STATIONARY RANDOM MEASURES AND RENEWAL THEORY

By

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1. Introduction. This paper attempts to apply the theory of Palm measures to renewal theory. Let P be a strictly stationary random measure on the real line and P^0 the Palm measure for P. Daley [3] and Vere-Jones [11] proved several properties of the covariance measure and the spectral measure of P. In §3 of this paper, applying Mecke's theory [8] of Palm measures, some results of [3] and [11] on the covariance measure V are proved under slightly different assumptions and a sufficient condition for a Blackwell type renewal theorem:

$$\lim_{t\to\infty} V(I+t)=\lambda|I|\,, \quad \text{for every bounded interval } I, \tag{1.1}$$

is given. The main result (Theorem 4.1) of the present paper concerning the 'shift' P^t of P^0 is proved in §4. This theorem states that P^t converges weakly to a stationary random measure iff (1.1) holds. In §5 we prove an extension of a theorem of Ryll-Nardzewski [9] which is needed to connect the theory of Palm measures with renewal theory. In §6 previous results are applied to renewal theory for sums of stationarily dependent sequences. It is well-known (say [4]) that an ordinary renewal process with i.i.d. positive aperiodic inter-renewal times tends to a 'steady state' as time goes on. Theorem 6.2 gives a precise meaning to this phenomenon and shows that the corresponding fact holds in more general situations.

2. Palm measure. Let $\mathfrak{R}$ be the σ-algebra of all Borel subsets of the real line R and $\mathfrak{R}_0$ the ring consisting of all bounded $A\in\mathfrak{R}$. Let $\mathfrak{B}$ be the class of all real-valued Baire function on R and $\mathfrak{C}_0$ the class of all continuous real-valued functions on R whose supports are compact. The subclasses of $\mathfrak{B}$ and $\mathfrak{C}_0$ consisting of all non-negative functions are denoted by $\mathfrak{B}^+$ and $\mathfrak{C}_0^+$ respectively.

Let M denote the set of all measures φ on R such that $\varphi(A)<\infty$ for $A\in\mathfrak{R}_0$. In what follows we write $z_f(\varphi)=z(f\,;\,\varphi)$ for the integral $\int f(t)\varphi(dt)$, $f\in\mathfrak{C}_0$, $\varphi\in M$, where an integral sign without limits means integration over the whole space. This notation is also used for $f\in\mathfrak{B}$ if this integral has meaning. Denoting by

$\chi_E=\chi(E;\ \cdot)$ the indicator of the set E we write $z_A(\varphi)=z(A;\ \varphi)$ for $z(\chi_A;\ \varphi)=\varphi(A)$, $A\in\mathfrak{R}$, $\varphi\in M$.

In this space M we introduce the vague topology. This is the coarsest topology with respect to which every function z_f, $f\in\mathfrak{C}_0$, is continuous. A base for the neighborhood system of $\varphi\in M$ is given by the class of all sets of the form

$$\begin{aligned}(2.1)\qquad & U(f_1,\cdots,f_n;\ \varepsilon_1,\cdots,\varepsilon_n;\ \varphi)\\ &=\{\psi;\ \psi\in M,\ |z(f_i;\ \psi)-z(f_i;\ \varphi)|<\varepsilon_i,\ 1\leq i\leq n\}\,,\end{aligned}$$

where $f_i\in\mathfrak{C}_0$, $\varepsilon_i>0$, $1\leq i\leq n$, $n\geq 1$.

It is known that M is a Polish space, i.e., M is homeomorphic to a complete separable metric space. The σ-algebra of all Borel subsets of M will be denoted by $\mathfrak{M}$. $\mathfrak{M}$ coincides with the smallest σ-algebra with respect to which every z_A, $A\in\mathfrak{R}$, is measurable. For every $f\in\mathfrak{B}^+$ z_f is $\mathfrak{M}$-measurable. The subset M_0 of M consisting of all integer valued measures is closed and therefore $M_0\in\mathfrak{M}$. Since M is Polish the product σ-algebra, say, $\mathfrak{R}\times\mathfrak{M}$ coincides with the σ-algebra of Borel sets of the product topological space $R\times M$.

For each $t\in R$ a homeomorphism T_t of M onto itself is defined by

$$(2.2)\qquad (T_t\varphi)(A)=\varphi(A+t)\,,\quad \varphi\in M\,,\quad A\in\mathfrak{R}\,.$$

The mapping from $R\times M$ onto M which sends (t,φ) to $T_t\varphi$ is continuous and therefore measurable with respect to the σ-algebra $\mathfrak{R}\times\mathfrak{M}$.

An $\mathfrak{M}$-measurable real-valued function u will be called invariant if

$$(2.3)\qquad u(T_t\varphi)=u(\varphi)\,,\quad \varphi\in M\,,\quad t\in R\,.$$

A measure P on $(M,\mathfrak{M})$ is called stationary if

$$(2.4)\qquad P(T_tE)=P(E)\,,\quad E\in\mathfrak{M}\,,\quad t\in R\,.$$

If P is a stationary measure and if u is a finite non-negative invariant function then the measure $u\cdot P$ defined by

$$(2.5)\qquad (u\cdot P)(E)=\int\chi_E(\varphi)u(\varphi)P(d\varphi)\,,\quad E\in\mathfrak{M}\,,$$

is stationary.

A probability measure P on $(M,\mathfrak{M})$ is called a random measure on R. A random measure concentrated on M_0 is called a point process on R. Sometimes these terms are used rather loosely to denote measures on $(M,\mathfrak{M})$ which are not necessarily probability measures.

Lemma 2.1 (*Mecke* [8]). *A σ-finite measure P on $(M,\ \mathfrak{M})$ is stationary iff*

$$\iiint w(t, s, T_s\varphi)\varphi(ds)P(d\varphi)dt = \iiint w(s, t, T_s\varphi)\varphi(ds)P(d\varphi)dt \tag{2.6}$$

for every non-negative $\mathfrak{R}\times\mathfrak{R}\times\mathfrak{M}$*-measurable function* w.

Lemma 2.2 (*Mecke* [8]). *If* P *is a* σ*-finite stationary measure on* $(M, \mathfrak{M})$ *then there exists a unique* σ*-finite measure* P^0 *on* $(M, \mathfrak{M})$ *such that*

$$\int u(\varphi)P^0(d\varphi) = \iint g(s)u(T_s\varphi)\varphi(ds)P(d\varphi) \tag{2.7}$$

for any non-negative $\mathfrak{M}$*-measurable functon* u *and any* $g\in\mathfrak{B}^+$ *such that* $\int g(t)dt=1$.

The measure P^0 defined by (2.7) is called Palm measure for P. Taking $u=1$ in (2.7) it is immediate that

$$P^0(M) = \int z_g(\varphi)P(d\varphi) \quad \text{if } g\in\mathfrak{B}^+, \quad \int g(t)dt=1\,. \tag{2.8}$$

If P is σ-finite stationary and if $u\geq 0$ is invariant then $u\cdot P$ is σ-finite stationary and it follows from (2.7) that

$$(u\cdot P)^0 = u\cdot P^0\,. \tag{2.9}$$

If $g\in\mathfrak{B}^+$ and $u\geq 0$ is $\mathfrak{M}$-measurable then it follows from Lemma 2.1 and Lemma 2.2 that

$$\begin{aligned}\iint g(t)u(T_{-t}\varphi)P^0(d\varphi)dt &= \iiint g(s)g_0(t)u(T_{s-t}\varphi)\varphi(ds)P(d\varphi)dt \\ &= \iiint g(s)g_0(t)u(\varphi)\varphi(ds)P(d\varphi)dt = \int u(\varphi)z_g(\varphi)P(d\varphi)\,,\end{aligned} \tag{2.10}$$

where $g_0\in\mathfrak{B}^+$, $\int g_0(t)dt=1$.

Lemma 2.3 (*Mecke* [8]). *Let* P^0 *be the Palm measure for a stationary* σ*-finite meassure* P *on* $(M, \mathfrak{M})$. *Then for every non-negative* $\mathfrak{M}$*-measurable function* u *with* $u(0)=0$, *where* 0 *on the left denotes the zero measure,*

$$\int u(\varphi)P(d\varphi) = \iint h(t, T_{-t}\varphi)u(T_{-t}\varphi)dtP^0(d\varphi)\,, \tag{2.11}$$

where h *is a non-negative* $\mathfrak{R}\times\mathfrak{M}$*-measurable function such that* $h(t, 0)=0$ *and*

$$\int h(t, \varphi)\varphi(dt) = 1\,, \quad \varphi\in M\,, \quad \varphi\neq 0\,. \tag{2.12}$$

This theorem shows that if we restrict ourselves to σ-finite stationary measures such that $P(\{0\})=0$ then P is uniquely determined by P^0.

