# FAKE SURFACES WHICH ARE SPINES OF 3-MANIFOLDS

By

Hiroshi Ikeda

(Received April 25, 1973)

## 1. Introduction

In this paper, we use the definitions and notations introduced in [1], [2] and [3]. For example, for a fake surface $P$ without $\mathfrak{S}_7(P)$ (for the numbering of the singularities of $P$, we use the definition made in [2]), let $B(P)$ and $+B(P)$ denote the set of singular block bundles over $P$ with fiberset $\Phi^1$ and the subset of $B(P)$ consisting of orientable 3-manifolds, respectively. Singular block bundles are defined in [2] and we obtained the following theorem in [2].

**Theorem.** *$B(P)$ consists of 3-manifolds, if it is non-empty.*

What we try to do in this paper is to give answers to the following two problems.

**Problem 1.** *How can we obtain a characterization of the fake surfaces which are spines of 3-manifolds?*

**Problem 2.** *How can we obtain a characterization of the fake surfaces which are spines of orientable 3-manifolds?*

In § 2, we study about block bundles over 2-manifolds and review some lemmas which are already proved.

In § 3, we obtain an answer to Problem 1 in Theorem 1, that is, a necessary and sufficient condition for closed fake surfaces to be spines of 3-manifolds

In § 4, we get a necessary and sufficient condition for closed fake surfaces to be spines of orientable 3-manifolds in Theorem 2 and Theorem 3 which gives an answer to Problem 2.

The author thanks all the members of All Japan Combinatorial Topology Study Group for many discussions.

## 2. Block bundles over 2-manifolds

The following proposition is already stated in [3].

**Proposition 1.** *For a 2-manifold $M$, $+B(M)$ consists of exactly one element.*

In this section, we prove another proposition without which we can not obtain an answer to Problem 1.

**Definition 2.** Let $P$ be a fake surface with boundary $\dot{P}=Q_1\cup\cdots\cup Q_n$, where $Q_i$ means a connected component of $\dot{P}$. Suppose that $P$ is contained in a 3-manifold $V$ properly. Then, we say that $Q_i$ is an *irregular boundary* of $P$ in $V$ if a regular neighborhood $N(Q_i, \dot{V})$ of $Q_i$ in $\dot{V}$ is non-orientable and is a *regular* one of $P$ in $V$ if $N(Q_i, \dot{V})$ is orientable. By $\mu(P, V)$, we denote the number of the irregular boundaries of $P$ in $V$.

**Proposition 2.** *Let $M$ be a 2-manifold with boundary $\dot{M}=b_1\cup\cdots\cup b_n$. Then, there exists an element $\eta$ in $B(M)$ such that $b_i$ is irregular for $1\leqq i\leqq\mu$ and is regular for $\mu+1\leqq i\leqq n$ in $\eta$, if and only if $\mu$ is even.*

**Proof of "Sufficiency" of Proposition 2.** Suppose that $\mu$ is even. We construct an element $\eta$ of $B(M)$ so that $\eta$ satisfies the required condition. Let $D$ be a punctured disk in $M$ with $\dot{D}=b_0\cup b_1\cup\cdots\cup b_\mu$, where $b_0$ is contained in the interior $\mathring{M}$ of $M$. Then, $D$ can be regarded as a 2-ball $B$ with $\mu$ untwisted bands $B_i$, $1\leqq i\leqq\mu$, as shown in Fig. 1. More presicely, we can assume the following. Put $B_i=C_i\times J$, where $C_i$ is a 1-ball and $J$ denotes the closed interval $[-1, 1]$. Then,

