D$ is a crossing point then M is obviously non-prime. Hence we suppose that there is no such a disk as A in Ext$M$.

Let $h: D\times I \to \mathrm{Int}M$ be an embedding of a 3-ball such that $h(D\times\{1/2\})=D$ and $h(D\times I)\cap M=h(\partial D\times I)$. $V_0=\mathrm{Int}M-h(D\times \mathring{I})$ is a 3-manifold with connected boundary $\partial V_0=M_0$ and $\pi_1(\mathrm{Int}M_0)=\pi_1(V_0)\cong K\not\cong Z$. Then there is a unique proper 2-disk E in ExtM_0 up to isotopy such that $\partial E\not\sim 0$ on M_0 and $\partial E\subset M\cap M_0$. If $\mathring{E}\cap\mathrm{Int}M$ $(=E\cap h(D\times\mathring{I}))=\emptyset$, M is non-prime by (2, 1).

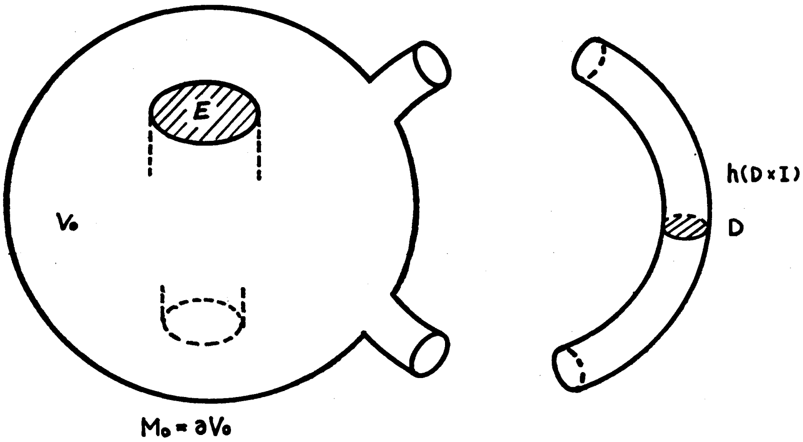

Fig. 6.

Hence we assume that $E\cap h(D\times I)$ consists of finite number $\neq 0$ of disjoint 2-disks which are isotopic to above D in Int$M$. Since $\pi_1(\mathrm{Ext}M)$ has an infinite cyclic group Z as a free factor, there exists a proper 2-disk C in Ext$M$ such that $\partial C\not\sim 0$ on M. If $\partial C\cap\partial D=\emptyset$ M is also non-prime by (2, 1). Hence we may assume that ∂C and $\partial D\cup\partial E$ are normal on M, and $C\cap E$ consists of finite number $\neq 0$ of arcs.

Now it is sufficient to prove the following;

(3, 6) *Suppose,* (i) $\mathring{E}\cap\mathrm{Int}M$ *consists of finite number of disjoint* 2*-disks* $H_1, H_2, \cdots, H_m$, *which are proper in* Int$M$, *such that some* H_i's *of them are isotopic to above* D *and the others* H_j's *are isotopic to each other and* $\partial H_j \not\simeq 1$ *on* M; $\partial D\simeq\partial H_i\not\sim 0$, $\partial H_j\sim 0$ *on* M,

(ii) ∂C *and* $\{\partial E, \partial H_1, \partial H_2, \cdots, \partial H_m\}$ *are normal on* M, *and*

(iii) $C\cap E=C\cap(E-\mathring{\mathrm{I}}\mathrm{nt}M)$ *consists of finite number* $\#(C\cap E)$ *of disjoint arcs* $\beta_1, \beta_2, \cdots, \beta_n$.

Then we can take another proper 2-disk E' in ExtM_0 *with* $\partial E' \subset M \cap M_0$ *and* $\partial E' \not\sim 0$ *on* M *such that* E' *has properties* (i), (ii) *and* (iii) *as above but* $\#(\mathring{E}' \cap \operatorname{Int} M) < \#(\mathring{E} \cap \operatorname{Int} M)$.

The H_j's in the above will appear by an operation for (case 1) in the later. The proof of (3, 6) will proceed as same as that of (2, 4) but slightly modified.