Lemma 2.4 (*Mecke* [8]). *A measure Q on $(M, \mathfrak{M})$ is the Palm measure of a σ-finite stationary measure on $(M, \mathfrak{M})$ iff it satisfies the following three conditions*:

(i) *Q is σ-finite,*

(ii) $Q(\{0\})=0$,

(iii) *for every $\mathfrak{R}\times\mathfrak{M}$-measurable $v\geq 0$,*

$$\iint v(-t, T_t\varphi)\varphi(dt)Q(d\varphi)=\iint v(t, \varphi)\varphi(dt)Q(d\varphi) . \tag{2.13}$$

3. **Covariance measure and spectral measure.** Throughout the rest of this paper P is a σ-finite stationary measure on $(M, \mathfrak{M})$ satisfying $P(\{0\})=0$ and P^0 is the Palm measure for P. Let V denote a not necessarily σ-finite measure on $\mathfrak{R}$ defined by

$$V(A)=\int\varphi(A)P^0(d\varphi) , \quad A\in\mathfrak{R} . \tag{3.1}$$

Let $(\mathscr{D})$ and $(\mathscr{S})$ be Schwartz's test function spaces on R. A measure $\varphi\in M$ is called positive definite if it is positive definite in the sense of Schwartz's distribution [10], i.e., if

$$\iint f(s)\overline{f(s+t)}ds\varphi(dt)\geq 0 , \quad f\in(\mathscr{D}) . \tag{3.2}$$

The following theorem is essentially due to Daley [3] and Vere-Jones [11].

Theorem 3.1. *The following three statements are equivalent*:

(i) $V\in M$, i.e. $V(A)<\infty$ *for* $A\in\mathfrak{R}_0$,

(ii) $V((-\varepsilon, \varepsilon))<\infty$ *for some* $\varepsilon>0$,

(iii) $\int(\varphi(A))^2P(d\varphi)<\infty$ *for* $A\in\mathfrak{R}_0$.

In this case V is positive definite and satisfies

$$\iint v(s, s+t)ds\, V(dt)=\iiint v(s, t)\varphi(ds)\varphi(dt)P(d\varphi) \tag{3.3}$$

for any Baire function $v\geq 0$ on R^2. In particular

$$\begin{aligned}\iint f(s)g(s+t)ds\, V(dt)&=\int z_f(\varphi)z_g(\varphi)P(d\varphi)\\&=\iint g(t)z_f(T_{-t}\varphi)P^0(d\varphi)dt\end{aligned} \tag{3.4}$$

for $f \in \mathfrak{B}^+$ *and* $g \in \mathfrak{B}^+$.

Proof. First assuming (iii) we prove (i) and the conclusion of the last half. Let m denote the masure on the σ-algebra $\mathfrak{R}^2$ of Borel sets of R^2 such that

$$m(A_1 \times A_2) = \iiint \chi(A_1; s)\chi(A_2; t-s)\varphi(ds)\varphi(dt)P(d\varphi) , \tag{3.5}$$

for $A_1 \in \mathfrak{R}$, $A_2 \in \mathfrak{R}$. It follows from (iii) that the value of m is finite for every bounded Borel set. By using (2.4) we have

$$\begin{aligned} m((A_1+u) \times A_2) &= \iiint \chi(A_1; s-u)\chi(A_2; t-s)\varphi(ds)\varphi(dt)P(d\varphi) \\ &= \iiint \chi(A_1; s)\chi(A_2; t-s)(T_u\varphi)(ds)(T_u\varphi)(dt)P(d\varphi) \\ &= \iiint \chi(A_1; s)\chi(A_2; t-s)\varphi(ds)\varphi(dt)P(d\varphi) \\ &= m(A_1 \times A_2) , \quad u \in R . \end{aligned}$$

Thus for any fixed $A_2 \in \mathfrak{R}_0$ the value of $m(A_1 \times A_2)$ is proportional to the Lebesgue measure $|A_1|$ of A_1:

$$m(A_1 \times A_2) = |A_1| \cdot V^*(A_2) , \quad A_1 \in \mathfrak{R} , \quad A_2 \in \mathfrak{R}_0 . \tag{3.6}$$

The set function V^* on $\mathfrak{R}_0$ is uniquely extended to a measure V^* on $\mathfrak{R}$ such that $V^* \in M$ and $m = |\cdot| \times V^*$. For any Baire function $v \geq 0$ on R^2 it follows from (3.5) and (3.6) that

$$\iint v(s, t) ds\, V^*(dt) = \iiint v(s, t-s)\varphi(ds)\varphi(dt)P(d\varphi) . \tag{3.7}$$

From (2.7) and (3.7) we have that

$$\begin{aligned} V(A) = \int z_A(\varphi)P^0(d\varphi) &= \iint g(s)\chi(A; t-s)\varphi(ds)\varphi(dt)P(d\varphi) \\ &= \iint g(s)\chi(A; t) ds\, V^*(dt) = V^*(A) , \quad A \in \mathfrak{R} . \end{aligned}$$

Thus $V = V^*$ and therefore (3.7) is equivalent to (3.3). It follows from (3.3) with $v(s, t) = f(s)\overline{f(s+t)}$, $f \in (\mathscr{D})$, that

$$\iint f(s)\overline{f(s+t)} ds\, V(dt) = \int |z_f(\varphi)|^2 P(d\varphi) \geq 0 ,$$

which shows that V is positive definite.

Thus we have proved that (iii) implies (i) and the conclusion of the last half. The implication (i)$\Longrightarrow$(ii) is obvious. To prove (ii)$\Longrightarrow$(iii) let $0<\varepsilon<1$ and $I=\left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$. It follows from (2.7) that

$$\begin{aligned}\int\varphi(I)^2P(d\varphi)&=\iiint\chi(I;s)\chi(I;t)\varphi(ds)\varphi(dt)P(d\varphi)\\&\leq\iiint\chi((-1,1);s)\chi((-\varepsilon,\varepsilon);t-s)\varphi(ds)\varphi(dt)P(d\varphi)\\&=\int\varphi((-\varepsilon,\varepsilon))P^0(d\varphi)=V((-\varepsilon,\varepsilon))<\infty\ .\end{aligned}$$

Together with (2.4) and Schwarz inequality this implies (iii). (3.4) is immediate from (3.3) and (2.10). This completes the proof.

Remark 3.1. The relation (3.3) indicates that when P is a probability measure V may be called the 'covariance measure' of P regarded as a stationary random distribution [6]. Thus, loosely speaking, Theorem 3.1 says that the intensity measure of the Palm measure P^0 coincides with the covariance measure of the original stationary measure P.

Throughout the rest of this section assume $V\in M$. Since V is positive definite it follows from Schwartz's extension of Bochner's theorem [10] that V is the Fourier transform of a tempered measure G: $V=\hat{G}$, i.e.,

$$\int f(t)V(dt)=\int\hat{f}(t)G(dt)\ , \tag{3.8}$$

where $f\in(\mathscr{S})$ and

$$\hat{f}(t)=\int e^{itx}f(x)dx\ ,\quad f(t)=\frac{1}{2\pi}\int e^{-itx}\hat{f}(x)dx\ .$$

Since V is a symmetric measure (3.8) implies that G is also a positive definite measure: $G=(2\pi)^{-1}\hat{V}$. G is called the spectral measure of P.

Let $a>0$ and let $f\in(\mathscr{D})$ be non-negative and satisfy $\int f(t)dt=1$. By (3.8) we have

$$\frac{1}{2a}\iint\chi(I_a;s)f(t-s)ds\,V(dt)=\int\frac{\sin at}{at}\hat{f}(t)G(dt)\ ,\quad I_a=(-a,a)\ ,$$

which converges to $\hat{f}(0)G(\{0\})=G(\{0\})$ as $a\to\infty$. This implies immediately that

$$\lim_{a\to\infty}\frac{1}{2a}V(I_a)=G(\{0\})\equiv\lambda\ . \tag{3.9}$$

Theorem 3.2. *If the spectral measure G is absolutely continuous with respect to the Lebesgue measure except for a possible atom at the origin, i.e., if*

$$G(A)=\lambda\cdot\chi(A;0)+\int_A\gamma(t)dt\,,\quad A\in\mathfrak{R}\,, \tag{3.10}$$

with a locally integrable γ, then

$$\lim_{t\to\infty}V(I+t)=\lambda\cdot|I|\,, \tag{3.11}$$

for every bounded interval I.

Proof. Approximate χ_I by non-negative functions in $(\mathscr{D})$ and apply Riemann-Lebesgue lemma.

Remark 3.2. Applying the argument of [4] p. 362 it can be shown that for every directly Riemann integrable function g on R

$$\lim_{t\to\infty}\int g(s-t)\,V(ds)=\lambda\int g(s)ds$$

holds iff (3.11) holds for every bounded interval I.

Remark 3.3. In addition to the assumptions of the theorem if γ is assumed to be of bounded variation in a neighborhood of the origin then it can be shown that

$$\lim_{a\to\infty}[V(I_a)-2a\lambda]=2\pi\lim_{t\to0}\gamma(t)\,. \tag{3.12}$$

4. Weak convergence of shift of Palm measure. Let P be a σ-finite stationary measure on $(M,\mathfrak{M})$ and $\alpha=P^0(M)$. In this section we prove the main results of the present paper assuming that

$$\int\varphi(A)P(d\varphi)<\infty\quad\text{for}\quad A\in\mathfrak{R}_0\,, \tag{4.1}$$

and

$$\int\varphi(A)^2P(d\varphi)<\infty\quad\text{for}\quad A\in\mathfrak{R}_0\,. \tag{4.2}$$

It is immediate from (2.8) that (4.1) holds iff $\alpha<\infty$ and in this case

$$\int z_f(\varphi)P(d\varphi)=\alpha\int f(t)dt \tag{4.3}$$

for integrable $f\in\mathfrak{B}$. Theorem 3.1 says (4.2) is equivalent to $V\in M$. Let $\lambda=$

$\lim_{a\to\infty}(2a)^{-1}V(I_a)$ where I_a denotes the interval $(-a, a)$. Note that (4.2) does not imply (4.1) in general.