(1) $B_i\cap B_j=\emptyset$, if $i\neq j$,

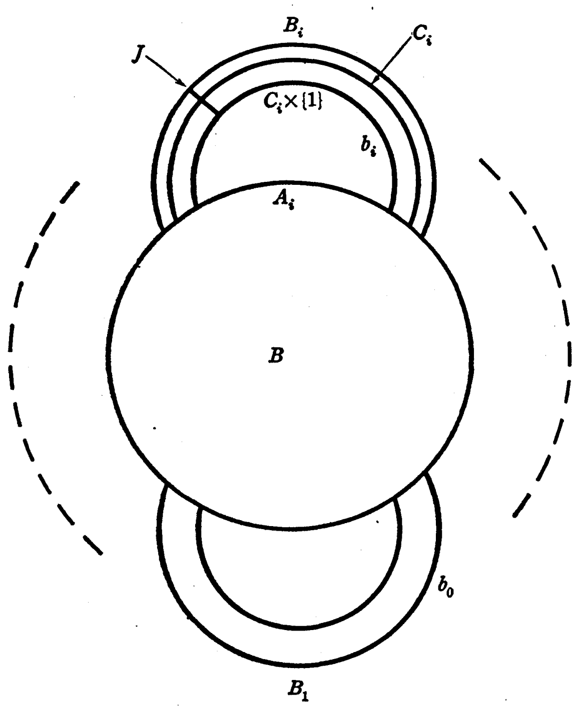


Fig. 1.

(2) $B_i \cap B = \dot{B}_i \cap \dot{B}$ consists of disjoint two 1-balls $\dot{C}_i \times J$.

(3) There exists a 1-ball $A_i$ in $\dot{B}$ such that $b_i$ is the union of $A_i$ and $C_i \times 1$. Now, let us consider the 3-balls $\tilde{B} = B \times J$ and $\tilde{B}_i = B_i \times J$ which are clearly block bundles over $B$ and $B_i$, respectively. Put $\dot{C}_i = c_1 \cup c_2$. We define a homeomorphism $h_i$ from $\dot{C}_i \times J \times J$ onto itself by

$$h_i((c, t_1, t_2)) = \begin{cases} (c, t_1, t_2), \text{ if } c = c_1 , \\ (c, t_1, -t_2), \text{ if } c = c_2 . \end{cases}$$

Note that $h_i$ is an equivalence of the block bundle $\dot{C}_i \times J \times J$ over $\dot{C}_i \times J$. Then, we obtain a block bundle $\eta_1$ over $D$ from $\tilde{B}$ by attaching all the $\tilde{B}_i$ by the homeomorphism $h_i$. Remember that $\mu$ is even and

$$b_0 = \bigcup_i C_i \times \{-1\} \cup (\dot{B} - (\bigcup_i (A_i \cup \dot{C}_i \times J))^\circ .$$

Then, the restriction $(\eta_1 | b_0)$ is a band from the definition of the attaching homeomorphisms $h_i$. On the other hand, let us consider the block bundle $\eta_2 = \overline{(M-D)} \times J$ over $\overline{M-D}$. Note that $b_0$ is also a boundary component of $\overline{M-D}$ and $(\eta_2 | b_0) = b_0 \times J$ is a band. Thus, we obtain an element $\eta$ of $B(M)$ from $\eta_1$ and $\eta_2$ by identifying $(\eta_1 | b_0)$ and $(\eta_2 | b_0)$ by an equivalence between them. Now, we have to show that $b_i$ is irregular for $1 \leqq i \leqq \mu$ and is regular for $\mu + 1 \leqq i \leqq n$ in $\eta$. It is easy to see

$$(\eta | b_i) = \begin{cases} (\eta_1 | b_i), \text{ if } 1 \leqq i \leqq \mu . \\ (\eta_2 | b_i), \text{ if } \mu + 1 \leqq i \leqq n . \end{cases}$$

Then, $b_i$ is regular in $\eta$ for $\mu + 1 \leqq i \leqq n$, because $(\eta_2 | b_i) = b_i \times J$ is a band. And for $1 \leqq i \leqq \mu$, it follows from the definition of $h_i$ that $(\eta_1 | b_i)$ is a Möbius band, so $b_i$ is irregular in $\eta$.