Proof of (3, 6). As (2, 4) $\mathring{E}$ separates Int$M$ into finite number of connected 3-manifolds $W_0, W_1, \cdots, W_p$, so that W_0 is isotopic to Int$M_0 = V_0$ in E^3 and the others are all 3-balls. β_i's also separate C into finite number of 2-disks $C_1, C_2, \cdots, C_{n+1}$, whose interiors are mutually disjoint. Then there must be a pair of outer most arc and 2-disk, say β_1 and C_1, so that $C_1 \cap E = \partial C_1 \cap E = \beta_1$. Hence following four cases are considerable as in (2, 4).

(**Case 1**) β_1 connects distinct components H_i and H_j in E.

In this case, the arc $\overline{\partial C_1 - \beta_1}$ must be contained in $\partial W_k \cap M$ for some k. If $\partial(\partial W_k \cap M) = \partial H_i \cup \partial H_j$ and $k \neq 0$, we take an operation as same as (case 1) in (2, 4), and we will get a desired 2-disk E'. Hence we suppose now that $\partial(\partial W_k \cap M) - \partial H_i - \partial H_j \neq \emptyset$ or $k = 0$. (If $k=0$, $\partial H_i \not\sim 0$ on M for all $i = 1, 2, \cdots, m$.)
Let $N = N(C_1 \cup H_i \cup H_j; E^3)$ be a regular neighborhood of $C_1 \cup H_i \cup H_j$ in E^3 relative to E so that $N \cap E = \partial N \cap E$ is a regular neighborhood of $\beta_1 \cup H_i \cup H_j$ in $\mathring{E}$ and $N \cap W_k$ is a regular neighborhood of $\overline{\partial C_1 - \beta_1} \cup H_i \cup H_j$ in W_k. N is a 3-ball. We take newly $E' = \overline{E - N} \cup \overline{\partial N - E}$. Note that $\overline{(\partial N - E) \cap W_k}$ is a proper 2-disk in Int$M$ whose boundary is non-trivial on M, $\partial E' = \partial E$ and $\mathring{E}' \cap \operatorname{Int} M = \{(\mathring{E} \cap \operatorname{Int} M) - H_i - H_j\} \cup \overline{(\partial N - E) \cap W_k}$. Obviously E' satisfy the conditions (i), (ii) and (iii), and $\#(\mathring{E}' \cap \operatorname{Int} M) \leqq \#(\mathring{E} \cap \operatorname{Int} M) - 1$. (See figure 7).

In the following three cases, we may assume that $\overline{\partial C_1 - \beta_1} \subset \partial W_0 \cap M$. For, if $\partial H_i \sim 0$ on M then the proofs are the same as cases 2 and 3 in (2, 4). So, in cases 2 and 3 there are 3-balls V_i and V_j in $h(D \times I)$ and intersection H_j such that $\overline{\partial V_i - M} \subset H_i \cup h(D \times \{0\})$ and $\overline{\partial V_j - M} \subset H_j \cup h(D \times \{1\})$. (If necessary we can change h to h': $D \times I \to \operatorname{Int} M$ with $h(\partial D) \simeq h'(\partial D)$ on M.)

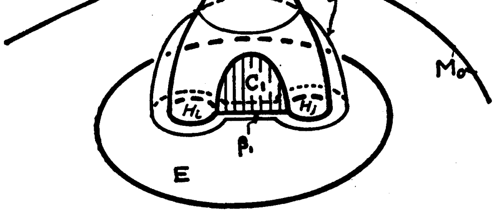

Fig. 7.

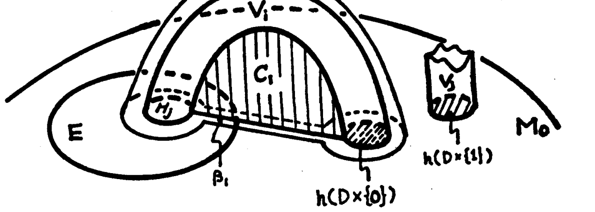

Fig. 8.

(**Case** 2) β_1 connects some H_i and ∂E.

Let $N=N(C_1\cup V_i;\,\mathrm{Ext}M_0)$ be a regular neighborhood of $C_1\cup V_i$ in $\mathrm{Ext}M_0$ relative to E so that $N\cap E=\partial N\cap E$ is a regular neighborhood of $(C_1\cup V_i)\cap E=(\beta_1\cup H_i)\cap E$ in E relative to ∂E (figure 8). The rest of proof is the same as that of case 2 in (2, 4).