Lemma 4.1. *If* (4.1) *holds then* $P(U^c)<\infty$ *for every neighborhood* U *of* $0\in M$.

Proof. If suffices to prove assuming that U is of the form

$$U=\{\varphi;\, z(f_i;\varphi)<\varepsilon_i,\ 1\leq i\leq n\}\ ,$$

where $f_i\in\mathfrak{C}_0^+$, $\varepsilon_i>0$, $1\leq i\leq n$. Then the lemma follows from the following:

$$P(U^c)\leq\sum_{i=1}^n P(\{\varphi;\, z(f_i;\varphi)<\varepsilon_i\})\leq\sum_{i=1}^n\varepsilon_i^{-1}\int z(f_i;\varphi)P(d\varphi)=\alpha\sum_{i=1}^n\varepsilon_i^{-1}\int f_i(t)dt<\infty\ .$$

In what follows P is assumed to satisfy (4.1) and (4.2). Let

$$\nu_a(\varphi)=(2a)^{-1}\varphi(I_a)\ ,\quad \varphi\in M\ ,\quad a>0\ .$$

Let Λ denote the set of all $\varphi\in M$ such that the finite limit $\lim_{a\to\infty}\nu_a(\varphi)$ do not exist. The set Λ is $\mathfrak{M}$-measurable and invariant. Let us define $\nu(\varphi)=0$ for $\varphi\in\Lambda$ and $\nu(\varphi)=\lim_{a\to\infty}\nu_a(\varphi)$ for $\varphi\in\Lambda^c$.

Lemma 4.2. $\nu\in L^1(P)\cap L^2(P)\cap L^1(P^0)$. *As* $a\to\infty$ ν_a *converges to* ν P-a.e., P^0-a.e., *in* $L^2(P)$ *and in* $L^1(P^0)$. *If* $P(M)<\infty$ *then* $\alpha=\int\nu dP$.

Proof. It follows from classical ergodic theorems that $\nu\in L^1(P)\cap L^2(P)$, $\nu_a\to\nu$ P-a.e. and in $L^2(P)$ and $\alpha=\int\nu dP$ when $P(M)<\infty$. Since Λ is invariant P-null it follows from (2.7) that

$$P^0(\Lambda)=\iint g(t)\chi(\Lambda;\, T_t\varphi)\varphi(dt)P(d\varphi)=\iint g(t)\chi(\Lambda;\,\varphi)\varphi(dt)P(d\varphi)=0\ .$$

Thus $\nu_a\to\nu$ P^0-a.e. Noting that

$$(a-1)\nu_{a-1}(\varphi)\leq a\nu_a(T_s\varphi)\leq(a+1)\nu_{a+1}(\varphi)\ ,\quad \varphi\in M\ ,\quad a>1\ ,\quad |s|\leq 1\ ,$$

we have

$$|\nu_a(T_s\varphi)-\nu(\varphi)|\leq\left|\frac{a-1}{a}\nu_{a-1}(\varphi)-\nu(\varphi)\right|+\left|\frac{a+1}{a}\nu_{a+1}(\varphi)-\nu(\varphi)\right|,\quad a>1\ ,\quad |s|\leq 1.$$

Hence if we choose $g(t)=\chi((0,1);t)$ then $z_g\in L^2(P)$ and

$$\int |\nu_a(\varphi)-\nu(\varphi)| P^0(d\varphi) = \iint g(s) |\nu_a(T_s\varphi)-\nu(T_s\varphi)| \varphi(ds) P(d\varphi)$$
$$\leq \int z_g(\varphi) \left| \frac{a-1}{a}\nu_{a-1}(\varphi)-\nu(\varphi)\right| P(d\varphi) + \int z_g(\varphi) \left| \frac{a+1}{a}\nu_{a+1}(\varphi)-\nu(\varphi)\right| P(d\varphi) .$$

Since $(1-a^{-1})\nu_{a-1}\to\nu$, $(1+a^{-1})\nu_{a+1}\to\nu$ in $L^2(P)$ we have $\nu_a\to\nu$ in $L^1(P^0)$.

Lemma 4.3. *If $u\geq 0$ is invariant P-integrable then*

$$\int u(\varphi) P^0(d\varphi) = \int u(\varphi)\nu(\varphi) P(d\varphi) . \tag{4.4}$$

Proof. Letting $g(t)=(2a)^{-1}\chi(I_a; t)$, $a>0$, in (2.7) we have

$$\int u(\varphi) P^0(d\varphi) = \iint g(t) u(\varphi)\varphi(dt) P(d\varphi) = \int u(\varphi)\nu_a(\varphi) P(d\varphi) .$$

By assumption $u\cdot P$ is a finite stationary measure and if u is bounded then $(u\cdot P)^0(M)=(u\cdot P^0)(M)<\infty$. Hence by Lemma 4.2

$$\int u(\varphi)\nu_a(\varphi) P(d\varphi) = \int u(\varphi)\nu(\varphi) P(d\varphi)$$

which proves (4.4) for bounded u. For unbounded u truncation and monotone convergence theorem can be used to obtain (4.4).

Lemma 4.4.

$$\int \nu(\varphi) P(d\varphi) \leq P^0(M) \equiv \alpha . \tag{4.5}$$

The equality holds iff $\nu>0$ P-a.e. or equivalently $\nu>0$ P^0-a.e.

Proof. The inequality is immediate from (4.3) and Fatou's lemma. Let $E=\{\varphi; \nu(\varphi)>0\}$ and $E_n=\{\varphi; \nu(\varphi)>n^{-1}\}$, $n\geq 1$. For every n $\chi(E_n; \cdot)$ is invariant and P-integrable and therefore

$$P^0(E_n) = \int \chi(E_n; \varphi)\nu(\varphi) P(d\varphi) , \quad n\geq 1 ,$$

by Lemma 4.3. Letting $n\to\infty$ we have

$$P^0(E) = \int \chi(E; \varphi)\nu(\varphi) P(d\varphi) ,$$

and therefore the equality in (4.5) holds iff $\nu>0$ P^0-a.e. If $P(E^c)=0$ then $\chi_E\cdot P=P$ and by (2.9) $P^0=(\chi_E\cdot P)^0=\chi_E\cdot P^0$. Hence $P^0(E^c)=0$. Conversely if $P^0(E^c)=0$ then by Lemma 2.3

$$P(E^c)=\iint h(t,\,T_{-t}\varphi)\chi(E^c;\,T_{-t}\varphi)dtP^0(d\varphi)$$
$$=\iint h(t,\,T_{-t}\varphi)\chi(E^c;\,\varphi)dtP^0(d\varphi)=0\ .$$

Thus $\nu>0$ P-a.e. iff $\nu>0$ P^0-a.e. This proves the lemma.

Lemma 4.5.

$$\int z_f(\varphi)\nu(\varphi)P(d\varphi)=\lambda\int f(t)dt \tag{4.6}$$

for integrable $f\in\mathfrak{B}$ *and*

$$\lambda=\int\nu(\varphi)P^0(d\varphi)=\int\nu(\varphi)^2P(d\varphi)\ . \tag{4.7}$$

Proof. From Lemma 4.2 we have

$$\lambda=\lim_{a\to\infty}\frac{1}{2a}V(I_a)=\lim_{a\to\infty}\int\nu_a dP^0=\int\nu dP^0\ .$$

Thus $\nu\cdot P$ is a stationary measure with $(\nu\cdot P)^0(M)=\lambda<\infty$. Hence (4.6) follows from (4.3). Let $f(t)=(2a)^{-1}\chi(I_a;t)$ in (4.6) and let $a\to\infty$ to obtain the last equality of (4.7).

Lemma 4.6. *Let* $g\in\mathfrak{B}^+$ *be bounded and satisfy* $\int g(t)dt=1$. *If* $\lim u_n=u$ *in* $L^2(P)$ *and if*

$$\lim_{\tau\to\infty}\iint g(t)u_n(T_{\tau-t}\varphi)P^0(d\varphi)dt=\int u_n(\varphi)\nu(\varphi)P(d\varphi)\ ,\quad n\geq 1\ ,$$

then

$$\lim_{\tau\to\infty}\iint g(t)u(T_{\tau-t}\varphi)P^0(d\varphi)dt=\int u(\varphi)\nu(\varphi)P(d\varphi)\ .$$

Proof. Apply (2.10) and Schwarz inequality.