In order to prove "Necessity" of Proposition 2, we need some lemmas.

**Lemma 1.** *Let $M$ be a Möbius band and $W$ a $k$-sheeted covering of $M$. If $k$ is even, then the number of the boundary components of $W$, denoted by $\# \dot{W}$, is even.*

**Proof.** Put $W = W_1 \cup \cdots \cup W_n$, where $W_i$ means a connected component of $W$. Then, $W_i$ is, naturally, a covering of $M$, say $k_i$-sheeted. We can assume that the number $k_i$ is odd for $1 \leqq i \leqq m$ and is even for $m+1 \leqq i \leqq n$ for some $m$. It is easy to see that $W_i$ is a Möbius band for $1 \leqq i \leqq m$ and a band form $m+1 \leqq i \leqq n$. Hence $\# \dot{W} = m + 2(n-m)$. From the assumption, $k = \sum k_i$ is even, so $m$ must be even. Thus, $\# \dot{W}$ is even. This completes the proof of Lemma 1.

And, we prove one more lemma from Lemma 1.

**Lemma 2.** *Let $M$ be a 2-manifold and $W$ a $k$-sheeted covering of $M$. If $k$ is even, then $\#\dot{W}$ is even.*

**Proof.** *Case 1.* Suppose that $M$ is an orientable 2-manifold. Let $\chi(M)$ denote the Euler characteristic of $M$. Then, $\chi(M)$ is given by

$$\chi(M)=2-2H_M-\#\dot{M} ,$$

wheae $H_M$ denotes the number of the handles of $M$. Since $W$ is also orientable and is a $k$-sheeted covering of $M$, we obtain the following easily.

$$\#\dot{W}=2-2H_W-k\chi(M) .$$

Thus, $\#\dot{W}$ is even, because $k$ is assumed to be even.

*Case 2.* Suppose that $M$ is a non-orientable 2-manifold. Then, we can write $M$ uniquely as

$$M=P_1\natural\cdots\natural P_n\natural D ,$$

where $P_i$ is a projective plane, $D$ a punctured disk and $\natural$ means the connected sum. The proof goes by induction on $n$. When $n=0$, it follows from Case 1 above, because $M=D$ is orientable. Let $A$ denote a Möbius band in $P_n$ and $M_1=M-\mathring{A}$. Then, again, we can write

$$M_1=P_1\natural\cdots\natural P_{n-1}\natural D_1 ,$$

where $D_1$ is a punctured disk with one more boundary components than $D$. Let $p$ be the covering projection from $W$ to $M$ and put $W_1=p^{-1}(M_1)$. Then, $W_1$ is clearly a $k$-sheeted covering of $M_1$. Thus, by the inductive hypothesis, $\#\dot{W}_1$ is even. Now, let us consider $W_2=p^{-1}(A)$. Again, $W_2$ is a $k$-sheeted covering of $A$. Since $A$ is a Mobius band, $\#\dot{W}_2$ is even by Lemma 1. We obtain $\#\dot{W}=\#\dot{W}_1-\#\dot{W}_2$, because $W=W_1\cup W_2$ and $W_1\cap W_2=\dot{W}_1\cap\dot{W}_2=\dot{W}_2$. Thus, $\#\dot{W}$ is even.

**Proof of "Necessity" of Proposition 2.** We can regard $V$ as a block bundle over $M$. Put $W=(V)^{\cdot}-(V|\dot{M})^{\circ}$. Then, $W$ is a double covering of $M$. It is not hard to see

$$\#\dot{W}=2(n-\mu)+\mu .$$

Since $\#\dot{W}$ is even by Lemma 2, $\mu$ has to be even. This completes the proof of Proposition 2.

## 3. An answer to Problem 1

Let $P$ be a closed fake surface and $\eta$ an element of $B(U(P))$. Note that $\eta$ is unique by the following proposition in [3].