(**Case** 3) $\partial\beta_1\subset\partial H_i$

$\beta_1\cup H_i$ cuts a 2-disk U off from $\overline{E-H_i}$ so that $\partial E\cap U=\emptyset$ and $\partial U\subset\beta_1\cup\partial H_i$. Since ∂C and ∂D are normal on M, $\partial(C_1\cup U)\not\simeq 1$ on M. If $\partial(C_1\cup U)\not\sim 0$ on $\partial(V_i\cup V_0)$ the proof is also the same as case 3 in (2, 4). Suppose $\partial(C_1\cup U)\sim 0$ on $\partial(V_i\cup V_0)$, then $\partial(U\cup C_1)$ bounds a 2-disk A on $\partial(V_i\cup V_0)-H_i$ and $\mathring{A}\supset h(D\times\{1\})$. Hence $\sharp(\mathring{U}\cap h(D\times I))\geqq 1$. (figure 9). $U\cup C_1\cup A$ is a non-singular 2-sphere in $\mathrm{Ext}\{\partial(V_i\cup V_0)\}$ and it bounds a 3-ball B^3_* in $\mathrm{Ext}\{\partial(V_i\cup V_0)\}$.
Let $N=N(B^3_*;\,E^3)$ be a regular neighborhood of B^3_* in E^3 relative to E so that $N\cap E=\partial N\cap\mathring{E}$ is a regular neighborhood of U in $\mathring{E}$ and that $N\cap\partial(V_i\cup V_0)$ is a regular neighborhood of A in $(V_i\cup V_0)$. Note that $\overline{\partial N-E}$ is a 2-disk and $\overline{\partial N-E}\cap\mathrm{Int}\,M=N\cap\partial(V_i\cup V_0)$. Now we set newly $E'=\overline{E-N}\cup\overline{\partial N-E}$. Then $\partial E'=\partial E$ and $\sharp(\mathring{E}'\cap\mathrm{Int}\,M)=\sharp(E\cap h(D\times I))-\sharp(\mathring{U}\cap h(D\times I))\leqq\sharp(E\cap\mathrm{Int}\,M)-1$.

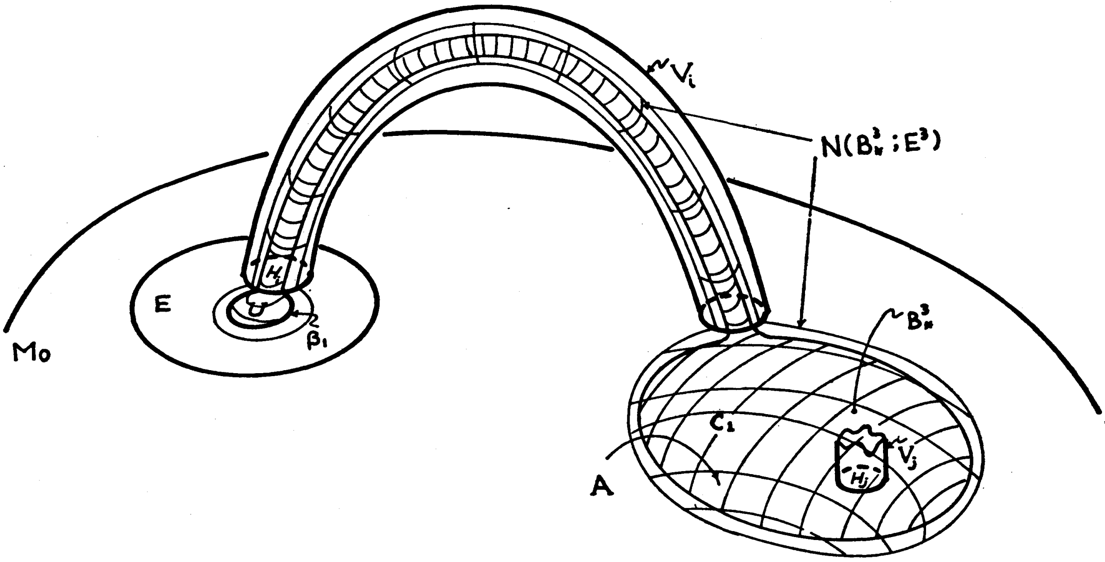

Fig. 9.