Lemma 4.7. *Let* $g\in\mathfrak{C}_0^+$ *satisfy* $\int g(t)dt=1$. *If*

$$\lim_{t\to\infty}V(I+t)=\lambda\cdot|I| \tag{4.8}$$

for every bounded interval I, *then for every* $u\in L^2(P)$

$$\lim_{\tau\to\infty}\iint g(t)u(T_{\tau-t}\varphi)P^0(d\varphi)dt=\int u(\varphi)\nu(\varphi)P(d\varphi) \tag{4.9}$$

Proof. If $f\in\mathfrak{C}_0$ then it follows from (4.8) that

$$\lim_{\tau\to\infty}\iint f(t-\tau)g(s+t)V(ds)dt=\lambda\int f(t)dt\ .$$

Thus in view of (3.4) and (4.6), (4.9) holds for $u=z_f$, $f\in\mathfrak{C}_0$. Let H denote the closed linear subspace of $L^2(P)$ spanned by z_f, $f\in\mathfrak{C}_0$, and let $H^\perp$ denote the orthogonal complement of H. It follows from Lemma 4.6 that (4.9) holds for every $u\in H$. For any $u\in L^2(P)$ let $u=u_1+u_2$, $u_1\in H$, $u_2\in H^\perp$. Since $z_g\in H$ and $\nu\in H$ we have

$$\int u_2(\varphi)\nu(\varphi)P(d\varphi)=0$$

and by (2.10)

$$\iint g(t)u_2(T_{\tau-t}\varphi)P^0(d\varphi)dt=\int u_2(T_\tau\varphi)z_g(\varphi)P(d\varphi)=0\ .$$

Hence

$$\begin{aligned}\lim_{\tau\to\infty}\iint g(t)u(T_{\tau-t}\varphi)P^0(d\varphi)dt&=\lim_{\tau\to\infty}\iint g(t)u_1(T_{\tau-t}\varphi)P^0(d\varphi)at\\&=\int u_1(\varphi)\nu(\varphi)P(d\varphi)=\int u(\varphi)\nu(\varphi)P(d\varphi)\ .\end{aligned}$$

This proves the lemma.

For $t\in R$ let $P^t=P^0T_{-t}$ denote the 'shift' of the Palm measure P^0:

$$\int u(\varphi)P^t(d\varphi)=\int u(T_t\varphi)P^0(d\varphi) \tag{4.10}$$

for bounded $\mathfrak{M}$-measurable U.

We are now in a position to prove the following:

Theorem 4.1. *Let P satisfy* (4.1) *and* (4.2). *Without loss of generality we assume that P^0 is a probability measure. In order that P^t converges weakly to a probability measure P^∞ as $t\to\infty$, i.e.,*

$$\lim_{t\to\infty}\int u(\varphi)P^t(d\varphi)=\int u(\varphi)P^\infty(d\varphi) \tag{4.11}$$

for every bounded continuous u, it is necessary and sufficient that (4.8) *holds for every bounded interval I. In this case P^∞ is stationary and*

$$P^\infty=\nu\cdot P+c\delta_0 \tag{4.12}$$

where δ_0 is the probability measure concentrated on $0\in M$ and $c=1-\int\nu dP$. In *particular* $P^\infty = \nu \cdot P$ *iff* $\nu > 0$ P^0-a.e. *or equivalently iff* $\nu > 0$ P-a.e.

Proof. (*Necessity*) Let $f \in \mathfrak{C}_0^+$ and $g \in \mathfrak{C}_0^+$ be arbitrary except that $\int g(t)dt = 1$. It suffices to prove that

$$\lim_{\tau\to\infty} \iint f(t-\tau)g(s+t)V(ds)dt = \lambda \int f(t)dt, \tag{4.13}$$

since (4.8) follows from (4.13) by a standard approximation. In view of (3.4) and (4.6), (4.13) is equivalent to

$$\lim_{\tau\to\infty} \iint g(t)z_f(T_{\tau-t}\varphi)P^0(d\varphi)dt = \int z_f(\varphi)\nu(\varphi)P(d\varphi). \tag{4.14}$$

By assumption $z_f \in L^2(P)$. If we define $z_f^n(\varphi) = \min(z_f(\varphi), n)$, $n \geq 1$, then $z_f^n \to z_f$ in $L^2(P)$. Since z_f is bounded continuous and $z_f^n(0) = 0$ we have by assumption (4.11) and (4.9) that

$$\lim_{\tau\to\infty} \int z_f^n(T_\tau\varphi)P^0(d\varphi) = \int z_f^n(\varphi)\nu(\varphi)P(d\varphi).$$

Hence (4.14) holds for z_f replaced by z_f^n, $n \geq 1$. Thus by Lemma 4.6 (4.14) holds for z_f, $f \in \mathfrak{C}_0^+$.

(*Sufficiency*) Let u be a bounded continuous function on M and let $K = \sup_{\varphi\in M} |u(\varphi)|$. At first we assume $u \in L^2(P)$. Let $g \in \mathfrak{C}_0^+$ be supported by the interval $[-1, 1]$ and satisfy $\int g(t)dt = 1$. Since $z_g \in L^1(P)$, for any $\varepsilon > 0$ one can choose $\eta > 0$ so that

$$\int \chi(\varphi)z_g(\varphi)P(d\varphi) < \frac{\varepsilon}{4K} \tag{4.15}$$

whenever $\Lambda \in M$, $P(\Lambda) < \eta$.

For each $\varphi \in M$, $u(T_t\varphi)$ is a continuous function of t and therefore for any δ, $0 < \delta < 1$, the subset

$$\Lambda_0 = \left\{\varphi;\ \sup_{|t|\leq\delta, |s|\leq 1} |u(T_{t+s}\varphi) - u(T_s\varphi)| > \frac{\varepsilon}{2}\right\}$$

of M belongs to $\mathfrak{M}$. One can show that $P(\Lambda_0) < \infty$ even if $P(M) = \infty$. In fact if $P(M) = \infty$ then by Lamma 4.1 $u \in L^2(P)$ implies $u(0) = 0$. Let U be a neighborhood of $0 \in M$ such that $|u(\varphi)| < \frac{\varepsilon}{4}$ for $\varphi \in U$. It is easy to see that there exists a neighborhood U_0 of $0 \in M$ such that $T_t\varphi \in U$ if $\varphi \in U_0$ and $|t| \leq 2$. Thus $\Lambda_0^c \supset U_0$ and by Lemma 4.1 one has $P(\Lambda_0) < \infty$.

Since for every $\varphi \in M$

$$\lim_{\delta\to 0} \sup_{|t|\le\delta, |s|\le 1} |u(T_{t+s}\varphi)-u(T_s\varphi)|=0 ,$$

one can choose $\delta>0$ so small that

$$P(\Lambda_0)<\eta .$$

Let $\Lambda_\tau = T_{-\tau}\Lambda_0$, $\tau\in R$, i.e.,

$$\Lambda_\tau=\left\{\varphi;\ \sup_{|t|\le\delta, |s|\le 1} |u(T_{\tau+t+s}\varphi)-u(T_{\tau+s}\varphi)|>\frac{\varepsilon}{2}\right\} \tag{4.16}$$

then by the stationarity of P one has

$$P(\Lambda_\tau)<\eta ,\quad \tau\in R . \tag{4.17}$$

Let $h\in\mathfrak{C}_0^+$ be supported by $[-\delta, \delta]$ and satisfy $\int h(t)dt=1$ and let

$$u_h(\varphi)=\int u(T_{-t}\varphi)h(t)dt .$$

It follows from (4.16) that

$$|u_h(T_{\tau+s}\varphi)-u(T_{\tau+s}\varphi)|<\frac{\varepsilon}{2} \quad \text{for } \varphi\in\Lambda_\tau^c ,\quad \tau\in R ,\quad |s|\le 1 . \tag{4.18}$$

We shall now evaluate

$$\begin{aligned}
&\int |u_h(T_\tau\varphi)-u(T_\tau\varphi)|P^0(d\varphi)\\
&\quad=\iint g(s)|u_h(T_{\tau+s}\varphi)-u(T_{\tau+s}\varphi)|\varphi(ds)P(d\varphi)\\
&\quad=\iint g(s)\chi(\Lambda_\tau^c;\varphi)|u_h(T_{\tau+s}\varphi)-u(T_{\tau+s}\varphi)|\varphi(ds)P(d\varphi)\\
&\quad+\iint g(s)\chi(\Lambda_\tau^c;\varphi)|u_h(T_{\tau+s}\varphi)-u(T_{\tau+s}\varphi)|\varphi(ds)P(d\varphi) .
\end{aligned}$$

Since g is supported by $[-1, 1]$ it follows from (4.18) that the first integral on the right does not exceed

$$\frac{\varepsilon}{2}\iint g(s)\varphi(ds)P(d\varphi)=\frac{\varepsilon}{2}\int g(s)\,ds=\frac{\varepsilon}{2} .$$

From (4.15) and (4.17) follows that the second integral is dominated by

$$2K\int \chi(\Lambda_\tau^c;\varphi)z_g(\varphi)P(d\varphi)\le 2K\frac{\varepsilon}{4K}=\frac{\varepsilon}{2} .$$

Thus we have

$$\left|\int u_h(T_\tau\varphi)P^0(d\varphi)-\int u(T_\tau\varphi)P^0(d\varphi)\right| \le\int|u_h(T_\tau\varphi)-u(T_\tau\varphi)|P^0(d\varphi)\le\varepsilon\,, \quad \tau\in R\,. \tag{4.19}$$

Since by Lemma 4.7

$$\lim_{\tau\to\infty}\int u_h(T_\tau\varphi)P^0(d\varphi)=\int u(\varphi)\nu(\varphi)P(d\varphi)$$

holds, it follows from (4.19) that

$$\lim_{\tau\to\infty}\int u(T_\tau\varphi)P^0(d\varphi)=\int u(\varphi)\nu(\varphi)P(d\varphi) \tag{4.20}$$

for bounded continuous $u\in L^2(P)$.