**Proposition 3.** *For a closed fake surface $P$, $B(U(P))$ consists of exactly one element.*

By $\mu_M(U(P))$, we denote the number of the boundary components of an element $M$ of $M(P)$ which are irregular in $\eta$ as boundary components of $U(P)$.

Then, we obtain a required theorem.

**Theorem 1.** *Let $P$ be a closed fake surface. Then, $B(P)$ is non-empty if and only if $\mu_M(U(P))$ is even for any element $M$ of $M(P)$.*

**Proof.** Put $\dot{M}=b_1\cup\cdots\cup b_n$, and let $\eta_U$ be the element of $B(U(P))$. First, we prove "Sufficiency". Suppose that $b_i$ is irregular for $1\leqq i\leqq\mu$ and is regular otherwise in $\eta_U$, where $\mu=\mu_M(U(P))$ is even. Then, by Proposition 2, we can find a block bundle $\eta_M$ in $B(M)$ such that $b_i$ is irregular for $1\leqq i\leqq\mu$ and is regular otherwise in $\eta_M$. That is, the regular neighborhood $(\eta_M|b_i)$ is a Möbius band if $b_i$ is irregular and is a band if $b_i$ is regular. On the other hand, it is known that $(\eta_U|b_i)$ is a Möbius band for $1\leqq i\leqq\mu$ and is a band for $\mu+1\leqq i\leqq n$. Then, it is not hard to obtain an element $\eta$ of $B(P)$ from $\eta_U$ and $\eta_M$ by identifyng them at $(\eta_U|b_i)$ and $(\eta_M|b_i)$ for all $M$ of $M(P)$. Next, we prove "Necessity". Suppose that $\eta$ is an element of $B(P)$. Let us cosider $\eta_M=(\eta|M)$ which is clearly an element of $B(M)$. Then, by Proposition 2, $\mu(M,\eta_M)$ is even. Since we can write

$$\eta=(\eta|U(P))\cup\bigcup_M(\eta|M)\ ,$$

and $(\eta|U(P))\cap(\eta|M)=(\eta|\dot{M})$, we see $\mu(M,\eta_M)=\mu_M(U(P))$. Thus, $\mu_M(U(P))$ must be even. This completes the proof of Theorem 1.

**Corollary to Theorem 1.** *Let $P$ be a closed fake surface and $\eta$ the element of $B(U(P))$. If $\mu(U(P),\eta)$ is odd, then $P$ can not be a spine of a 3-manifold.*

## 4. An answer to Problem 2

Let $P$ be a closed fake surface. In this section, we use the concept of a decomposition $U(P)=E_1\cup\cdots\cup E_n$ of $U(P)$. For the definition of a decomposition of $U(P)$, see Definition 3 [1]. And, as is seen in [1], we can assume that $E_i$ is a $T$-bundle embedded in $U(P)$.

First of all, we prove the following theorem.

**Theorem 2.** *Let $P$ be a closed fake surface and $\eta$ the element of $B(U(P))$. Then, $+B(P)$ is non-empty if and only if $\eta$ is a solid torus with certain genus.*

And, in Theorem 3, we obtain a characterization of $U(P)$ so that the element $\eta$ of $B(U(P))$ is a solid torus with certain genus.

**Theorem 3.** *Let $P$ be a closed fake surface and $\eta$ the element of $B(U(P))$. Then, $\eta$ is a solid torus with certain genus if and only if, for any decomposition $U(P)=E_1\cup\cdots\cup E_n$, $E_i\neq S\times\tau T$ holds for any $i$, $1\leqq i\leqq n$.*