(**Case** 4) $\partial\beta_1\subset\partial E$.

β_1 cuts E into two 2-disks U_1 and U_2 so that $U_1\cup U_2=E$ and $U_1\cap U_2=\partial U_1\cap\partial U_2=\beta_1$. Since $\partial E\not\sim 0$ and $\partial(U_i\cup C_1)\not\simeq 1$ on M, one of $\partial(U_1\cup C_1)$ and $\partial(U_2\cup C_1)$, say $\partial(U_1\cup C_1)$, is not null-homologues and another is null-honologues; $\partial(U_1\cup C_1)\not\sim 0$

and $\partial(U_2\cup C_1)\sim 0$ on M_0. Hence $\partial(U_2\cup C_1)\simeq 1$ on M_0 and $\partial(U_2\cup C_1)$ must bound a 2-disk A on M_0. Since the normality of ∂C and ∂E, $\mathring{A}\cap h(D\times\partial I)\neq\emptyset$ and $\#(\mathring{U}_2\cap h(D\times I))\geqq 1$. And the rest of proof is the same as case 4 in (2, 4).

Hence these complete the proof of (3, 6) and so that of (3, 1). If a surface M is non-prime then obviously both $\pi_1(\mathrm{Int}M)$ and $\pi_1(\mathrm{Ext}M)$ are non-trivial free products. So, theorem 1 is trivial from (3, 1).

4. Homeomorphic splitting of S^3 by a prime surface

Now we will extend the primeness to surfaces in 3-manifolds. Suppose V and M be a 3-manifold and its boundary component, a loop L on M is said to be V-unknotted if L bounds a non-singular proper 2-disk in V. Let N be a connected orientable 3-manifold with boundary (may empty) and M be a surface in $\mathring{N}$ which separates N into two components V and W so that $V\cup W=N$ and $V\cap W=\partial V=M$.

Definition (4, 1). *A surface M is said to be prime if L is trivial on M for any V- and W- unknotted (abbreviately, bi-unknotted) loop L such that $L\sim 0$ in M.*

This definition coincides with the primeness defined before for surfaces in E^3. Through the natural inclusion $i: E^3\to S^3$ such that $S^3-i(E^3)=\infty$ is an infinite point of E^3, the phenomena of surfaces in S^3 are similar to that in E^3. In this section we will consider with prime surfaces in S^3.

There is only one isotopy class of prime surfaces of genus 1 which split S^3 homeomorphically. For such a property of prime surfaces of genus greater than 1, following is a special case which is easily derived from theorem 1.

Corollary (4, 2). *There is no prime surface of genus 2 in S^3 which separates S^3 homeomorphically.*

Proof. Suppose a surface M of genus 2 splits S^3 homeomorphically. $S^3-p\cong E^3$ for any point p in S^3-M, and we can consider M a surface in E^3. From [2, (4, 3)] there is a non-trivial I- or E- unknotted loop, say I-unknotted, and $\pi_1(\mathrm{Int}M)$ is non-trivial free product. Hence $\pi_1(\mathrm{Ext}M)$ is so, since $\pi_1(\mathrm{Int}M)\cong\pi_1(\mathrm{Ext}M)$. By theorem 1 M is non-prime and also non-prime in S^3.

For prime surfaces of genus greater than 2, we have following.

Theorem 2. *For any positive integer $n\neq 2$, there exists a prime surface of genus n which separates S^3 homeomorphically.*

The theorem is obviously true for $n=1$ and we prove the theorem by constructing examples. At first we recall a group theoretic theorem which is derived from the "*Kurosh subgroup theorem*" [1, p. 246].

Proposition (4, 3). *If A and B are indecomposable groups and H and K are non-trivial normal subgroups of A and B, respectively, such that $H \cong K$, then the free product of A and B with amalgamation $H(\cong K)$ is also indecomposable.*

The auther wish to thank *S. Suzuki* who informed me the existence of this proposition.