Now let u be any bounded continuous function on M. For any $\varepsilon>0$ one can choose an open neighborhood U of $0\in M$ such that

$$|u(\varphi)-u(0)|<\varepsilon \quad \text{for } \varphi\in U\,. \tag{4.21}$$

Let U_0 be an open neighborhood of $0\in M$ such that $\bar{U}_0\subset U$ and α a continuous function on M such that $0\le\alpha(\varphi)\le 1$, $\varphi\in M$, $\alpha(\varphi)=0$ on $\bar{U}_0$ and $\alpha(\varphi)=1$ on U^c. Let us write

$$u(\varphi)=u(0)+u_1(\varphi)+u_2(\varphi)\,,$$

where

$$u_1=\alpha\cdot(u-u(0))\,,\quad u_2=(1-\alpha)\cdot(u-u(0))\,.$$

Since u_1 is bounded and supported by U_0^c it follows from Lemma 4.1 that $u_1\in L^2(P)$ and therefore (4.20) holds with u replaced by u_1. On the other hand it follows from (4.21) that $|u_2(\varphi)|<\varepsilon$, $\varphi\in M$, and therefore

$$\sup_{t\in R}\left|\int u_2(T_t\varphi)P^0(d\varphi)\right|<\varepsilon\,.$$

Since $\int\nu dP\le P^0(M)=1$ we have also

$$\left|\int u_2(\varphi)\nu(\varphi)P(d\varphi)\right|<\varepsilon\,.$$

Consequently

$$\limsup_{t\to\infty}\int u(T_t\varphi)P^0(d\varphi)\le u(0)+\lim_{t\to\infty}\int u_1(T_t\varphi)P^0(d\varphi) =u(0)+\int u_1\cdot\nu dP+\varepsilon<\left\{\left(1-\int\nu dP\right)\right\}u(0)+\int u\cdot\nu dP+2\varepsilon\,.$$

The same argument gives

$$\liminf_{t\to\infty}\int u(T_t\varphi)P^0(d\varphi)\geq\left(1-\int\nu dP\right)\cdot u(0)+\int u\cdot\nu dP-2\varepsilon .$$

Hence we have

$$\lim_{t\to\infty}\int u(T_t\varphi)P^0(d\varphi)=\left(1-\int\nu dP\right)\cdot u(0)+\int u\cdot\nu dP , \tag{4.22}$$

which proves (4.11) with (4.12). The last assertion of the theorem follows from Lemma 4.4. This completes the proof.

Corollary 4.1. *Let P satisfy* (4.1) *and* (4.2). *Assume P^0 is a probability measure. Let I_i, $1\leq i\leq n$, be bounded intervals. If* (4.8) *holds then the random vector $(\varphi(I_1+t),\cdots,\varphi(I_n+t))$ on the probability space $(M,\mathfrak{M},P^0)$ converges in distribution as $t\to\infty$ to the random vector $(\varphi(I_1),\cdots,\varphi(I_n))$ on the probability space $(M,\mathfrak{M},P^\infty)$.*

Proof. Let h be the mapping from M to R^n which sends φ to $(\varphi(I_1),\cdots,\varphi(I_n))$ and D_h the set of discontinuities of h. We have $D_h\subset\{\varphi;\varphi(J)>0\}$, where J is the set of all end points of I_i, $1\leq i\leq n$. In fact if $\varphi(J)=0$, $1\leq i\leq n$, then for any $\varepsilon>0$ and for each i there exist $f_{i1}\in\mathfrak{C}_0^+$ and $f_{i2}\in\mathfrak{C}_0^+$ such that $f_{i1}\leq\chi(I_i;)\leq f_{i2}$, and

$$\int(f_{i2}(t)-f_{i1}(t))\varphi(dt)<\varepsilon .$$

Let

$$U=\{\psi;\ |z(f_{ij};\psi)-z(f_{ij};\varphi)|<\varepsilon\ ,\quad 1\leq i\leq n\ ,\quad j=1,2\} .$$

If $\psi\in U$ then it is easy to see that $|\psi(I_i)-\varphi(I_i)|<\varepsilon$, $1\leq i\leq n$. This shows that h is continuous at φ. Since $\int\varphi(J)P(d\varphi)=\lambda\cdot|J|=0$, $1\leq i\leq n$, we have

$$P(D_h)\leq P(\{\varphi;\varphi(J)>0\})=0 ,$$

and therefore $P^\infty(D_h)=0$. Thus from Theorem 5.1 of [1] and the preceding theorem we have that P^th^{-1} converges weakly to $P^\infty h^{-1}$. This proves the corollary.

5. A theorem of Ryll-Nardzewski. Throughout the rest of this paper we shall consider a strictly stationary two-sided sequence $\cdots,X_{-1},X_0,X_1,\cdots$ of real random variables defined on a probability space $(\Omega,\mathfrak{F},\mathbf{P})$. Let us denote

$$S_n=\sum_{k=1}^n X_k\quad\text{for } n\geq1\ ,\quad S_0\equiv0\ ,\quad S_n=-\sum_{k=n+1}^{0}X_k\quad\text{for } n\leq-1\ , \tag{5.1}$$

and for each $\omega\in\Omega$ let $\Phi(\omega)=\Phi(\cdot;\omega)$ be a measure on $\mathfrak{R}$ defined by

$$\Phi(\cdot\,;\omega)=\sum_{n=-\infty}^{\infty}\chi(\cdot:S_n(\omega))\,. \tag{5.2}$$

Let K and L be two independent random variables having the common geometric distribution:

$$\mathbf{P}\{K=j\}=\mathbf{P}\{L=j\}=(1-\alpha)\alpha^j\,,\quad j=0,1,\cdots,$$

where $0<\alpha<1$ and the pair (K, L) is assumed to be independent of X_n's. For every $\omega\in\Omega$ let $\Phi_\alpha(\omega)=\Phi_\alpha(\cdot\,;\omega)$, $0<\alpha<1$, be a measure defined by

$$\Phi_\alpha(\cdot\,;\omega)=\sum_{n=-K}^{L}\chi(\cdot\,;S_n(\omega))\,. \tag{5.3}$$

Let us assume that

$$\Phi(A;\omega)<\infty\ \mathbf{P}\text{-a.e. for every } A\in\mathfrak{R}_0\,. \tag{5.4}$$

If we denote by E the set of all ω such that $\Phi(A;\omega)<\infty$, $A\in\mathfrak{R}_0$, then $E=\bigcap_{n=1}^{\infty}\{\omega;\ \Phi((-n,n);\omega)<\infty\}$ and by (5.4) $\mathbf{P}(E)=1$. Thus $\Phi(\omega)\in M_0$ $\mathbf{P}$-a.e. Since $(z_A\circ\Phi)(\omega)=z_A(\Phi(\omega))=\Phi(A;\omega)=\sum_{n=-\infty}^{\infty}\chi(A;S_n(\omega))$, $A\in\mathfrak{R}$, $z_A\circ\Phi$ is measurable for every $A\in\mathfrak{R}$ and therefore Φ is a measurable mapping from $(\Omega,\mathfrak{F})$ into $(M,\mathfrak{M})$. Similarly every Φ_α, $0<\alpha<1$, is measurable even if (5.4) does not hold. Let $Q=\mathbf{P}\Phi^{-1}$ and $Q_\alpha=\mathbf{P}\Phi_\alpha^{-1}$, $0<\alpha<1$, be probability measures on $(M,\mathfrak{M})$ induced by Φ and Φ_α respectively. Obviously $Q(M_0)=Q_\alpha(M_0)=1$, $Q(\{0\})=Q_\alpha(\{0\})=0$.

The following theorem admits to connect the theory of stationary random measures to renewal theory. This theorem was first proved by Ryll-Nardzewski [9] for Q assuming that X_n's are positive and integrable. The present proof is an application of a result of [8].

Theorem 5.1. *The measure Q_α, $0<\alpha<1$, and if (5.4) is satisfied then the measure Q are the Palm measures for some σ-finite stationary measures P_α and P on $(M,\mathfrak{M})$ respectively. P_α and P are concentrated on M_0 and they may be assumed to satisfy $P_\alpha(\{0\})=0$ and $P(\{0\})=0$.*

Proof. We prove only the assertion on Q_α since a similar and easier argument can be applied to prove the assertion on Q.