**Proof of Theorem 2.** "Necessity" is trivial, for $\eta$ is unique. So, we prove just "Sufficiency". Suppose that $\eta$ is a solid torus with certain genus. For an element $M$ of $M(P)$, take the unique element $\eta_M$ of $+B(M)$ by Proposition 1. If $M$ is with boundary, there exists some disjoint proper 1-balls $A_i$, $i=1,\cdots,n$, in $M$ such that $M_1=\overline{M-N(A_M)}$ is a 2-ball, where $A_M$ means the union of the 1-balls $A_i$ and $N(A_M)$ is a regular neighborhood of $A_M$ in $M$ meeting the boundary regularly. Now, we have

$$\eta_M=(\eta_M|A_M)\cup(\eta_M|M_1)\ .$$

Then, it is not hard to attach $\bigcup_M(\eta_M|A_M)$ to $\eta$ so that the block bundle $\eta_1=\eta\cup\bigcup_M(\eta_M|A_M)$ over $U(P)\cup\bigcup_M A_M$ is a solid torus, because we can regard each connected component of $\bigcup_M(\eta_M|A_M)$ to be a 1-handle to $\eta$. Then, attaching $\bigcup_M(\eta_M|M_1)$ to $\eta_1$ by the natural way, we obtain a required element of $+B(P)$. Thus, Theorem 2 is established.

**Proof of Theorem 3.** "Necessity" is trivial from Lemma 24 [1]. So, we prove just "Sufficiency". Let $r$ be the rank of $H_1(U(P))$. The proof goes by induction on $r$. When $r=1$, it is known by Lemma 5 [1] and the hypothesis that $U(P)=E_1$ is either $S\times T$ or $S\times\sigma T$. Then, $B(U(P))$ consists of a solid torus with genus 1. Let us consider $U_1=E_1\cup\cdots\cup E_{n-1}$. Then, it is not hard to see that there exists a closed fake surface $P_1$ with $U(P_1)=U_1$. Since any decomposition of $U(P)$ contains no $S\times\tau T$, so is with one of $U_1=U(P_1)$. Thus, by the inductive hypothesis, $B(U_1)$ consists of a solid torus, say $\eta_1$, because rank of $H_1(U_1)\leqq r-1$ is clear. Since we can write

$$A=\overline{E_n-\bigcup_x(st(x,U(P))\cap E_n)}=\bigcup_j(T\times I)_j$$

as in [1], where $x$ is a point of $\mathfrak{S}_3(P)$, we can regard $(\eta|A)$ as 1-handles $H_j$ attached to $(\eta|\overline{U(P)-A})$. It is easy to see that $\eta_1$ and $(\eta|\overline{U(P)-A})$ are homeomorphic. Then, we can write

$$\eta=\eta_1 \cup \bigcup_j H_j .$$

We have to show that $\eta$ is a solid torus. Suppose not. Then, there must be a non-orientable handle $H_s$ for some $s$. Let $\alpha$ denote the 1-ball $(o(T)\times I)_s$. Here, we can assume

$$\eta_s=\eta_1 \cup \bigcup_{j<s} H_j$$

to be orientable, i.e. a solid torus. Then, there exists a 1-ball $\beta$ in $\eta_s \cap \mathfrak{S}_2(P)$ such that $\gamma=\alpha\cup\beta$ is a 1-sphere. Then, we obtain a decomposition $U(P)=E'_1\cup\cdots\cup E'_m$ of $U(P)$ such that $\gamma$ is the base space of some $E'_k$. Now, the regular neighborhood of $\gamma$ in $\eta$ is a solid Klein bottle, because $H_s$ is a non-orientable handle, so $E'_k$ must be $S\times\tau T$. This is a contradiction. This completes the proof.

## REFERENCES

[1] H. Ikeda, *Acyclic fake surfaces*, Topology 10 (1971), p. 9-36.
[2] ———, *Singular block bundles*, The Yokohama Math. J. 22 (1974), 79-100.
[3] ———, *Orientable 3-manifolds as singular block bundles*, The Yokohama Math. J. 22 (1974), 141-149.

Kobe University,
Nada, Kobe,
Japan