Lemma (4, 4). *There exists a proper embedding $g: I \to H_n$ of an arc I into a solid torus H_n of genus n for any n such that $\pi_1(H_n - g(I))$ is indecomposable.*

Proof. Figure 2 in [2] is an example satisfying above conditions for $n=1$. We will proceed by induction on genus n.
Suppose $g_1: I \to H_1$ and $g_n: I \to H_n$ are embeddings of arc such that $\pi_1(H_1 - g_1(I))$ and $\pi_1(H_n - g_n(I))$ are both indecomposable. Let D_1 and D_n be 2-disks on boundaries ∂H_1 and ∂H_n, respectively, such that $g_1(\{1\}) = \mathring{D}_1 \cap g_1(I)$, $g_n(\{0\}) = \mathring{D}_n \cap g_n(I)$ and $hg_1(\{1\}) = g_n(\{0\})$ for an orientation reversing homeomorphism $h: D_1 \to D_n$. Then $H_1 \underset{h}{\cup} H_n \cong H_{n+1}$ and $g_1(I) \underset{h}{\cup} g_n(I)$ is a proper arc in H_{n+1} so that $\pi_1(H_{n+1} - (g_1(I) \underset{h}{\cup} g_n(I))) \cong \pi_1(H_1 - g_1(I)) * \pi_1(H_n - g_n(I)) / \langle \partial D_1 \rangle$, where $\langle \partial D_1 \rangle$ means a normal subgroup generated by a homotopy class of ∂D_1 which is infinite cyclic (figure 10). Hence by (4, 3) $\pi_1(H_{n+1} - (g_1(I) \underset{h}{\cup} g_n(I)))$ is indecomposable.

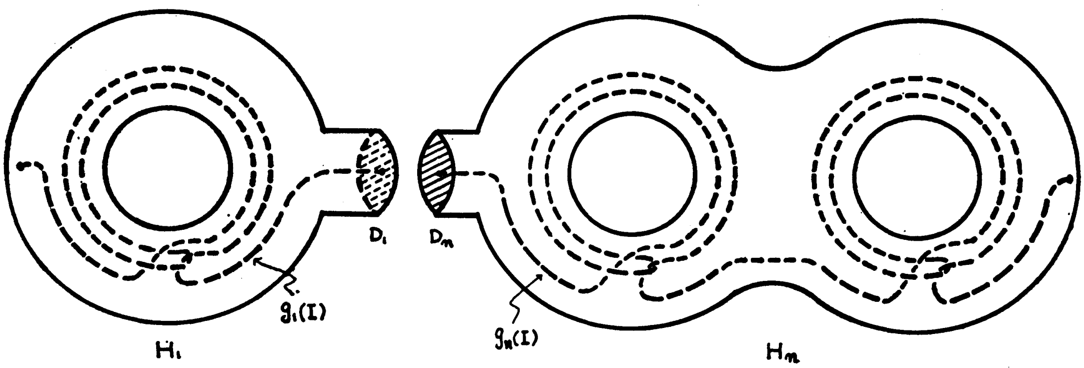

Fig. 10.

Proof of theorem 2. Let $g_n: I \to H_n$ be an embedding of an arc I into solid torus H_n of genus n which satisfies (4, 4), and let $g'_n: I \to H'_n$ be a copy of g_n and H_n. We choose a homeomerphism $\varnothing: \partial H_n \to \partial H'_n$ so that $(H_n \underset{\varnothing}{\cup} H'_n) \cong S^3$ and $\varnothing g_n(\partial I) \cap g'_n(\partial I) = \varnothing$. Let $N = N(g_n(I); H_n)$ and $N' = N(g'_n(I); H'_n)$ be regular neighborhoods of $g_n(I)$ and $g'_n(I)$ in H_n and H'_n, respectively. Set $V = (\overline{H_n - N}) \underset{\varnothing}{\cup} N'$. Obviously

$V \cong S^3 - \mathring{V}$ and $M \equiv \partial V$ is of genus $n+2$. It will complete the proof if we show M prime. For this, note that $\pi_1(V) \cong \pi_1(H_n - g_n(I)) * Z$, where $\pi_1(H_n - g_n(I))$ is indecomposable. We can see M a surface in E^3 by removing a point from $S^3 - M$. Suppose M has a non-trivial decomposition $M \approx M_1 \# M_2$, then the genus of M_1 or M_2 is 1, say the genus of M_1 equal 1. Hence $\pi_1(\mathrm{Int} M_2) \cong \pi_1(\mathrm{Ext} M_2) \cong \pi_1(H_n - g_n(I))$, which is indecomposable. It means that there is no I- or E-unknotted loop on M_2. This is a contradiction and completed the proof of theorem 2. Figure 11 shows an example of this theorem for $n=5$.