For a proof it suffices to verify the conditions of Lemma 2.4. Conditions (i) and (ii) are obviously satisfied. Let us verify

$$\iint v(-t,\,T_t\varphi)\varphi(dt)Q_\alpha(d\varphi)=\iint v(t,\,\varphi)\varphi(dt)Q_\alpha(d\varphi) \tag{5.5}$$

for non-negative $\mathfrak{R}\times\mathfrak{M}$-measurable v. The right side of (5.5) may be written as follows:

$$\text{(5.6)}\qquad \iint v(t,\varphi)\varphi(dt)Q_\alpha(d\varphi)$$

$$=\int \sum_{j=-K(\omega)}^{L(\omega)} v(S_j(\omega),\ \Phi_\alpha(\cdot\,;\omega))\mathbf{P}(d\omega)$$

$$=\sum_{m=0}^{\infty}\ \sum_{n=0}^{\infty}\int \chi(E_{mn};\omega)\sum_{j=-m}^{n} v(S_j(\omega),\ \sum_{k=-m}^{n}\chi(\cdot\,;S_k(\omega)))\mathbf{P}(d\omega)$$

$$=\sum_{m=0}^{\infty}\ \sum_{n=0}^{\infty}(1-\alpha)^2\alpha^{m+n}\int \sum_{j=-m}^{n} v(S_j,\ \sum_{k=-m}^{n}\chi(\cdot\,;S_k))d\mathbf{P}$$

$$=\sum_{\nu=0}^{\infty}(1-\alpha)^2\alpha^{\nu}\int \sum_{m=0}^{\nu}\ \sum_{j=-m}^{\nu-m} v(S_j,\ \sum_{k=-m}^{\nu-m}\chi(\cdot\,;S_k))d\mathbf{P}\ ,$$

where $E_{mn}=\{\omega;\ K(\omega)=m,\ L(\omega)=n\}$. Similarly

$$\text{(5.7)}\qquad \iint v(-t,\ T_t\varphi)\varphi(dt)Q_\alpha(d\varphi)$$

$$=\sum_{m=0}^{\infty}\ \sum_{n=0}^{\infty}(1-\alpha)^2\alpha^{m+n}\int \sum_{j=-m}^{n} v(-S_j,\ \sum_{k=-m}^{n}\chi(\cdot\,;S_k-S_j))d\mathbf{P}$$

$$=\sum_{m=0}^{\infty}\ \sum_{n=0}^{\infty}(1-\alpha)^2\alpha^{m+n}\int \sum_{j=-m}^{n} v(S_{-j},\ \sum_{k=-m}^{n}\chi(\cdot\,;S_{k-j}))d\mathbf{P}$$

$$=\sum_{\nu=0}^{\infty}(1-\alpha)^2\alpha^{\nu}\int \sum_{m=0}^{\nu}\ \sum_{j=-m}^{\nu-m} v(S_j,\ \sum_{k=-m}^{\nu-m}\chi(\cdot\,;S_{j-k}))d\mathbf{P}\ ,$$

where the second equality follows from the stationarity of X_n. On the other hand for every $\nu\geq 0$ we have

$$\sum_{m=0}^{\nu}\ \sum_{j=-m}^{\nu-m} v(S_j,\ \sum_{k=-m}^{\nu-m}\chi(\cdot\,;S_{j-k}))=\sum_{m=0}^{\nu}\ \sum_{j=-m}^{\nu-m} v(S_j,\ \sum_{k=j+m-\nu}^{j+m}\chi(\cdot\,;S_k))$$

$$=\sum_{l=0}^{\nu}\ \sum_{j=-l}^{\nu-l} v(S_j,\ \sum_{k=-l}^{\nu-l}\chi(\cdot\,;S_k))\ .$$

This proves that the right sides of (5.6) and (5.7) coincide and proves the existence of P.

Since Q_α is concentrated on M_0 so is P_α. If $P_\alpha(\{0\})\neq 0$ then by modifying this value to be zero we obtain a stationary measure P_α satisfying the assertion of the theorem.

Let F_n denote the distribution of S_n:

$$F_n(A)=\mathbf{P}\{S_n\in A\}\ ,\quad A\in\mathfrak{R}\ ,\quad n=\cdots,-1,0,1,\cdots\ ,$$

and let V and V_α, $0<\alpha<1$, denote measures defined by

$$\text{(5.8)}\qquad V(A)=\sum_{n=-\infty}^{\infty}F_n(A)\ ,\quad A\in\mathfrak{R}\ ,$$

and

$$V_\alpha(A)=\sum_{n=-\infty}^{\infty}\alpha^{|n|}F_n(A)\,,\quad A\in\mathfrak{R}\,, \tag{5.9}$$

respectively. These notations are justified by the following:

Lemma 5.1.

$$V_\alpha(A)=\int\varphi(A)Q_\alpha(d\varphi)\,,\quad A\in\mathfrak{R}\,,\quad 0<\alpha<1\,. \tag{5.10}$$

If $V\in M$ *then* (5.4) *is satisfied and*

$$V(A)=\int\varphi(A)Q(d\varphi)\,,\quad A\in\mathfrak{R}\,. \tag{5.11}$$

Proof.

$$\begin{aligned}\int\varphi(A)Q_\alpha(d\varphi)&=\int\Phi_\alpha(A;\,\omega)\mathbf{P}(d\omega)\\&=\int\sum_{n=-K}^{L}\chi(A;\,S_n(\omega))\mathbf{P}(d\omega)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\mathbf{P}\{K=j,\,L=k\}\int\sum_{n=-j}^{k}\chi(A;\,S_n)d\mathbf{P}\\&=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(1-\alpha)^2\alpha^{j+k}\sum_{n=-j}^{k}F_n(A)=\sum_{n=-\infty}^{\infty}\alpha^{|n|}F_n(A)=V_\alpha(A)\,.\end{aligned}$$

If $V\in M$ then $\int\Phi(A;\,\omega)\mathbf{P}(d\omega)=V(A)<\infty$ for every $A\in\mathfrak{R}_0$ and therefore $\Phi(A;\,\cdot)<\infty$ $\mathbf{P}$-a.e. The equality (5.11) is easy.

Let M_1 denote the subset of M_0 consisting of all φ such that $\varphi((-\infty,a))=\infty$, $\varphi((a,\infty))=\infty$ and $\Phi(\{a\})=0$ or 1 for every $a\in R$. It is easy to see that M_1 is $\mathfrak{M}$-measurable and invariant. It follows from (2.9) that a stationary measure P on $(M,\mathfrak{M})$ satisfying $P(\{0\})=0$ is concentrated on M_1 iff so is the Palm measure P^0. For each $\varphi\in M_1$ let

$$\begin{aligned}&\zeta_0(\varphi)=\inf\{t;\,t\geq 0,\ \varphi([0,t])>0\}\\&\zeta_n(\varphi)=\inf\{t;\,t>\zeta_{n-1}(\varphi),\ \varphi((\zeta_{n-1}(\varphi),t])>0\}\,,\quad n\geq 1\,,\\&\zeta_{-1}(\varphi)=\sup\{t;\,t<0,\ \varphi([t,0))>0\}\,,\\&\zeta_n(\varphi)=\sup\{t;\,t<\zeta_{n+1}(\varphi),\ \varphi([t,\zeta_{n+1}(\varphi)))>0\}\,,\quad n\leq -2\,.\end{aligned}$$

and

$$\begin{aligned}&\xi_0'(\varphi)=-\zeta_{-1}(\varphi)\,,\quad \xi_0''(\varphi)=\zeta_0(\varphi)\,,\\&\xi_0(\varphi)=\xi_0'(\varphi)+\xi_0''(\varphi)\,,\\&\xi_n(\varphi)=\zeta_n(\varphi)-\zeta_{n-1}(\varphi)\,,\quad\text{for}\quad n\neq 0\,.\end{aligned}$$

All of these functions are $\mathfrak{M}$-measurable on M_1.

Next let the stationary sequence $\{X_n\}$ satisfy $X_n>0$ $\mathbf{P}$-a.e. Then (5.4) is satisfied and therefore by Theorem 5.1 there exists a stationary σ-finite measure P on $(M, \mathfrak{M})$ such that $P(\{0\})=0$ and its Palm measure P^0 coincides with $\mathbf{P}\Phi^{-1}$. Let Ω_0 be the set of ω such that $X_n(\omega)>0$ for all n, $\lim_{n\to\infty} S_n(\omega)=\infty$, and $\lim_{n\to-\infty} S_n(\omega)=-\infty$. Then $\mathbf{P}(\Omega_0)=1$ and $\Phi(\cdot\,;\omega)\in M_1$ if $\omega\in\Omega_0$. Hence P^0 and therefore P are concentrated on M_1.

The following theorem is a slight extension of a result stated in [7].

Theorem 5.2. *Assume the stationary sequence $\{X_n\}$ satisfy $X_n>0$ $\mathbf{P}$-a.e. Let $u\geq 0$ be an $\mathfrak{M}$-measurable invariant function and let $Z(\omega)=u(\Phi(\omega))$ $\mathbf{P}$-a.e. Then for any $m\geq 0$, $n\geq 0$ and any Baire function $f\geq 0$ on R^{m+n+2}*

$$(5.12)\qquad \int f(\xi_{-m}(\varphi),\cdots,\xi_{-1}(\varphi),\xi_0'(\varphi),\xi_0''(\varphi),\cdots,\xi_n(\varphi))u(\varphi)P(d\varphi)$$

$$=\int\left\{\int_0^{X_0} f(X_{-m},\cdots,X_{-1},X_0-t,t,X_1,\cdots,X_n)dt\right\}\cdot Z d\mathbf{P}\,.$$

In particular for $t'\geq 0$, $t''\geq 0$ we have

$$(5.13)\qquad P(\{\varphi;\,\xi_0'(\varphi)\geq t',\ \xi_0''(\varphi)\geq t''\})=\int_{t'+t''}^{\infty}\{1-F(t)\}dt\,,$$

where $F(t)$ is the distribution function of X_0.

Proof. For simplicity write

$$v(\varphi)=f(\xi_{-m}(\varphi),\cdots,\xi_{-1}(\varphi),\xi_0'(\varphi),\xi_0''(\varphi),\xi_1(\varphi),\cdots,\xi_n(\varphi))\,.$$

For every $\varphi\in M_1$ let $h(t,\varphi)=1$ if $t>0$ and $\varphi((0,t))=0$, $h(t,\varphi)=0$ otherwise. Then

$$\int h(t,\varphi)\varphi(dt)=1\,,\quad \varphi\in M_1\,.$$

For $\varphi\in M_1$, $h(t, T_{-t}\varphi)=1$ iff $0<t\leq\xi_0'(\varphi)$. If $\omega\in\Omega_0$ then $\Phi(\omega)\in M_1$ and $\xi_0'(\Phi(\omega))=X_0(\omega)$. If $\omega\in\Omega_0$ and $0<t<X_0(\omega)$ then

$$\xi_0'(T_{-t}(\Phi(\omega)))=X_0(\omega)-t\,,\quad \xi_0''(T_{-t}(\Phi(\omega)))=t\,,$$

$$\xi_k(T_{-t}(\Phi(\omega)))=X_k(\omega)\,,\quad k\neq 0\,.$$

Thus it follows from Lemma 2.3 that

$$\int v(\varphi)u(\varphi)P(d\varphi)=\iint v(T_{-t}\varphi)u(\varphi)h(t,T_{-t}\varphi)dtP(d\varphi)$$

$$=\int\left\{\int_0^{\xi_0'(\varphi)} v(T_{-t}\varphi)u(\varphi)dt\right\}P^0(d\varphi)$$

$$=\int\left\{\int_0^{X_0} f(X_{-m},\cdots,X_{-1},X_0-t,t,X_1,\cdots,X_n)dt\right\}Z d\mathbf{P}\ .$$

Let $t'\geq 0$, $t''\geq 0$, $m=n=0$ and $f(x,y)=1$ if $x\geq t'$, $y\geq t''$, $f(x,y)=0$ otherwise. Then it is easy to see that

$$\int_0^{X_0} f(X_0-t,t)dt=(X_0-(t'+t''))^+$$

Hence by letting $u\equiv 1$ in (5.12) we have

$$P\{\xi_0'(\varphi)\geq t',\xi_0''(\varphi)\geq t''\}=\int(X_0-(t'+t''))^+ d\mathbf{P}=\int_{t'+t''}^\infty\{1-F(t)\}dt\ .$$

6. Renewal theory for sums of stationary sequences. For two-sided stationary sequence $\{X_n\}$ let V and V_α be measures defined by (5.8) and (5.9) respectively. It is obvious that for every $A\in\mathfrak{R}$, $V_\alpha(A)$ tends to $V(A)$ as $\alpha\to 1-0$. Assume $V\in M$ and let G and G_α be Fourier transforms of V and V_α respectively. It is easy to see that

$$\lim_{\alpha\to 1-0}\int f(t)V_\alpha(dt)=\int f(t)V(dt)\ ,\quad f\in(\mathscr{S})\ , \tag{6.1}$$

and

$$\lim_{\alpha\to 1-0}\int f(t)G_\alpha(dt)=\int f(t)G(dt)\ ,\quad f\in(\mathscr{S})\ . \tag{6.2}$$

Let f_n be the characteristic function of S_n:

$$f_n(t)=\int e^{itx}F_n(dx)\ ,\quad n=\cdots,-1,0,1,\cdots\ .$$

Then it follows from (5.9) that G_α is represented as

$$G_\alpha(A)=\int_A \gamma_\alpha(t)dt\ ,\quad A\in\mathfrak{R}\ , \tag{6.3}$$

where

$$\gamma_\alpha(t)=\frac{1}{2\pi}[1+2\sum_{n=1}^\infty \alpha^n\,\mathrm{Re}\,f_n(t)]\ .$$

From (3.8), (6.1) and (6.3) it is easy to prove the following lemma which reduces to the Chung-Fuchs criterion [2] when X_n's are i.i.d.

Lemma 6.1. *$V\in M$ iff for some non-zero $f\in(\mathscr{S})$ such that $f\geq 0$ and $\hat{f}\geq 0$*

(6.4)
$$\limsup_{\alpha\to 1-0}\int f(t)\gamma_\alpha(t)dt<\infty\ ,$$

or equivalently iff for some $\varepsilon>0$,

(6.5)
$$\limsup_{\alpha\to 1-0}\int_{-\varepsilon}^{\varepsilon}\gamma_\alpha(t)dt<\infty\ .$$

It is immediate from Lemma 4.2 and Theorem 5.1 that if $V\in M$ then $(2a)^{-1}\times\Phi(I_a)$ converges to a random variable N **P**-a.e. and in $L^1(\mathbf{P})$. When $Y=\lim_{n\to\infty} n^{-1}S_n$ exists **P**-a.e., in particular when $\mathbf{E}|X_0|<\infty$, it can be shown that $N=|Y|^{-1}$, **P**-a.e.

As an immediate consequence of Theorem 3.2 we obtain the following renewal theorem for sums of stationary sequences.

Theorem 6.1. *If* $V\in M$ *then*

(6.6)
$$\lim_{a\to\infty}\frac{1}{2a}V(I_a)=\mathbf{E}N\equiv\lambda<\infty\ .$$

If, in addition, γ_α, $0<\alpha<1$, *are uniformly bounded on every compact interval excluding the origin and converges a.e. as* $\alpha\to 1-0$, *then*

(6.7)
$$\lim_{t\to\infty}V(I+t)=\lambda\cdot|I|$$

for every bounded interval I.

Example 6.1 (*Gaussian random variables*). Let $\{X_n\}$ be stationary Gaussian with $\mathbf{E}X_n=\mu$ and $\mathrm{Var}\,(S_n)=s_n^2$. Applying Lemma 6.1 with $f(t)=\exp(-t^2/2)$ it is found that $V\in M$ iff

$$\limsup_{\alpha\to 1-0}\sum_{n=1}^{\infty}\alpha^n\int e^{-t^2/2}e^{(-s_n{}^2/2)t^2}\cos n\mu t\,dt<\infty\ .$$

It is easy to see that this is equivalent to

(6.8)
$$\sum_{n=1}^{\infty}\exp\left(-\frac{n^2\mu^2}{2(1+s_n^2)}\right)<\infty\ .$$

From Theorem 6.1 it is found that (6.7) holds if (6.8) and

(6.9)
$$\sum_{n=1}^{\infty}\exp(-s_n^2t)<\infty\ ,\quad\text{for}\quad t>0\ ,$$

are satisfied.

Example 6.2 (*identical random variables*). Let X be a r.v. with distribution F and characteristic function f. If $X_n=X$ for every n then $S_n=nX$ and

$f_n(t)=f(nt)$. It is easy to see that $V\in M$ iff $\mathbf{E}(|X|^{-1})<\infty$ and that $\lambda=\mathbf{E}(|X|^{-1})$.

Let H denote the distribution of X^{-1} and $K(dy)=|y|H(dy)$. Assume

$$\mathbf{E}(|X|^{-1})=\int|x|^{-1}F(dx)=\int|x|H(dx)=\int K(dx)<\infty .$$

It follows from (6.3) that for $g\in(\mathscr{S})$

$$\int g(t)G_\alpha(dt)=\frac{1}{2\pi}\int\Big[1+2\sum_{n=1}^{\infty}\alpha^n \operatorname{Re} f(nt)\Big]\cdot g(t)dt$$
$$=\int F(dx)\frac{1}{2\pi|x|}\int_{-\pi}^{\pi}\frac{1-\alpha^2}{1-2\alpha\cos u+\alpha^2}\sum_{k=-\infty}^{\infty}g\Big(\frac{u+2k\pi}{|x|}\Big)du .$$

Letting $\alpha\to 1-0$ we have

$$\int g(t)G(dt)=\int\sum_{k=-\infty}^{\infty}|x|^{-1}g\Big(\frac{2k\pi}{x}\Big)F(dx)=\sum_{k=-\infty}^{\infty}\int g(2k\pi y)K(dy) .$$

Thus

$$G(\cdot)=\lambda\cdot\chi(\cdot\,;0)+\sum_{k\neq 0}K\Big(\frac{\cdot}{2k\pi}\Big) . \tag{6.10}$$

This shows that G is absolutely continuous except for an atom at the origin iff F is absolutely continuous. Hence from Theorem 6.1 (6.7) holds if $\mathbf{E}(|X|^{-1})<\infty$ and F is absolutely continuous.