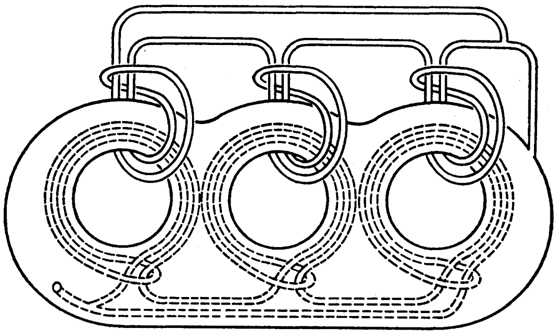

Fig. 11.

5. Homeomorphic complements and isomorphism

It is a problem if knot types of two knots K and K' are the same, for $S^3 - K \cong S^3 - K'$. Analogous problem for links have been solved negatively. The following example (figure 12) show the situation of surfaces in E^3 for above problem.

It is obvious that $\mathrm{Int} M_1 \cong \mathrm{Int} M_2$ are solid tori of genus 3 for surfaces M_1 and M_2 in figure 12. Also $\mathrm{Ext} M_1 \cong \mathrm{Ext} M_2$ and M_2 is non-prime. We will show M_1

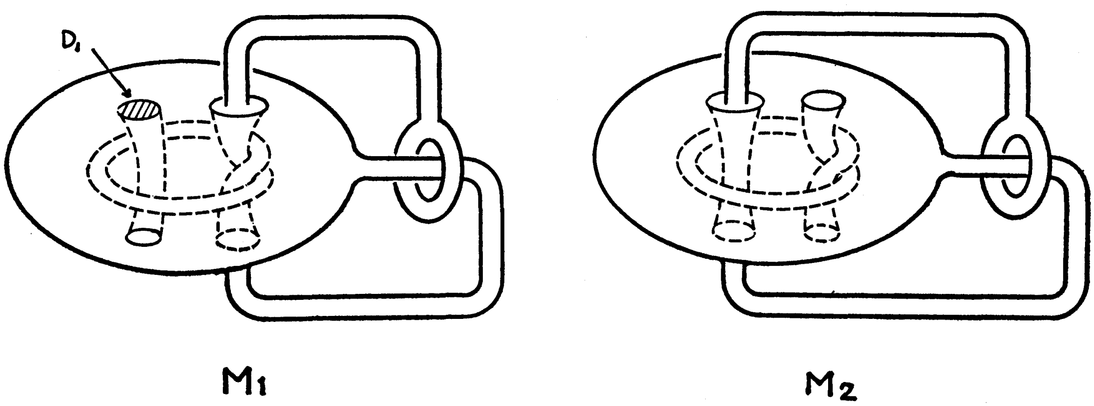

Fig. 12.

prime. It is noted that $\pi_1(\mathrm{Ext}M_1)\cong\pi_1(\mathrm{Ext}H_2)*Z$, where H_2 is "*Homma's example*" of genus 2 [2, p. 99], and $\pi_1(\mathrm{Ext}H_2)$ is indecomposable. By (2, 3) non-trivial E-unknotted loop of M_1 is unique up to isotopy. Hence, if M_1 is non-prime, $M_1\approx M_1'\#T$, where T is a unique bi-free surface of genus 1, and $M_1'\approx\partial(\mathrm{Int}M_1\cup N(D_1;\mathrm{Ext}M_1))$. ($D_1$ is as in the figure and $N(D_1;\mathrm{Ext}M_1)$ is a regular neighborhood.) But we can see from the figure that $(\mathrm{Int}M_1\cup N(D_1;\mathrm{Ext}M_1))$ is not a solid torus. This is a contradiction to $\pi_1(\mathrm{Int}M_1')\cong Z$ and M_1 is prime. Hence, M_1 and M_2 are not isomorphic.

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