If $F(dy)=p(y)dy$ then $K(dy)=|y|^{-1}p(y^{-1})dy$ and from (6.10) we have

$$\gamma(t)=|t|^{-1}\sum_{k\neq 0}p(2k\pi t^{-1}) \text{ a.e.} \tag{6.11}$$

This may be regarded as a variant of Poisson's summation formula.

Throughout the rest we assume $V\in M$. Let Φ be a random element of M defined in §5 and $\Phi_t(\omega)=\Phi_t(\cdot\,;\omega)=\Phi(\cdot+t;\omega)$, $t\in R$. Let P be the stationary measure on $(M,\mathfrak{M})$ defined in Theorem 5.1 and $P^0=\mathbf{P}\Phi^{-1}$ the Palm measure for P. It is obvious that $P^t=P^0T_{-t}=\mathbf{P}\Phi_t^{-1}$. Thus by Theorem 4.1 we have immediately the following:

Theorem 6.2. *In order that the random element Φ_t converge in distribution as $t\to\infty$ it is necessary and sufficient that the limit*

$$\lim_{t\to\infty}\mathbf{E}\Phi_t(I)=\lim_{t\to\infty}\mathbf{E}\Phi(I+t)=\lim_{t\to\infty}\sum_{n=-\infty}^{\infty}\mathbf{P}(S_n\in I+t) \tag{6.12}$$

exist for every bounded interval I. The limit distribution of Φ_t is stationary and given by $P^\infty=\nu\cdot P+c\delta_0$, $c=1-\int\nu dP$.

The following corollary is immediate from Corollary 4.1.

Corollary 6.1. *Let* I_i, $1\leq i\leq m$, $m\geq 1$, *be bounded intervals. If* (6.12) *holds then the random vector* $(\Phi(I_1+t),\cdots,\Phi(I_m+t))$ *converges in distribution as* $t\to\infty$.

Let Φ^+ and Φ^- be random elements of M defined by

$$\Phi^+(\omega)=\Phi^+(\cdot\,;\omega)=\sum_{n=0}^{\infty}\chi(\cdot\,;S_n(\omega))\ ,\quad \Phi^-(\omega)=\Phi(\omega)-\Phi^+(\omega)\ ,$$

and let $\Phi_t^+(\cdot\,;\omega)=\Phi^+(\cdot+t;\omega)$ and $\Phi_t^-(\cdot\,;\omega)=\Phi^-(\cdot+t;\omega)$, $t\in R$.

Corollary 6.2. *Let* $\{X_n\}$ *be i.i.d.,* $\{S_n\}$ *a transient random walk. If the distribution of* X_0 *is aperiodic then* Φ_t *and* Φ_t^+ *converge in distribution as* $t\to\infty$.

Proof. The assertion on Φ_t follows from Theorem 6.2 and Feller-Orey's renewal theorem [5]. Let U be any neighborhood of $0\in M$. It follows from Feller-Orey's theorem that $\lim_{t\to\infty}\mathbf{P}\{\Phi_t^+\in U\}=1$ if either $\mathbf{E}|X_0|=\infty$ or $-\infty<\mathbf{E}X_0<0$, and $\lim_{t\to\infty}\mathbf{P}\{\Phi_t^-\in U\}=1$ if $0<\mathbf{E}X_0<\infty$. These facts prove the assertion of Φ_t^+.

For the rest we assume $X_0>0$ $\mathbf{P}$-a.e. and write $N(t)=\Phi((0,t))$, $t>0$. Let $Z_0'(t)=t-S_{N(t)}$, $Z_0''(t)=S_{N(t)+1}-t$, and $Z_k(t)=S_{N(t)+k+1}-S_{N(t)+k}$ for $k\neq 0$. Then we have

Corollary 6.3. *Assume* $X_0>0$ $\mathbf{P}$-a.e. *If* (6.12) *holds and if* $Y=\lim_{n\to\infty}n^{-1}S_n<\infty$ $\mathbf{P}$-a.e., *then for any* $m\geq 0$, $n\geq 0$, *and for any bounded continuous function* f *on* R^{m+n+2} *we have*

$$\lim_{t\to\infty}\mathbf{E}f(Z_{-m}(t),\cdots,Z_{-1}(t),Z_0'(t),Z_0''(t),Z_1(t),\cdots,Z_n(t))$$
$$=\mathbf{E}\left\{Y^{-1}\cdot\int_0^{X_0}f(X_{-m},\cdots,X_{-1},X_0-s,s,X_1,\cdots,X_n)ds\right\}\ .$$

Proof. Let us use notations in §5 and write for $\varphi\in M_1$

$$u(\varphi)=f(\xi_{-m}(\varphi),\cdots,\xi_{-1}(\varphi),\xi_0'(\varphi),\xi_0''(\varphi),\xi_1(\varphi),\cdots,\xi_n(\varphi))\ .$$

Then u has a bounded continuous extension on M and

$$\mathbf{E}f(Z_{-m}(t),\cdots,Z_{-1}(t),Z_0'(t),Z_0''(t),Z_1(t),\cdots,Z_n(t))$$
$$=\int u(\Phi_t)d\mathbf{P}=\int u(T_t\varphi)P^0(d\varphi)\ .$$

Since $Y^{-1}=N=\nu(\Phi)$ $\mathbf{P}$-a.e. the assumption $Y<\infty$ implies that $\nu>0$ P^0-a.e. Thus it follows from Theorem 4.1 and Theorem 5.2 that

$$\lim_{t\to\infty}\int u(T_t\varphi)P^0(d\varphi)=\int u(\varphi)\nu(\varphi)P(d\varphi)$$
$$=\int Y^{-1}\cdot\left\{\int_0^{X_0} f(X_{-m},\cdots,X_{-1},X_0-s,s,X_1,\cdots,X_n)ds\right\}d\mathbf{P}\,.$$

Corollary 6.4. *Assume* $X_n>0$ *and* $\lim_{n\to\infty} n^{-1}S_n=Y<\infty$ **P**-a.e. *If* (6.12) *holds then for any* $y'>0$ *and* $y''>0$

$$\lim_{t\to\infty}\mathbf{P}\{Z_0'(t)\geq y',\,Z_0''(t)\geq y''\}=\mathbf{E}\{Y^{-1}(X_0-(y'+y''))^+\}\,.$$

In particular if $0<\mu=\mathbf{E}X_0<\infty$ *and* $Y=\mu$ **P**-a.e. *then*

$$\lim_{t\to\infty}\mathbf{P}\{Z_0'(t)\geq y',\,Z_0''(t)\geq y''\}=\mu^{-1}\cdot\int_{y'+y''}^{\infty}\{1-F(t)\}dt\,,$$

where $F(t)$ *is the distribution function of* X_0.

Proof. The boundary of the set $\Lambda=\{\varphi;\,\xi_0'(\varphi)\geq y',\,\xi_0''(\varphi)\geq y''\}$ is contained in the set $\{\varphi;\,\varphi(\{y'\})>0$ or $\varphi(\{y''\})>0\}$ which has P-measure zero. Hence by Theorem 4.1 $\mathbf{P}\{Z_0'(t)\geq y',\,Z_0''(t)\geq y''\}=P^t(\Lambda)$ converges to $P^\infty(\Lambda)=\int_\Lambda \nu\cdot dP$, which is identical with $\mathbf{E}\{Y^{-1}\cdot(X_0-(y'+y''))^+\}$ by Theorem 5.2.

REFERENCES

[1] Billingsley, P.: *Convergence of Probability Measures*, Wiley, New York, 1968.

[2] Chung, K. L. and Fuchs, W. H. J.: *On the distribution of values of sums of random variables*, Mem. Amer. Math. Soc. No. 6, 1951.

[3] Daley, D. J.: *Weakly stationary point processes and random measures*, J. Roy. Statist. Soc. B, 33 (1971), 406–428.

[4] Feller, W.: *An Introduction to Probability Theory and its Applications*, Vol. II, Wiley, New York, 1966.

[5] Feller, W. and Orey, S.: *A renewal theorem*, J. Math. Mech. 10 (1961), 619–624.

[6] Ito, K.: *Stationary random distributions*, Mem. Coll. Sci. Univ. Kyoto, 28 (1954), 209–223.

[7] Matthes, K.: *Stationäre zufällige Punktfolgen I*, Jber. Deutsch Math.-Verein., 66 (1963), 66–79.

[8] Mecke, J.: *Stationäre zufällige Masse auf lokalkompakten Abelschen Gruppen*, Z. Wahrscheinlichkeitstheorie verw. Geb. 9 (1967), 36–58.

[9] Ryll-Nardzewski, C.: *Remarks on processes of calls*, Proc. Fourth Berkley Symp. Math. Statist. Probab., 2 (1961), 455–465.

[10] Schwartz, L.: *Théorie des Distributions*, Hermann, Paris, 1966.

[11] Vere-Jones, D.: Appendix to [3].

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